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APPROXIMATE RELIABILITIES OF m-CONSECUTIVE-k-OUT-OF-n: FAILURE SYSTEMS

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Abstract: An m-consecutive-k-out-of-n: F reliability system consists of n linearly arranged, independently functioning (or Markov-dependent) components, and fails iff k consecutive component failures occur (disjointly) m or more times. We employ the Stein-Chen method to obtain Poisson approximations for the reliability of such systems. The i.i.d., independent but non-identical, and stationary (rth order, $r \leq k$) Markov dependent cases are considered.

Key words and phrases: Consecutive k-out-of-n: F systems, Stein-Chen method, failure runs, rth order Markov dependence.

1. Introduction

A consecutive k-out-of-n: F reliability system consists of n linearly arranged, independently functioning components, and fails if and only if at least k of these fail in succession. Extensions have been given to the case where the components exhibit some form of Markov dependence (see Chiang and Niu (1981), Fu (1986) and the references cited therein). A generalization of this set-up was formulated by Griffith (1986), who considered a system for which $m(\geq 2)$ strings of consecutive failures were needed for system failure. In this paper, we will focus on a study of these systems, called *m-consecutive-k-out-of-n*: F systems, and shall refer to them, for brevity, as $m/C/k-n/F$ systems.

Denote the number of recurrent and non-overlapping failure runs by $N = N(n,k)$; the reliability of an $m/C/k-n/F$ system clearly equals $P(N(n,k) \leq m-1)$. An exact formula for $P(N(n,k) \leq m-1)$ was obtained independently by Godbole (1990a) and Papastavridis (1990). Our main goal in this paper is to derive Poisson approximations for the distribution of $N(n,k)$ which are valid when the maximal component failure probability is small.

In the i.i.d. case, Godbole (1990b) and Fu (1993) proved that the system reliability $P(N(n,k) \leq m-1)$ converges to the cumulative Poisson probability $\sum_{j=0}^{m-1} \exp(-\lambda)\lambda^j/j!$, provided that $nq_n^k \to \lambda$, where q_n is the component failure probability. We shall extend the i.i.d. case to the case of (i) independent but not necessarily identical components and (ii) stationary rth-order ($r \leq k$) Markov-Bernoulli chains in Sections 2 and 3 respectively. For $m = 1$ and the independent case, our results coincide with those obtained by Chryssaphinou and Papastavridis (1990).

2. Independent Components

We consider, in this section, components that operate independently of each other. The limit theorem of Godbole (1990b) will follow from Theorem 2.1 below, but we will be more interested in the problem of deriving total variation error bounds for a Poisson approximation. Let $R = R(n,k)$ be the number of occurrences of the pattern $SFF\dots FF$ (k F's preceded by an S) and $Q = Q(n,k)$ be the number of occurrences of failure runs of length k or more; note that

$$\begin{aligned} d(Q,R) &= \sup_{A\subset Z^+} |P(Q\in A) - P(R\in A)| \leq P(Q\neq R) \\ &= P(X_1 = X_2 = \cdots = X_k = 1) = q_1 q_2 \cdots q_k \leq q^k, \end{aligned} \tag{2.1}$$

where $X_1, X_2, \dots$ is Bernoulli sequence with parameters $q_1, q_2, \dots$, and $q = \max_{j\geq 1} q_j$. Also,

$$\begin{aligned} d(N,Q) &\leq P(N\neq Q) = P(\text{there is a failure run of length } 2k) \\ &\leq \sum_{j=2k}^{n} q_{j-2k+1}\cdots q_j \leq nq^{2k}. \end{aligned} \tag{2.2}$$

The inequality $d(X,Y) \leq P(X \neq Y)$ follows since for any subset A of Z^+, $|P(X\in A) - P(Y\in A)| \leq P(X\neq Y)$. Since

$$R = \sum_{j=k+1}^{n} I_j, \tag{2.3}$$

where $I_j = 1$ iff the trials $j-k, j-k+1, \dots, j$ consist of the pattern $SFFF\dots FFF$, it follows that

$$E(R) = \sum_{j=k+1}^{n} p_{j-k} q_{j-k+1}\cdots q_j \quad (= (n-k)p(1-p)^k \text{ in the i.i.d. case}) \tag{2.4}$$

