Random Weighting and Edgeworth Expansion for the Nonparametric Time-Dependent AUC Estimator

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Supplementary Material

Let $\tau < \sup\{u : S_X(u) > 0\}$ with $S_X(u) = P(X > u)$ and $\tau_0 = \inf\{u : P(T > u) < 1, u \in [\tau_0, \tau]\}$. The second condition (**A2**: the cumulative hazard function $\Lambda_C(t)$ of C is bounded for $t \in [0, \tau]$) is further assumed throughout the rest of this paper. Some concise notations below are used to simplify the complicated mathematical expressions: $U_{nt} = \sum_{i \neq j} H_{ijt}/[n(n-1)]$ with $H_{ijt} = (S_C(t) - D^{*c}_{it})D^{*c}_{jt}(\phi_{ij} - \theta_t) - \eta_t S_C(t) \int_0^t dM_i(u)/S_X(u^-)$, $\nu_t = S^2_C(t)S_T(t))(1 - S_T(t))$, and η_t being defined in Section 2.2.

S1. Asymptotic Normality of $\widehat{\theta}_t$

From (1.1), one has

$$n^{1/2}(\widehat{\theta}_t - \theta_t) = n^{1/2}(\frac{U_{nt}}{V_{nt}} + r_{1nt}), \tag{S1.1}$$

where V_{nt} is defined in Section 2.2 and

$$r_{1nt} = \frac{\eta_t S_C(t)}{V_{nt}}[\int_0^t \frac{d\bar{M}.(u)}{S_X(u^-)} - (1 - \frac{\widehat{S}_C(t)}{S_C(t)})\frac{\sum_{i \neq j} D^{*c}_{jt}(\phi_{ij} - \theta_t)}{n(n-1)\eta_t}].$$

By the boundedness of D^{*c}_{it}'s and the consistency of $\widehat{S}_C(t)$ and a U-statistic, it follows that

$$V_{nt} \xrightarrow{p} \nu_t \text{ as } n \to \infty. \tag{S1.2}$$

Again, the consistency of a U-statistic and the martingale representation

$$n^{1/2}(\widehat{S}_C(t) - S_C(t)) = -n^{-1/2} S_C(t) \sum_{i=1}^{n} \int_0^t \frac{dM_i(u)}{S_X(u)} + o_p(1), \tag{S1.3}$$

where $M_i(t) = I(X_i \leq t)(1-\delta_i) - \int_0^t I(X_i \geq u) d\Lambda_C(u)$, entail that

$$n^{1/2}|r_{1nt}| \xrightarrow{p} 0. \tag{S1.4}$$

From (S1.2) and (S1.4), we get

$$n^{1/2}(\widehat{\theta}_t - \theta_t) = \frac{n^{1/2} U_{nt}}{\nu_t} + o_p(1). \tag{S1.5}$$

It follows immediately from Hoeffding (1948) that

$$n^{1/2} U_{nt} = n^{-1/2} \sum_{i=1}^{n} \Psi_{it} + o_p(1) \text{ with } \Psi_{it} = E(H_{ijt} + H_{jit} | X_i, Y_i, \delta_i). \tag{S1.6}$$

Together with (S1.5), $n^{1/2}(\widehat{\theta}_t - \theta_t)$ is shown to converge weakly to a normal distribution with mean zero and variance $\sigma_t^2 = \nu_t^{-2} E(\Psi_{it}^2)$.

S2. Normality Approximated Confidence Interval for θ_t

The asymptotic normality of $\widehat{\theta}_t$ and the consistent estimator $\widehat{\sigma}_t^2$ of σ_t^2 enable us to construct an approximated $(1-\alpha)$, $0 < \alpha < 1$, confidence interval for θ_t via

$$(\widehat{\theta}_t - n^{-1/2}\widehat{\sigma}_t z_{\alpha/2}, \widehat{\theta}_t + n^{-1/2}\widehat{\sigma}_t z_{\alpha/2}), \tag{S2.1}$$

where z_p is the $100p$ percentile point of a standard normal distribution. It is naturally to estimate the asymptotic variance by

$$\widehat{\sigma}_t^2 = \frac{\sum_{i=1}^{n} \widehat{\Psi}_{it}^2}{n \widehat{\nu}_t^2} \tag{S2.2}$$

where $\widehat{\nu}_t = \widehat{S}_C^2(t) \widehat{S}_T(t)(1 - \widehat{S}_T(t))$ and $\widehat{\Psi}_{it} = n^{-1} \sum_{j \neq i} (\widehat{H}_{ijt} + \widehat{H}_{jit})$ with

$$\widehat{H}_{ijt} = (\widehat{S}_C(t) - D_{it}^{*c}) D_{jt}^{*c} (\phi_{ij} - \widehat{\theta}_t) - \widehat{\eta}_t \widehat{S}_C(t) \int_0^t \frac{d\widehat{M}_i(u)}{\widehat{S}_X(u)}, \tag{S2.3}$$

