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# RUN LENGTH PROPERTIES OF THE CUSUM AND EWMA SCHEMES FOR A STATIONARY LINEAR PROCESS

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*Abstract:* This paper presents the asymptotic expressions of the average run length (ARL) for the cumulative sum (CUSUM) and exponentially weighted moving average (EWMA) control charts in detecting an unknown mean shift in a stationary linear process. Based on the ARL expressions, we compare the detection performance of the two popular charts in monitoring the mean shifts in such autocorrelated processes. Both theoretical analysis and numerical simulation results show that auto-covariance can play an important role in the detection performance of the two charts.

*Key words and phrases:* Autocorrelated stationary processes, average run length, change point detection.

## 1. Introduction

The average run length (ARL) is an extensively used measure in statistical process control (SPC) for evaluating and comparing the detection performance of various control charts. Since the asymptotic expression of an ARL can establish clear relations between the ARL, the control limit, and various statistical properties (the expectation, variance and covariance, etc.) of the observation processes, much effort has been expended on estimating the ARL by various methods. Asymptotic expressions of the ARL for the cumulative sum (CUSUM), exponentially weighted moving average (EWMA), Shiryayev-Robert, generalized likelihood ratio (GLR), and reference-free cumulative score (RFCuscore) control charts in detecting mean changes in an i.i.d. observation processes, have been studied by Lorden (1971), Taylor (1975), Pollak and Siegmund (1985), Pollak (1987), Novikov (1990), Srivastava and Wu (1993, 1997), Siegmund and Venkatraman (1995) and Han and Tsung (2004, 2006).

As automated sampling technology develops and high-volume production processes become more common, the need to monitor autocorrelated processes also increases. To detect mean changes in autocorrelated processes, a variety of

CUSUM and EWMA methods, enhancements, and new charting schemes have been developed. See, for example, Alwan and Roberts (1988), Alwan (1992), Harris and Ross (1991), Montgomery and Mastrangelo (1991), Wardell, Moskowitz and Plante (1992, 1994), Yashchin (1993) Woodall and Faltin (1993), Apley and Shi (1999), Runger and Willemain (1995), Schmid and Schöne (1997), Zhang (1998), Lu and Reynolds (1999), Luceño (1999), Jiang, Tsui and Woodall (2000), Jiang (2001), Apley and Tsung (2002), Shu, Apley and Tsung (2002), Apley and Lee (2003), Schmid and Rosolowski (2003), Knoth and Schmid (2002, 2004), and Ben-Gal and Morag (2003). However, the studies of control charts in detecting the mean changes in autocorrelated processes in the above papers are mainly based on numerical simulations of ARL.

In studying optimal detection in autocorrelated processes, Bansal and Papantoni-Kazakos (1986) and Lai (1998) generalized Lorden's (1971) asymptotic theory to the case of stationary ergodic and even more general stochastic processes for the CUSUM chart. However, their studies involved some strong assumptions and did not discuss the asymptotic estimation of the $\mathrm{ARL}_0$ (the in-control ARL). The basic idea of Yashchin's method (1993) to estimate the ARL involves replacing the sequence of serially correlated observations by a sequence of i.i.d. observations for which the run length distribution is approximately the same. Although Wardell, Moskowitz and Plante (1994) presented a closed form of run-length distributions of special-cause control charts for correlated processes, the asymptotic estimations of the ARL was not tackled as well. Neither was this done in Schmid and Schöne (1997) work, in which they only compared the tail probabilities of the run length of the EWMA chart in both stationary and i.i.d. Gaussian cases.

Here, the main purpose of the paper is to consider the asymptotic expressions of the ARL for the two popularly used control charts, the CUSUM and EWMA charts, in detecting an unknown mean shift of stationary autocorrelated processes. By the comparison of the ARLs based on the asymptotic expressions, we will see how the auto-covariance of the processes plays an important role on the detection performance of the two charts. The main method of estimating the ARL in this paper is the large deviation technique (see Durrett (2005)), which is in fact different from that used by Pollak and Siegmund (1985), Pollak (1987), Novikov (1990), Srivastava and Wu (1993, 1997), Siegmund and Venkatraman (1995) and Han and Tsung (2004).