and, by (2.1) and (2.2) that

$$\begin{aligned} d(N,\text{Po}(E(R))) &\leq d(N,Q) + d(Q,R) + d(R,\text{Po}(E(R))) \\ &\leq d(R,\text{Po}(E(R))) + q^k + nq^{2k}; \end{aligned} \tag{2.5}$$

so that it remains to estimate $d(R, \mathrm{Po}(E(R)))$. It follows from Theorem 2 in Barbour and Eagleson (1984) (and a result in their earlier paper (1983)) that

$$\begin{aligned} & d(R, \mathrm{Po}(E(R))) \\ \leq\ & \frac{1-e^{-E(R)}}{E(R)} \sum_{j \in \mathcal{T}} \{P^2(I_j = 1) \\ & + \sum_{s=1}^{r-1} \sum_{k} P(I_j = 1) P(I_k = 1) + P(I_j I_k = 1)\}; \end{aligned} \tag{2.6}$$

the last summation above is taken over the set of all k's in $\mathcal{T}$ satisfying the condition $\#(A(j) \cap A(k)) = s$, where $A(j)$ denotes the set of integers associated with the index j: in our case, we have $A(j) = \{j-k, \ldots, j\}$ and $r = k+1$. Thus,

$$\begin{aligned} & d(R, \mathrm{Po}(E(R))) \\ & \leq \frac{1-e^{-E(R)}}{E(R)} \sum_{j=k+1}^{n} \left\{ p_{j-k}^2 q_{j-k+1}^2 \cdots q_j^2 + \sum_{r=1}^{k} \sum_{i} p_{j-k} q_{j-k+1} \cdots q_j p_{i-k} q_{i-k+1} \cdots q_i \right\} \\ & \leq 1 - e^{-E(R)}(q^k + 2kq^k) \leq (2k+1) q^k, \end{aligned} \tag{2.7}$$

as $P(I_i I_j = 1) = 0$ (the pattern $SFFF \ldots FFF$ is non-overlapping), so that by (2.5),

$$d(N, \mathrm{Po}(E(R))) \leq (2k + 2 + nq^k) q^k. \tag{2.8}$$

We have proved the following theorem.

Theorem 2.1. *Consider an $m/C/k - n/F$ system where the jth component has reliability $p_j = 1 - q_j$, $j = 1, 2, \ldots, n$, and the components operate independently. Then the system reliability $\mu_n = P(N(n,k) \leq m-1)$ satisfies*

$$\left| \mu_n - \sum_{x=0}^{m-1} \exp(-\lambda_n) \lambda_n^x / x! \right| \leq (2k + 2 + nq^k) q^k, \tag{2.9}$$

with $q = \max_{j \geq 1} q_j$ and $\lambda_n = E(R)$ given by (2.4).

3. Markov-Dependent Components

We start by assuming that the behaviour of the n components is governed by the sequence of ergodic Markov-Bernoulli random variables $X_1, X_2, \ldots, X_n$ that satisfy

$$P(X_j = 1) = q = 1 - p, \quad j = 1, 2, \ldots, n \tag{3.1}$$

and which evolve according to the stationary transition matrix $\mathbf{P}$ determined by

$$P(X_{j+1} = 1 | X_j = 1) = \alpha; \quad P(X_{j+1} = 1 | X_j = 0) = \beta; \tag{3.2}$$

ergodicity implies that

$$q = \beta/(1-\alpha+\beta); \quad p = (1-\alpha)/(1-\alpha+\beta). \tag{3.3}$$

We assume, without loss of generality, that $\alpha > \beta$. It is easy to verify that the correlation between successive trials is $\alpha - \beta$. Under the above conditions, it will follow from Theorem 3.1 below that the system reliability approaches the corresponding Poisson (λ) probability if β and α both tend to zero so that $n\beta\alpha^{k-1} \to \lambda$. We will always be assuming, in the reliability framework, that q (and thus β) are small, but there is no reason to suppose *a priori*, that α is small as well. If α is not small, and k is not large, a Poisson approximation is no longer valid, and the system reliability is better estimated by a compound Poisson r.v. (with geometric compounding distribution); general results along these lines are proved in Geske et al. (1993), while Godbole and Schaffner (1993) address the question of Poisson approximation for non-overlapping occurrences of word patterns under similar Markovian hypotheses as the ones considered in this section.

In Godbole (1991), a result similar to Theorem 3.1 below was proved, but only yielded an approximation of order $O(\max\{\alpha, \beta\})$; we will improve this rate to $O(\max\{\beta\alpha^{k-1}, \alpha^k\})$.