$\widehat{M}_i(t) = I(X_i \le t)(1-\delta_i) - \int_0^t I(X_i \ge u) d\widehat{\Lambda}_C(u)$, $\widehat{\Lambda}_C(t) = \widehat{S}_{X\delta}(t)\widehat{S}_X^{-1}(t)$, $\widehat{S}_X(t) = n^{-1}\sum_{i=1}^n I(X_i > t)$, and $\widehat{S}_{X\delta}(t) = n^{-1}\sum_{i=1}^n I(X_i > t)(1-\delta_i)$. By the consistency of $\widehat{S}_C(t)$, $\widehat{S}_T(t)$, $\widehat{\theta}_t$, and $\widehat{\eta}_t$, and the uniform convergence of $\widehat{S}_X(t)$ and $\widehat{S}_{X\delta}(t)$, we have

$$\widehat{\sigma}_t^2 = \frac{\sum_{i,j,k}(H_{ijt} + H_{jit})(H_{ikt} + H_{kit})}{n^3 \nu_t^2} + o_p(1). \tag{S2.4}$$

Finally, the consistency of a U-statistic ensures that the dominating term in (S2.4) converges to $\nu_t^{-2} E((H_{ijt} + H_{jit})(H_{ikt} + H_{kit})) = \sigma_t^2$.

S3. Proof of Main Results

Proof of Theorem 2.1. Let the corresponding random weighting analogues of U_{nt} and V_{nt} be separately denoted by U_{nt}^w and V_{nt}^w. Thus, $(\widehat{\theta}_t^w - \widehat{\theta}_t)$ can be expressed as $(U_{nt}^w - U_{nt})/V_{nt}^w - r_{1nt} + r_{2nt} + r_{3nt}$ with

$$r_{2nt} = U_{nt}(\frac{1}{V_{nt}^w} - \frac{1}{V_{nt}}), r_{3nt} = \frac{\eta_t S_C(t)}{V_{nt}^w}[\int_0^t \frac{dM_\cdot^w(u)}{S_X(u^-)} - (1 - \frac{\widehat{S}_C^w(t)}{S_C(t)})\frac{\sum_{i\neq j} w_i w_j D_{jt}^{*c}(\phi_{ij} - \theta_t)}{\eta_t}].$$

It is implied from $P(n^{1/2}|r_{1nt}| > \varepsilon | D_n) = I(n^{1/2}|r_{1nt}| > \varepsilon)$ and (S1.4) that

$$P(n^{1/2}|r_{1nt}| > \varepsilon | D_n) \xrightarrow{p} 0. \tag{S3.1}$$

As for the convergence of V_{nt}^w to V_{nt} in r_{2nt} and r_{3nt}, a direct calculation first shows that

$$V_{nt}^w - V_{nt} = \sum_{i\neq j}(w_i w_j - \frac{1}{n(n-1)})(\widehat{S}_C(t) - D_{it}^{*c})D_{jt}^{*c} + (\widehat{S}_c^w(t) - \widehat{S}_c(t))(\sum_i w_i D_{it}^{*c}). \tag{S3.2}$$

The convergence property of Hoeffding (1961) yields that

$$n(n-1)\sum_{i\neq j}(w_i w_j - \frac{1}{n(n-1)})^2 \xrightarrow{p} \rho^{-2}(\rho^{-2} + 2). \tag{S3.3}$$

Using the boundedness of $(\widehat{S}_C(t) - D_{it}^{*c})D_{jt}^{*c}$ and (S3.3), one has

$$|\sum_{i\neq j}(w_i w_j - \frac{1}{n(n-1)})(\widehat{S}_C(t) - D_{it}^{*c})D_{jt}^{*c}| \xrightarrow{p} 0. \tag{S3.4}$$