In the next section, we start with some notes and assumptions for a stationary linear process. After that, two theorems on the asymptotic expressions of the ARL are proposed. In Section 3, based on the two theorems, the theoretical analysis and numerical simulation comparison of the ARLs in detecting unknown mean shifts in both i.i.d. normal data and the first-order autoregressive ($AR(1)$) model are presented. The proofs of the two theorems are put on www.stat.sinica.edu.tw as an on-line supplement. The paper concludes in Section 4 with discussion on further studies.

## 2. Asymptotic Expressions of the ARL for the CUSUM and EWMA Charts

In this section, we first introduce a discrete-time stationary process and the popular CUSUM and EWMA charts to monitor it. After that, we present three conditions on the process and propose two theorems on asymptotic expressions of the ARL for the CUSUM and EWMA control charts.

Let $X_i$ $(i = 0, 1, 2, \ldots)$ be the $i$th observation on a discrete stationary process with a constant mean and auto-covariance function. That is, $\mathbf{E}(X_i) = \mu_0$ for all $i \geq 0$ and

$$\mathrm{Cov}\,(X_i, X_{i+m}) = \mathbf{E}[(X_i - \mu_0)(X_{i+m} - \mu_0)],$$

where the auto-covariance function, $\mathrm{Cov}\,(X_i, X_{i+m})$ depends only on the lag $m$. Such a stationary process is usually presented in the linear form

$$X_i = \sum_{j=0}^{\infty} a_j \xi_{i-j}, \tag{2.1}$$

where the sequence of random variables, $\xi_j$, $-\infty < j < +\infty$, are i.i.d. with finite variance. It is also assumed that

$$A = \sum_{j=0}^{\infty} a_j \neq 0, \qquad \|A\| = \sum_{j=0}^{\infty} |a_j| < \infty.$$

The stationary linear process in (2.1) is referred to as a general linear process in Chatfield (1996) and as a linear filter model in Box, Jenkins and Reinsel (1994). The stationary linear process model is quite general and included such popular autocorrelated processes as stationary autoregressive (AR(p)), moving-average (MA(q)) and MA($\infty$)), and autoregressive moving-average (ARMA(p, q)). Without loss of generality, we further assume that $a_0 = 1$. Thus, $X_i = \xi_i$ when $a_i = 0$, $i \geq 1$.

Suppose that, at some time $\tau \geq 1$ , the mean of $X_i$ changes from $\mu_0$ to $\mu$. That is, from time period $\tau$ onwards, the mean of $X_i$ undergoes persistent shifts of sizes $\mu - \mu_0$, where $\mu_0$ is known, and $\mu$ and $\tau$ are unknown. Meanwhile, the variance, $\sigma_X^2$ of $X_i$ does not change. We let $\bar{\xi} = \mathbf{E}(\xi_1)$, so that $\mu = \bar{\xi}A$. Without loss of generality, we also assume that $\mu_0 = 0$, $\sigma_X^2 = 1$ and $\mu > 0$. Obviously, $\mathbf{E}(X_i) = 0$ for $i < \tau$ and $\mathbf{E}(X_i) = \mu = \bar{\xi}A > 0$ for $i \geq \tau$. That is, the out-of-control process is

$$X_i = \mu + \sum_{j=0}^{\infty} a_j \xi_{i-j},$$

for $i \geq \tau$ with $\mathbf{E}(\xi_k) = 0$.

We introduce two popular control charts designed to detect the change in the process. The popular upward-sided CUSUM chart, $T_C$, is defined as

$$T_C(c) = \inf\left\{n : \max_{1\leq k\leq n}\left[\sum_{i=n-k+1}^{n}\delta\Big(X_i - \frac{\delta}{2}\Big)\right] \geq c\right\}, \tag{2.2}$$

where $c > 0$ is a control limit and $\delta/2 > 0$ is the reference value. When $\{X_i\}$ is i.i.d., Moustakides (1986) and Ritov (1990) proved that the performance of the CUSUM control chart with a reference value of $\delta/2$ is optimal if the real mean shift is $\delta\sigma_X$.

The EWMA is a control chart for detecting a mean shift that relies on

$$T_E(r) = \inf\{n \geq 1 : E_n(r) \geq \tilde{c}\},$$

where $r$ is a weighting parameter ($0 < r \leq 1$), $\tilde{c}$ is the control limit and $E_n(r) = rX_n + (1-r)E_{n-1}(r)$ with $E_0(r) = 0$.