Theorem 3.1. *The reliability μ_n of a $m/C/k-n/F$ system governed by a stationary two-state Markov chain (with initial distribution and transition structure determined by* (3.1) *through* (3.3)) *satisfies*

$$\begin{aligned}
&\left|\mu_n - \sum_{x=0}^{m-1} \exp(-\lambda_n)(\lambda_n)^x/x!\right| \\
&\le (1-e^{-E(R)})p\beta\alpha^{k-1}\left\{(2k+1) + 2\frac{\alpha-\beta}{1-\alpha+\beta}\right\} + \frac{\beta\alpha^{k-1}}{1-\alpha+\beta} + \frac{n\beta\alpha^{k-1}}{1-\alpha+\beta}\alpha^k \\
&\approx (1-e^{-E(R)})p\beta\alpha^{k-1}\left\{(2k+1) + 2\frac{\alpha-\beta}{1-\alpha+\beta}\right\} + \frac{\beta\alpha^{k-1}}{1-\alpha+\beta} + E(R)\alpha^k, \qquad (3.4)
\end{aligned}$$

where $\lambda_n = E(R) = (n-k)p\beta\alpha^{k-1}$.

Proof. If $\mathbf{Q}^{(r)}$ denotes the r-step transition matrix, $q_{ij}^{(r)}$ its (i,j)th element, and μ the stationary distribution of the above chain, then it follows from Theorem 8.H in Barbour, Holst and Janson (1992) that, with $A = \{SFFF\dots FFF\}$ and $E(R) = (n-k)p\beta\alpha^{k-1}$, we have

$$d(R, \mathrm{Po}(E(R))) \le (1-e^{-E(R)})\left\{p\beta\alpha^{k-1} + 2\sum_{j\ge1}|q_{rr}^{(j)} - p\beta\alpha^{k-1}|\right\}, \tag{3.5}$$

so that we need to estimate the probabilities $q_{rr}^{(j)}$. It is clear that $q_{rr}^{(j)} = 0$ for each $j \leq k$; furthermore, for $j \geq k+1$,

$$q_{rr}^{(j)} = p_{10}^{(j-k)} \beta \alpha^{k-1} \tag{3.6}$$

where $p_{10}^{(i)}$ denotes the $(1,0)$th element of the i-step transition matrix $\mathbf{P}^{(i)}$. Now, a straightforward transition matrix diagonalization reveals that for $i \geq 1$,

$$p_{10}^{(i)} = (1-\alpha)/(1-\alpha+\beta)\{1-(\alpha-\beta)^i\} = p\{1-(\alpha-\beta)^i\}, \tag{3.7}$$

so that by (3.6), for $j \geq k+1$,

$$q_{rr}^{(i)} = p(1-(\alpha-\beta)^{j-k})\beta\alpha^{k-1}. \tag{3.8}$$

Substituting (3.8) and utilizing the fact that $q_{rr}^{(j)} = 0$ $(j \leq k)$ in (3.5) yields

$$\begin{aligned} &d(R, \mathrm{Po}(E(R))) \\ &\leq (1-e^{-E(R)})\Big\{p\beta\alpha^{k-1} + 2kp\beta\alpha^{k-1} + 2p\beta\alpha^{k-1} \sum_{j \geq k+1} (\alpha-\beta)^{j-k}\Big\} \\ &= (1-e^{-E(R)})\Big\{(2k+1)p\beta\alpha^{k-1} + 2p\beta\alpha^{k-1}(\alpha-\beta)/(1-\alpha+\beta)\Big\}. \end{aligned} \tag{3.9}$$

Now, as in (2.1) and (2.2), we have $d(R,Q) \leq \beta\alpha^{k-1}/(1-\alpha+\beta)$ and $d(N,Q) \leq \{n\beta\alpha^{k-1}/(1-\alpha+\beta)\}\alpha^k$, which, together with (3.9), yield the required result.