Thus, the properties of $\widehat{S}_C^w(t) \xrightarrow{p} S_C(t)$, $\sum_{i=1}^n w_i D_{it}^{*c} \xrightarrow{p} S_X(t)$, and (S3.4) imply that

$$V_{nt}^w - V_{nt} \xrightarrow{p} 0 \text{ as } n \to \infty. \tag{S3.5}$$

It is entailed by the convergence of a U-statistic, (S3.5), and the Slutsky's theorem that

$$P(n^{1/2}|r_{2nt}| > \varepsilon | D_n) \xrightarrow{p} 0 \text{ as } n \to \infty. \tag{S3.6}$$

By Lemma S3.1, (S3.5), and $\sum_{i \neq j} w_i w_j D_{jt}^{*c}(\phi_{ij} - \theta_t) \xrightarrow{p} \eta_t$, we further derive that

$$P(n^{1/2}|r_{3nt}| > \varepsilon | D_n) \xrightarrow{p} 0. \tag{S3.7}$$

It is shown from (S3.1) and (S3.6)-(S3.7) that (2.2) is ascertained if

$$\sup_{x \in R} |P(\frac{n^{1/2}\rho(U_{nt}^w - U_{nt})}{V_{nt}^w} \leq x | D_n) - P(\frac{n^{1/2}U_{nt}}{V_{nt}} \leq x)| \xrightarrow{p} 0 \text{ as } n \to \infty. \tag{S3.8}$$

From the result of Janssen (1994), one can ensure that

$$\sup_{x \in R} |P(n^{1/2}\rho(U_{nt}^w - U_{nt}) \leq x | D_n) - P(n^{1/2}U_{nt} \leq x)| \xrightarrow{p} 0 \text{ as } n \to \infty. \tag{S3.9}$$

Together with (S1.2) and (S3.5), (S3.8) is derived by applying the Slutsky's theorem.

Lemma S3.1. Suppose that assumptions (A1)-(A2) are satisfied. Then, for any $\varepsilon > 0$,

$$P(n^{1/2}|1 - \frac{\widehat{S}_C^w(t)}{S_C(t)} - \int_0^t \frac{dM_\cdot^w(u)}{S_X(u^-)}| > \varepsilon | D_n) \xrightarrow{p} 0 \text{ as } n \to \infty. \tag{S3.10}$$

Proof. By the integration by parts and the right-continuity of $\widehat{S}_C^w(t)$, one has

$$1 - \frac{\widehat{S}_C^w(t)}{S_C(t)} = \int_0^t \frac{\widehat{S}_C^w(u^-)I(R_\cdot^w(u) > 0)}{S_C(u)R_\cdot^w(u)} dM_\cdot^w(u) - B^w(t), \tag{S3.11}$$

where $B^w(t) = \int_0^t (\widehat{S}_C^w(u^-)/S_C(u))I(R_\cdot^w(u) = 0)d\Lambda_C(u)$. Thus, the conditional probability in (S3.10) is shown to satisfy the following probability inequality:

$$\begin{aligned} P(n^{1/2}|1 - \frac{\widehat{S}_C^w(t)}{S_C(t)} - \int_0^t \frac{dM_\cdot^w(u)}{S_X(u^-)}| > \varepsilon | D_n) &\leq P(|n^{1/2}B^w(t)| > \frac{\varepsilon}{2} | D_n) \\ &+ P(|n^{1/2} \int_0^t [\frac{\widehat{S}_C^w(u^-)I(R_\cdot^w(u) > 0)}{S_C(u)R_\cdot^w(u)} - \frac{1}{S_X(u^-)}] dM_\cdot^w(u)| > \frac{\varepsilon}{2} | D_n). \end{aligned} \tag{S3.12}$$

Paralleling the argument of Fleming and Harrington (1991), we can derive that

$$\sup_{0\le u\le t} |n^{1/2}B^w(u)| \le n^{1/2}(1-S_C(t))I(R_{\cdot}^{w}(t)=0). \tag{S3.13}$$

It is further implied that

$$E(P(|n^{1/2}I(R_{\cdot}^{w}(t)=0)| > \varepsilon|D_n)) = (P(\xi_1=0)S_X(t))^n. \tag{S3.14}$$

Combining (S3.13)-(S3.14), we have

$$P(|n^{1/2}B^w(t)| > \frac{\varepsilon}{2}|D_n) \xrightarrow{p} 0 \text{ as } n\to\infty. \tag{S3.15}$$