In this paper we only consider the upper-sided CUSUM and EWMA charts, since it is more difficult to estimate the ARLs of the two-sided CUSUM and EWMA charts.

Let $\mathbf{P}_0(.)$ and $\mathbf{E}_0(.)$ denote the probability and expectation when there is no change in the mean and variance, that is, $\mu_0 = 0$ and $\sigma_X^2 = 1$ and the change point is $\tau = \infty$. Write $\mathbf{P}_\mu(.)$ and $\mathbf{E}_\mu(.)$ as the probability and expectation, respectively, when the mean shift value is $\mu > 0$ after the change point $\tau$. Here we assume $\tau = 1$. For a stopping time, $T$, as the alarm time with a detecting procedure, the two most frequently used operating characteristics are the in-control ARL ($\mathbf{ARL}_0$) and the out-of-control ARL ($\mathbf{ARL}_\mu$), defined by

$$\mathbf{ARL}_0(T) = \mathbf{E}_0(T), \qquad \mathbf{ARL}_\mu(T) = \mathbf{E}_\mu(T).$$

To obtain the asymptotic ARL for the two control charts, we need three conditions. Let $h(\theta) = \mathbf{E}(e^{\theta\xi_j})$ denote the moment-generating functions of $\xi_j$. We suppose that the white noise $\{\xi_j\}$ satisfies the following two conditions.

(I) The distribution of $\xi_1$ is not a point mass at $\mathbf{E}(\xi_1)$.

(II) The moment-generating function of $\xi_1$ satisfies $h(\theta) < \infty$ for some $\theta > 0$ and $\bar{h} = \sup\{h'(\theta)/h(\theta) : \theta < \bar{\theta}\} > 0$, where $\bar{\theta} = \sup\{\theta : h(\theta) < \infty\}$.

Note that, from condition II, it follows that $h(\theta)$ is an analytic function for $|\theta| < \bar{\theta}$. It can be shown that many distributions, such as normal, exponential, uniform and Poisson, satisfy conditions I and II.

Another condition is that

(III) $\sum_{k=1}^{\infty} k|a_k| < \infty$. This condition implies that

$$\lim_{n\to\infty} n \sum_{k=n+1}^{\infty} |a_k| = 0. \tag{2.3}$$

Let $\eta_j = \delta(A\xi_j - \delta/2)$ and $h_\eta(\theta) = \mathbf{E}(e^{\theta\eta_1})$ denote the moment-generating functions of $\eta_1$. Let $\theta(y)$ satisfy $y = h'_\eta(\theta(y))/h_\eta(\theta(y))$ since $h'_\eta(\theta)/h_\eta(\theta)$ is strictly increasing (see Durrett (2005, p.60)).

**Theorem 1.** *Suppose conditions* (I), (II) *and* (III) *hold. Let* $\hat\mu = \delta(\mu - \delta/2)$.

(i) *If* $0 \le \mu < \delta/2$, *then*

$$\frac{1}{bc} e^{c(\theta^*+o(1))} \le \mathbf{ARL}_\mu(T_C(c)) \le \frac{2c}{u} e^{c(\theta^*+o(1))} \tag{2.4}$$

*for a large control limit* $c$, *where* $\theta^* > 0$ *is a unique positive root of the equation* $\log h(\delta A\theta) - \delta^2\theta/2 = 0$ *on* $\theta > 0$, $u = \delta A h'(\delta A\theta^*)/h(\delta A\theta^*) - \delta^2/2 > 0$, *and* $b$ *is a positive constant defined by*

$$b = \inf\Big\{ x > \frac{1}{u} : \theta\Big(\frac{1}{x}\Big) - x \log h_\eta\Big(\theta\Big(\frac{1}{x}\Big)\Big) \ge 2\theta^* \Big\}. \tag{2.5}$$

(ii) *If* $\mu > \delta/2$, *then*

$$-(1+o(1))\frac{3\sqrt{c}\log c}{(\hat\mu)^{\frac{3}{2}}} + \frac{c}{\hat\mu} \le \mathbf{ARL}_\mu(T_C(c)) \le \frac{c}{\hat\mu} + \frac{2\sqrt{c}\log c}{(\hat\mu)^{\frac{3}{2}}} + \frac{e^{(\delta\sigma A)^2/2}}{\hat\mu c^{\sqrt{2}-1}}(1+o(1)) \tag{2.6}$$

*for large* $c$.