Next consider rth order Markov dependence $(r \leq k)$. We analyze the rth order one-dimensional process as a simple $(r+1)$-dimensional Markov chain by using the following device (see Cinlar (1975, p.142)): Given a discrete process that satisfies for each $j \geq r+1$

$$P(X_j = i | X_0, X_1, \ldots, X_{j-1}) = P(X_j = i | X_{j-r}, \ldots, X_{j-1}), \tag{3.10}$$

we consider the associated process $\{Z_j\}_{j \geq r+1}$ defined by

$$Z_j = (X_{j-r}, \ldots, X_{j-1}, X_j). \tag{3.11}$$

Assume, without loss of generality, that $r = k$. It is clear that $\{Z_j\}$ is simple Markov with state space consisting of 2^{k+1} points and we need to analyze the number of visits by it to the state $(0, 1, \ldots, 1)$ (k ones) under the hypothesis that the chain is stationary. Observe that the only possible transition from a state $(a_1, a_2, \ldots, a_{k+1})$ is to either $(a_2, \ldots, a_{k+1}, 0)$ or $(a_2, \ldots, a_{k+1}, 1)$ and that each state is immediately accessible from just two others, so that the transition probabilities form a $2^{k+1} \times 2^{k+1}$ matrix $\mathbf{P}$ with each row (or column) containing just two non-zero entries. Denote the stationary distribution of $\{Z_j\}$ by

$(\pi_a, \pi_b, \pi_c, \pi_d, \pi_e, \ldots, \pi_s)$, $s = 2^{k+1}$, where $a = (0,1,\ldots,1)$, $b = (1,0,1,\ldots,1)$, $c = (0,0,1,\ldots,1)$, $d = (1,1,\ldots,1,0)$ and $e = (1,1,\ldots,1,1)$ will be counted, respectively, as the first five elements of the stationary distribution.

We now exhibit the fact that the salient information contained in $\mathbf{P}$ can be expressed in terms of the 2×2 transition matrix $\mathbf{Q}$ of a Markov chain with state space $\{a, a'\}$, with a as above and a' representing all other states. Clearly, $q_{aa} = 0$, $q_{aa'} = 1$ and

$$\begin{aligned} q_{a'a} &= P(Z_{j+1} = a | Z_j \neq a) = \frac{P(Z_{j+1} = a, Z_j = b) + P(Z_{j+1} = a, Z_j = c)}{P(Z_j \neq a)} \\ &= (\pi_b p_{ba} + \pi_c p_{ca})/P(Z_j \neq a) = \{(\pi_b + \pi_c) p_{\cdot a}\}/(1 - \pi_a) = \beta \qquad (3.12) \end{aligned}$$

since $p_{ba} = p_{ca} = p_{\cdot a}$ by k-dependence. Since we need to obtain a Poisson approximation for the number of visits to state a, it suffices to use the matrix

$$\mathbf{Q} = \begin{pmatrix} 0 & 1 \\ \beta & 1-\beta \end{pmatrix} \qquad (3.13)$$

with stationary distribution $(\beta/(1+\beta), 1/(1+\beta))$ and satisfying

$$q_{aa}^{(i)} = \beta/(1+\beta)\{1 - (-\beta)^{i-1}\}. \qquad (3.14)$$

As before,

$$\begin{aligned} d(R, \mathrm{Po}(E(R))) &\leq (1 - e^{-E(R)}) \Big\{ \beta/(1+\beta) + 2 \sum_{i \geq 1} \Big| q_{aa}^{(i)} - \frac{\beta}{1+\beta} \Big| \Big\} \\ &= (1 - e^{-E(R)}) \Big\{ \frac{\beta}{1+\beta} + \frac{\beta}{1-\beta^2} \Big\} \qquad (3.15) \end{aligned}$$

by (3.14), where $E(R) = (n-k)\beta/(1+\beta)$. Since $d(Q,R) \leq \pi_d + \pi_e$ and $d(Q,N) \leq n\pi_e(p_{ee})^{k-1}$, the following result holds:

Theorem 3.2. *The reliability μ_n of an $m/C/k - n/F$ system operating under the rth order Markovian hypotheses ($r \leq k$) of this section satisfies*

$$\Big| \mu_n - \sum_{x=0}^{m-1} e^{-\lambda_n} \lambda_n^x / x! \Big| \leq (1 - e^{-E(R)}) \Big\{ \frac{\beta}{1+\beta} + \frac{\beta}{1-\beta^2} \Big\} + \pi_d + \pi_e + n\pi_e(p_{ee})^{k-1},$$

where β is defined by (3.12) *and $\lambda_n = E(R) = (n-k)\beta/(1+\beta)$.*

Remark. Theorem 3.2 can be extended to the case where the Markov chain is non-stationary (but time homogeneous). We omit the cumbersome details.

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