The Lenglart's inequality yields that for any ε_1, $\varepsilon_2 > 0$,

$$\begin{aligned} &E(P(n^{1/2}\sup_{0\le s\le\tau}|\int_0^s [\frac{\widehat{S}_C^w(u^-)I(R_{\cdot}^w(u)>0)}{S_C(u)R_{\cdot}^w(u)} - \frac{1}{S_X(u^-)}]dM_{\cdot}^w(u)| > \varepsilon_1|D_n)) \\ &< \frac{\varepsilon_2}{\varepsilon_1^2} + P(\Delta_n\int_0^\tau [\frac{\widehat{S}_C^w(u^-)I(R_{\cdot}^w(u)>0)}{S_C(u)R_{\cdot}^w(u)} - \frac{1}{S_X(u^-)}]^2 d\Lambda_C(u) > \varepsilon_2), \end{aligned} \tag{S3.16}$$

where $\Delta_n = n\sum_{i=1}^n w_i^2$. By applying the uniform convergence of $\widehat{S}_C(t)$ (Shorack and Wellner (1986)) and the uniform strong law of large numbers for $R_{\cdot}(t)$ (Pollard (1990)) with respect to t to $\widehat{S}_C^w(t)$ and $R_{\cdot}^w(t)$, we derive that

$$\Delta_n\int_0^\tau [\frac{\widehat{S}_C^w(u^-)I(R_{\cdot}^w(u)>0)}{S_C(u)R_{\cdot}^w(u)} - \frac{1}{S_X(u^-)}]^2 d\Lambda_C(u) \xrightarrow{p} 0 \text{ as } n\to\infty. \tag{S3.17}$$

From (S3.16)-(S3.17), it follows that

$$P(|n^{1/2}\int_0^t [\frac{\widehat{S}_C^w(u^-)I(R_{\cdot}^w(u)>0)}{S_C(u)R_{\cdot}^w(u)} - \frac{1}{S_X(u^-)}]dM_{\cdot}^w(u)| > \frac{\varepsilon}{2}|D_n) \xrightarrow{p} 0 \text{ as } n\to\infty. \tag{S3.18}$$

Together with (S3.12) and (S3.15), the proof of (S3.10) is completed . □

Proof of Theorem 2.2. From (2.6), an alternative expression of $\widehat{\theta}_t^{(s)}$ is derived to be

$$\widehat{\theta}_t^{(s)} = \frac{n^{1/2}(U_{nt}^{(s)} + 4n^{-1}S_C(t)E(h_{1t}^{(0)}h_{1t}^{(\eta)}) + \sum_{j=0}^3 r_{jnt}^{(s)})}{\widehat{\sigma}_{nt}}, \tag{S3.19}$$

where

$$r_{0nt}^{(s)} = \frac{\sum_{i\neq j}[2(H_{ijt}^{(km)}h_{it}^{(\eta)} + H_{ijt}^{(km)}h_{jt}^{(\eta)}) - 4S_C(t)E(h_{1t}^{(0)}h_{1t}^{(\eta)})]}{n^2(n-1)}, r_{1nt}^{(s)} = r_{nt}^{(km)}U_{nt}^{(2)},$$

$$r_{2nt}^{(s)} = \frac{2\sum_{i\neq j\neq l}H_{ijt}^{(km)}h_{lt}^{(\eta)}}{n^2(n-1)}, r_{3nt}^{(s)} = U_{nt}^{(km)}\Psi_{nt}^{(\eta)} \text{ with } \Psi_{nt}^{(\eta)} = U_{nt}^{(2)} - 2\bar{h}_{\cdot t}^{(\eta)} \text{ and } \bar{h}_{\cdot t}^{(\eta)} = \frac{\sum_{l=1}^n h_{lt}^{\eta}}{n}.$$

It is entailed from $E(r_{0nt}^{(s)2}) = O(n^{-3})$ that

$$P(n^{1/2}|r_{0nt}^{(s)}| \geq n^{-1/2}(\ln n)^{-1}) = O(n^{-2}\ln n)^2). \quad \text{(S3.20)}$$

Lemma S3.2 below and the boundedness of D_{1t}^{*c}, ϕ_{12}, and θ_t ensure that

$$P(n^{1/2}|r_{1nt}^{(s)}| > n^{-1/2}(\ln n)^{-1}) = o(n^{-1/2}). \quad \text{(S3.21)}$$