For the EWMA chart, we let the control limit $\tilde c$ be fixed, and the weight parameter $r$ be so small that the $\mathrm{ARL}_0$ becomes large. In the following theorem, we see that the role of the control limit in the EWMA chart is the same as the reference value $\delta/2$ in the CUSUM chart, and the weight parameter in the EWMA chart is like the control limit in the CUSUM chart.

**Theorem 2.** *Suppose that conditions* (I), (II) *and* (III) *hold.*

(i) *If* $0 \le \mu < \tilde c$, *then*

$$e^{\frac{1}{r}(\theta^*(\tilde c)+o(1))} \le \mathbf{ARL}_\mu(T_E(r)) \le \frac{3\log r^{-1}}{r} e^{\frac{1}{r}(\theta^*(\tilde c)+o(1))} \tag{2.7}$$

*for a small weighting parameter* $r$, *where* $\theta^*(\tilde c) = \tilde c\theta_{\tilde c} - \log h_\zeta(\theta_{\tilde c})$, $\theta_{\tilde c}$ *is a unique positive root of the equation* $\tilde c\theta - \log h(A\theta) = 0$ *on* $\theta > 0$, *and* $h_\zeta(\theta)$ *is defined by*

$$h_\zeta(\theta) = \exp\Big\{ \int_0^\theta \frac{\log h(Ax)}{x} dx \Big\}. \tag{2.8}$$

(ii) *If $\mu > \tilde{c}$, then*

$$(1+o(1))\Big(1-\frac{1}{(\log r^{-1})^p}\Big)\frac{1}{r}\log\frac{\mu}{\mu-\tilde{c}} \le \mathbf{ARL}_\mu(T_E(r)) \le (1+o(1))\frac{1}{r}\log\frac{\mu}{\mu-\tilde{c}} \tag{2.9}$$

*for small $r$, where $p$ is a positive number.*

**Remark 1.** It is convenient to rewrite the results of the two theorems. For large $c$ and small $r$ we have

$$\mathbf{ARL}_\mu(T_C(c)) = L_C e^{c(\theta^*+o(1))}, \qquad \mathbf{ARL}_\mu(T_E(r)) = L_E e^{\frac{1}{r}(\theta^*(\tilde{c})+o(1))} \tag{2.10}$$

for $0 \le \mu < \delta/2$ and $0 \le \mu < \tilde{c}$, respectively, and

$$\mathbf{ARL}_\mu(T_C(c)) = (1+o(1))\frac{c}{\delta(\mu-\delta/2)}, \ \mathbf{ARL}_\mu(T_E(r)) = (1+o(1))\frac{1}{r}\log\frac{\mu}{\mu-\tilde{c}} \tag{2.11}$$

for $\mu > \delta/2$ and $\mu > \tilde{c}$, respectively. Here $c$ and $\tilde{c}$ are the control limits of the CUSUM and EWMA, and $L_C$ and $L_E$ satisfy $1/(bc) \le L_C \le 2c/u$ and $1 \le L_E \le 3\log r^{-1}/r$, respectively.

**Remark 2.** It can be seen that the results in (2.11) are the same as those in the case of $X_i$, $i \ge 1$, being i.i.d. normal variables (see Novikov (1990), Srivastava and Wu (1993, 1997) and Wu (1994)) as $c \to \infty$ and $r \to 0$.

As an application of the theorems, we give two examples as follows.

**Example 1.** Let $\xi_i \sim N(\bar{\xi}, \sigma)$ be a normal distribution so that

$$h(\theta) = \exp\Big\{\bar{\xi}\theta + \frac{\sigma^2\theta^2}{2}\Big\},$$

$$\log h_\zeta(\theta) = \int_0^\theta \frac{\log h(Ax)}{x}dx = \mu\theta + \frac{A^2\sigma^2}{4}\theta^2.$$