Since the projection of a U-statistic $r_{2nt}^{(s)}$ is 0 and $E(r_{2nt}^{(s)2}) = O(n^{-2})$ (Hoeffding (1948)), we have

$$P(n^{1/2}|r_{2nt}^{(s)}| > n^{-1/2}(\ln n)^{-1}) = o(n^{-1/2}). \quad \text{(S3.22)}$$

The probability inequality and $P(n^{1/2}|U_{nt}^{(km)}| > (\ln n)^{1/2}) = o(n^{-1/2})$ (Malevich and Abdalimov (1979)) yield

$$P(n^{1/2}|r_{3nt}^{(s)}| > (n\ln n)^{-1/2}) \leq P(|\Psi_{nt}^{(\eta)}| > n^{-1/2}(\ln n)^{-1}) + o(n^{-1/2}). \quad \text{(S3.23)}$$

The Chebyshev's inequality and $E(\Psi_{nt}^{(\eta)2}) = O(n^{-2})$ imply that $P(|\Psi_{nt}^{(\eta)}| > n^{-1/2}(\ln n)^{-1})$ $= o(n^{-1/2})$ and, hence,

$$P(n^{1/2}|r_{3nt}^{(s)}| > (n\ln n)^{-1/2}) = o(n^{-1/2}). \quad \text{(S3.24)}$$

From (S3.20)-(S3.22) and (S3.24), it follows that

$$P(|\widehat{\theta}_t^{(s)} - \frac{n^{1/2}(U_{nt}^{(s)} + 4n^{-1}S_C(t)E(h_{1t}^{(0)}h_{1t}^{(\eta)}))}{\widehat{\sigma}_{nt}}| > (n\ln n)^{-1/2}) = o(n^{-1/2}). \quad \text{(S3.25)}$$

Similar to the proof steps for the approximation of the numerator term of $\widehat{\theta}_t^{(s)}$, $\widehat{\sigma}_{nt}$ can be substituted via the square root of $\sigma_{nt}^2 = 4(n-1)(n-2)^{-2}\sum_{i=1}^n[\sum_{j=1}^n H_{ijt}^{(s)}/(n-1) - U_{nt}^{(s)}]^2$ in (S3.25). A further application of Lemma 2 in Chang and Rao (1989) entails that

$$\sup_x |F_n^{(s)}(x) - \widehat{F}_n^{(s)}(x)| = \sup_x |P(\frac{n^{1/2}U_{nt}^{(s)} + 4n^{-1/2}S_C(t)E(h_{1t}^{(0)}h_{1t}^{(\eta)})}{\sigma_{nt}} \leq x) - \widehat{F}_n^{(s)}(x)| + o(n^{-1/2}). \tag{S3.26}$$

It can be shown as Helmers (1991) that

$$\frac{2\sigma_t^{(s)}}{\sigma_{nt}} = 1 - \frac{\bar{f}_{\cdot t}}{8\sigma_t^{(s)2}} + R_{nt}^*,$$

where $\bar{f}_{\cdot t}$ is the mean of $f_{it} = 8E(h_{jt}^{(s)}(H_{ijt}^{(s)} - h_{it}^{(s)} - h_{jt}^{(s)})|X_i, \delta_i, Y_i) + 4(h_{it}^{(s)} - \sigma_t^{(s)2}), i = 1, \cdots, n$, and R_{nt}^* satisfies $P(|R_{nt}^*| \geq n^{-1/2}(\ln n)^{-1}) = o(n^{-1/2})$. Thus,

$$\frac{n^{1/2}U_{nt}^{(s)} + 4n^{-1/2}S_C(t)E(h_{1t}^{(0)}h_{1t}^{(\eta)})}{\sigma_{nt}} = \frac{n^{1/2}U_{nt}^{(s)}}{2\sigma_t^{(s)}}(1 - \frac{\bar{f}_{\cdot t}}{8\sigma_t^{(s)2}}) + \frac{2n^{-1/2}S_C(t)E(h_{1t}^{(0)}h_{1t}^{(\eta)})}{\sigma_t^{(s)}} + R_{nt}^{**} \tag{S3.27}$$

with $R_{nt}^{**} = -S_C(t)E(h_{1t}^{(0)}h_{1t}^{(\eta)})\bar{f}_{\cdot t}/(4n^{1/2}\sigma_t^{(s)3}) + n^{1/2}R_{nt}^*U_{nt}^{(s)}/(2\sigma_t^{(s)})$ such that $P(|R_{nt}^{**}| > (n\ln n)^{-1/2}) = o(n^{-1/2})$. Similar to the proofs of Theorem 1 in Helmers (1991), one derives that