It can be shown that $u = \delta(\delta/2 - \mu), b = (4 + \sqrt{15})/u$ and

$$\theta^* = \frac{1 - 2\mu/\delta}{\sigma^2 A^2}, \qquad \theta^*(\tilde{c}) = \frac{(\tilde{c} - \mu)^2}{\sigma^2 A^2}, \tag{2.12}$$

respectively, for $\mu < \delta/2$ and $\mu < \tilde{c}$. When $a_i = 0$, $i \ge 1$, i.e. $A = 1$, the above results become the ones for the i.i.d. Gaussian process. Note that, to our knowledge, we only find the result $\theta^*(\tilde{c}) = \tilde{c}^2/\sigma^2$ given in Novikov (1990) for EWMA charts with $\mu = 0$. When the mean shift is $0 < \mu < \tilde{c}$ in an i.i.d. Gaussian process, the number $\theta^*(\tilde{c}) = (\tilde{c} - \mu)^2/\sigma^2$ for the EWMA chart appears to be a new result.

**Example 2.** Let $\xi_i \sim \exp(\lambda)$ be an exponential distribution with parameter $\lambda > 0$. Then, $h(\theta) = 1/(1-\theta/\lambda)$ for $\theta < \lambda$,

$$\log h_\zeta(\theta) = -\int_0^\theta \frac{\log(1 - Ax/\lambda)}{x} dx$$

for $A\theta < \lambda$, and $\theta^*$ and $\theta_{\tilde{c}}$ satisfy, respectively,

$$-\log(1 - \frac{\delta A \theta^*}{\lambda}) - \frac{\delta^2 \theta^*}{2} = 0, \qquad \tilde{c} + \frac{\log(1 - A\theta_{\tilde{c}}/\lambda)}{\theta_{\tilde{c}}} = 0,$$

for $A/\lambda = \mu < \delta/2$ and $\mu < \tilde{c}$, where $\mu > A/\lambda_0$. Though it is difficult to obtain the closed forms of $\theta^*, \theta_{\tilde{c}}$ and $\theta^*(\tilde{c})$ for the exponential distribution, one can find the values from numerical calculation. For instance, taking $\lambda = 1, \delta = 1, A = 1/4$ and $\tilde{c} = 2$ for the EWMA, we have $\theta^* = 3.1872, \theta_{\tilde{c}} = 3.9987$ and $\theta^*(\tilde{c}) = 2\theta_{\tilde{c}} - \log h_\zeta(\theta_{\tilde{c}}) = 6.3554$.

## 3. Theoretical Analysis and Numerical Comparison of the ARLs

As an application of Theorems 1 and 2, we first compare the ARLs of the i.i.d. Gaussian and the stationary Gaussian processes, and between CUSUM and EWMA charts. After that, we illustrate the numerical simulation results of the ARLs for the one-sided CUSUM and EWMA control charts.

### 3.1. Theoretical analysis and comparisons of the ARLs

Usually, comparisons of the performance of control charts are made by designating the common $\mathbf{ARL}_0$ and comparing the $\mathbf{ARL}'_\mu$s of the control charts for a given shift, $\mu$. The chart with the smaller $\mathbf{ARL}_\mu$ is considered to have better performance.

Let $\mathbf{ARL}_\mu^{(i.i.d.)}(T_C(c'))$ and $\mathbf{ARL}_\mu^{(i.i.d.)}(T_E(r'))$ denote, respectively, the ARL of the CUSUM and EWMA charts corresponding to the case of $\{X_i\}$ being an i.i.d. Gaussian process, that is, $a_i = 0$ for $i \geq 1$ and $\xi \sim N(\bar{\xi}, \sigma)$, where $c'$ and $r'$ are the control limit and weighting parameter, respectively, for CUSUM and EWMA charts.

**Corollary 1.** *Suppose $\{X_i\}$ is a stationary Gaussian process defined in* (2.1)*. Let $\mu > \delta/2$ for the CUSUM chart and $\mu > \tilde{c}$ for the EWMA chart. Take large $c'$ and $c$ such that $\boldsymbol{ARL}_0^{(i.i.d.)}(T_C(c')) = \boldsymbol{ARL}_0(T_C(c))$, and small $r$ and $r'$ such that $\boldsymbol{ARL}_0^{(i.i.d.)}(T_E(r')) = \boldsymbol{ARL}_0(T_E(r))$. Then,*

$$\boldsymbol{ARL}_\mu^{(i.i.d.)}(T_C(c')) < \boldsymbol{ARL}_\mu(T_C(c)), \qquad \boldsymbol{ARL}_\mu^{(i.i.d.)}(T_E(r')) < \boldsymbol{ARL}_\mu(T_E(r))$$

*for $|A| > 1$, and*

$$\boldsymbol{ARL}_\mu^{(i.i.d.)}(T_C(c')) > \boldsymbol{ARL}_\mu(T_C(c)), \qquad \boldsymbol{ARL}_\mu^{(i.i.d.)}(T_E(r')) > \boldsymbol{ARL}_\mu(T_E(r))$$

*for* $|A| < 1$.

**Proof.** By (2.10) and (2.12), the equality $\mathbf{ARL}_0^{(i.i.d.)}(T_C(c')) = \mathbf{ARL}_0(T_C(c))$ means that $c' = (1+o(1))c/A^2$ for large $c$ and $c'$. Thus, by (2.6) or (2.11), we have

$$\begin{aligned}\mathbf{ARL}_\mu^{(i.i.d.)}(T_C(c')) &= (1+o(1))\frac{c}{A^2\delta(\mu-\delta/2)} < (1+o(1))\frac{c}{\delta(\mu-\delta/2)}\\ &= \mathbf{ARL}_\mu(T_C(c)),\end{aligned}$$

for $|A| > 1$, and

$$\begin{aligned}\mathbf{ARL}_\mu^{(i.i.d.)}(T_C(c')) &= (1+o(1))\frac{c}{A^2\delta(\mu-\delta/2)} > (1+o(1))\frac{c}{\delta(\mu-\delta/2)}\\ &= \mathbf{ARL}_\mu(T_C(c)),\end{aligned}$$

for $|A| < 1$. Similarly, the inequalities above for the EWMA chart can be checked by using $1/r' = (1+o(1))1/(rA^2)$ for small $r$ and $r'$.

**Remark 3.** It has been proved by Schmid and Schöne (1997) that the ARL of the stationary Gaussian process is larger than that of for the case of the i.i.d. Gaussian process provided all auto-covariances are nonnegative and at least one is greater than zero. The result is for the case of $|A| = |1+\sum_{i=1}^{\infty} a_i| > 1$, since all auto-covariances are nonnegative and at least one is greater than zero, which implies that $a_i \geq 0$ for $i \geq 1$ and at least one $a_j > 0$. Obviously, it cannot be ensured that $|A| < 1$ if some auto-covariances are negative. However, this is true in some special cases. For example, let $\{X_i\}$ be the AR(1) model; that is, $X_{i+1} = \rho X_i + \xi_{i+1}, a_i = \rho^i$, where $|\rho| < 1$. We have $A = (1-\rho)^{-1}$. Obviously, $0 < A < 1$ for $\rho < 0$. If $\mathbf{E}(X_i) = 0$, then the number $\rho^k, k \geq 1$, is the covariance of $X_{i+k}$ and $X_i$. In this case, the $\mathbf{ARL}_\mu$ of the stationary Gaussian process is larger (smaller) than in the case of the i.i.d. Gaussian process if $\rho > 0$ ($\rho < 0$).

**Remark 4.** Moustakides (1986) and Ritov (1990) have proved that the performance of the one-sided CUSUM control chart with the reference value $\delta/2$ in detecting the mean shift of i.i.d. Gaussian processes is optimal if the real mean shift is $\delta$; that is, $\mathbf{ARL}_\mu^{(i.i.d.)}(T_C(c')) = 2c'/\mu^2$ attains its minimal value when $\mu = \delta$. However, by Corollary 1, we see that the optimal case, $\mu = \delta$, does not hold for the CUSUM chart in detecting the mean shift for autocorrelated processes. It follows from (2.10), (2.11) and (2.12) that

$$\begin{aligned}&\mathbf{ARL}_\mu^{(i.i.d.)}(T_E(r'))\\ &\quad= (1+o(1))\frac{1}{r'}\log\frac{\mu}{\mu-\tilde{c}} = (1+o(1))\frac{\log[\mathbf{ARL}_0(T_E(r'))]}{\tilde{c}^2}\log\Big(\frac{\mu}{\mu-\tilde{c}}\Big)\\ &\quad= (1+o(1))\frac{\log[\mathbf{ARL}_0(T_E(r'))]}{\mu^2}\frac{1}{x^2}\log\Big(\frac{1}{1-x}\Big), \qquad (3.1)\end{aligned}$$