$$\sup_x |P(\frac{n^{1/2}U_{nt}^{(s)}}{2\sigma_t^{(s)}}(1 - \frac{\bar{f}_{\cdot t}}{8\sigma_t^{(s)2}}) + \frac{2n^{-1/2}S_C(t)E(h_{1t}^{(0)}h_{1t}^{(\eta)})}{\sigma_t^{(s)}} \leq x) - \widehat{F}_n^{(s)}(x)| = o(n^{-1/2}) \tag{S3.28}$$

as $n \to \infty$. Together with (S3.26), (2.7) is obtained.

Lemma S3.2. Suppose that assumptions (A1)-(A2) are satisfied. Then,

$$P(|n^{1/2}r_{nt}^{(km)}| > n^{-1/2}(\ln n)^{-1}) = o(n^{-1/2}) \text{ as } n \to \infty. \tag{S3.29}$$

Proof. It was shown by Chang (1991) that

$$P(|cn^{1/2}r_{nt}^{(0)}| > n^{-1/2}(\ln n)^{-1}) = o(n^{-1/2}) \text{ for any constant } c, \tag{S3.30}$$

where $r_{nt}^{(0)} = (\widehat{\Lambda}_C(t) - \Lambda_C(t)) - (U_{nt}^{(0)} + 2n^{-1}\sigma_{0t}^2)$. The consistency of a U-statistic and (S3.30) imply that $U_{nt}^{(0)} \xrightarrow{p} 0$ and $r_{nt}^{(0)} \xrightarrow{p} 0$. Taking the Taylor expansion with respect to $\Lambda_C(t)$, one has $r_{nt}^{(km)} = O_p(\sum_{j=0}^4 r_{nt}^{(j)})$, where $r_{nt}^{(1)} = \Psi_{nt}^{(0)}(\Psi_{nt}^{(0)} + 4\bar{h}_{\cdot t}^{(0)})$, $r_{nt}^{(2)} = U_{nt}^{(0)}(\sum_{i \neq j} h_{it}^{(0)} h_{jt}^{(0)})/[n(n-1)]$, $r_{nt}^{(3)} = (\sum_{i=1}^n h_{it}^{(0)2}/n - \sigma_{0t}^2)/n$, and $r_{nt}^{(4)} = (\Psi_{nt}^{(0)} + 2\bar{h}_{\cdot t}^{(0)})/n$ with $\Psi_{nt}^{(0)} = U_{nt}^{(0)} - 2\bar{h}_{\cdot t}^{(0)}$ and $\bar{h}_{\cdot t}^{(0)}$ being the sample mean of $h_{it}^{(0)}$'s. From the result of Malevich and Abdalimov (1979), it follows that

$$P(n^{1/2}|U_{nt}^{(0)}| > \sqrt{\ln n}) = o(n^{-1/2}) \text{ and } P(n^{1/2}|\Psi_{nt}^{(0)} + 4\bar{h}^{(0)}| > (\ln n)^{1/2}) = o(n^{-1/2}). \quad \text{(S3.31)}$$

Using the probability inequality, $E(\Psi_{nt}^{(0)2}) = O(n^{-2})$ (Hoeffding (1948)), and (S3.31), one has

$$P(n^{1/2}|r_{nt}^{(1)}| > n^{-1/2}(\ln n)^{-1}) \leq P(|\Psi_{nt}^{(0)}| > n^{-1/2}(\ln n)^{-3/2}) + o(n^{-1/2}) = o(n^{-1/2}). \quad \text{(S3.32)}$$

Since the conditional expectation of $\sum_{i \neq j} h_{it}^{(0)} h_{jt}^{(0)}/[n(n-1)]$ is zero, it can be derived in the same way as (S3.32) that

$$P(n^{1/2}|r_{nt}^{(1)}| > n^{-1/2}(\ln n)^{-1}) = o(n^{-1/2}). \quad \text{(S3.33)}$$

By the Chebyshev's inequality, it is further implied that

$$P(n^{1/2}|r_{nt}^{(k)}| > n^{-1/2}(\ln n)^{-1}) = o(n^{-1/2}), k = 3, 4. \quad \text{(S3.34)}$$

Finally, from (S3.30), (S3.31)-(S3.34), (S3.29) is obtained. □

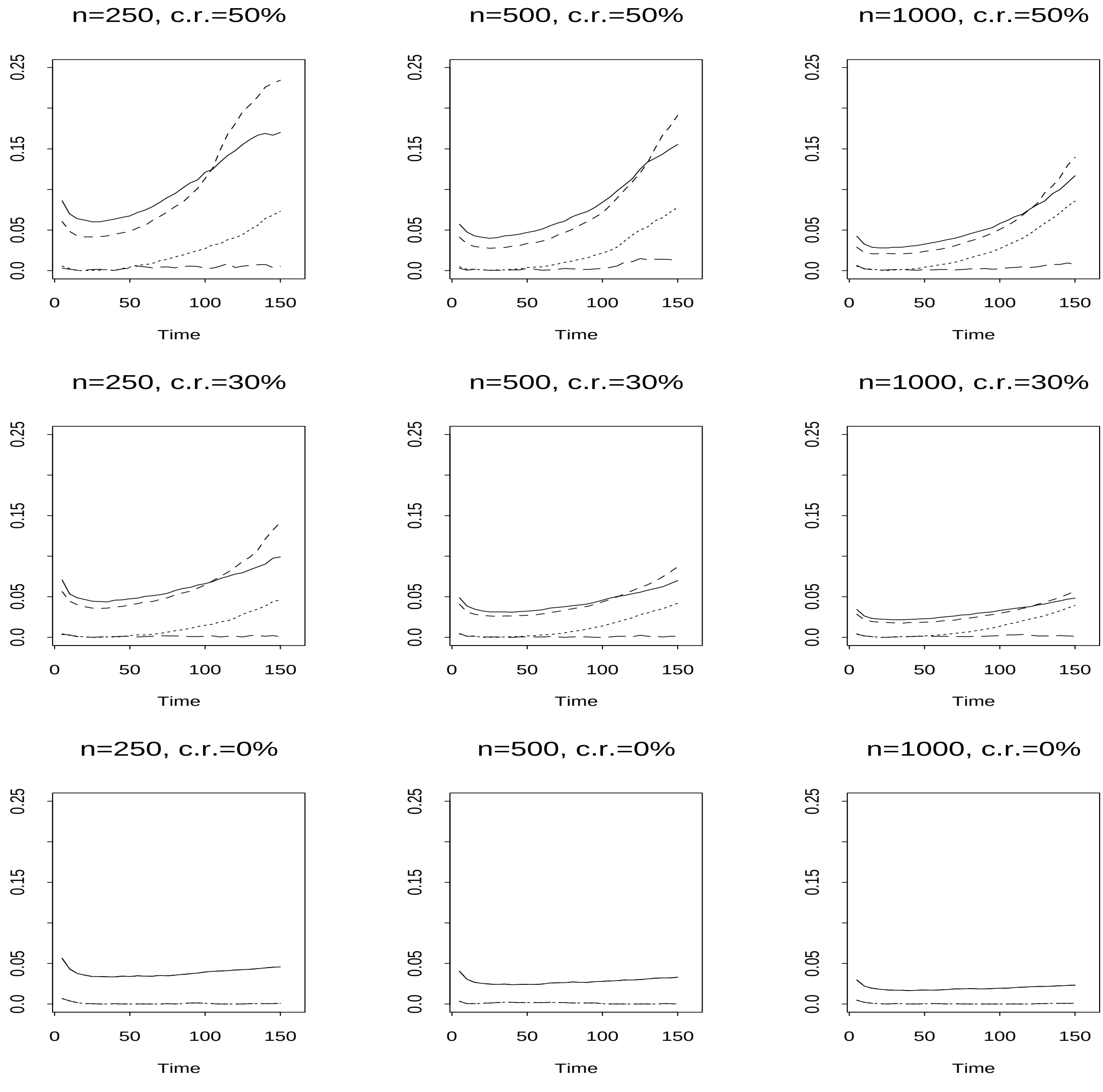

Figure 1: The biases of $\widehat{\theta}_t$ (dashed-dotted curve) and $\widehat{\theta}_{cdt}$ (dotted curve), and the standard errors of $\widehat{\theta}_t$ (solid curve) and $\widehat{\theta}_{cdt}$ (dashed curve) for the sample sizes (n) of 250, 500, and 1000, and the censoring rates (c.r.) of 0%, 30%, and 50%.