where $0 < x = \tilde{c}/\mu < 1$. Since the function $x^{-2}\log(1-x)^{-1}$ attains its minimal value, 2.45541..., at $x = 0.71533...$, we have the smallest $\mathbf{ARL}_{\mu}^{(i.i.d.)}(T_E(r'))$ if the control limits are chosen to be $\tilde{c} = 0.71533\mu$, where $\mu$ is the real mean shift. While the EWMA chart has the best performance in detecting the mean shift of i.i.d. processes when $\tilde{c} = 0.71533\mu$, Corollary 1 has it that the optimal case, $\tilde{c} = 0.71533\mu$, does not hold for the EWMA in detecting the mean shift of the autocorrelated processes.

The following corollary lists the results of a comparison of ARLs between CUSUM and EWMA charts for large $\mathbf{ARL}_0$. We first introduce a notation. Let

$$f_a(x) = \log\Big(\frac{1}{1-ax}\Big) - \frac{a^2x}{2(1-x)}, \tag{3.2}$$

where $a > 0$, $0 < x < 1$, and $0 < ax < 1$. It can be shown that $f_a(x) > 0$ for all $0 < ax < 1$ when $1 \le a \le 2$. Moreover, if $0 < a < 1$, there is a number $a_1^* \in (0,1)$ such that $f_a(x) > 0$ for $0 < x < a_1^*$, and $f_a(x) < 0$ for $a_1^* < x < 1$; if $a > 2$, there is a number $a_2^* \in (0, 1/a)$ such that $f_a(x) < 0$ for $0 < x < a_2^*$ and $f_a(x) > 0$ for $a_2^* < x < 1/a$. For instance, $a_1^* = 0.85, a_1^* = 0.750, a_2^* = 0.2473$ and $a_2^* = 0.27$, respectively, correspond to the cases $a = 0.357665, a = 0.71533, a = 2.14599$ and $a = 3$.

**Corollary 2.** *Suppose that $\{X_i\}$ is a stationary Gaussian process defined in* (2.1). *Take large $c$ and small $r$ such that $\boldsymbol{ARL}_0(T_C(c)) = \boldsymbol{ARL}_0(T_E(r))$.*

(i) *If $\tilde{c} < \mu < \delta/2$, then $\boldsymbol{ARL}_\mu(T_C(c)) > \boldsymbol{ARL}_\mu(T_E(r))$.*

(ii) *If $\tilde{c} > \mu > \delta/2$, then $\boldsymbol{ARL}_\mu(T_C(c)) < \boldsymbol{ARL}_\mu(T_E(r))$.*

(iii) *If $\mu < \delta/2$ and $\mu < \tilde{c}$, then*

$$\mathbf{ARL}_\mu(T_C(c)) > \mathbf{ARL}_\mu(T_E(r)), \quad \text{for } \ \tilde{c} < \min\Big\{\delta,\ \frac{\delta(1+\sqrt{1-2\mu/\delta})}{2}\Big\};$$

$$\mathbf{ARL}_\mu(T_C(c)) < \mathbf{ARL}_\mu(T_E(r)), \quad \text{for } \ \tilde{c} > \max\Big\{\delta,\ \frac{\delta(1+\sqrt{1-2\mu/\delta})}{2}\Big\}.$$

(iv) *If $\mu > \delta/2$ and $\mu > \tilde{c}$, then*

$$\boldsymbol{ARL}_\mu(T_C(c)) < \boldsymbol{ARL}_\mu(T_E(r)), \quad \textit{for } \ \frac{\delta}{2} \le \tilde{c} \le \delta;$$

$$\boldsymbol{ARL}_\mu(T_C(c)) < \boldsymbol{ARL}_\mu(T_E(r)), \quad \textit{for } \ \tilde{c} < \frac{\delta}{2},\ \frac{\delta}{\mu} < 2a_1^*;$$

$$\boldsymbol{ARL}_\mu(T_C(c)) > \boldsymbol{ARL}_\mu(T_E(r)), \quad \textit{for } \ \tilde{c} < \frac{\delta}{2},\ \frac{\delta}{\mu} > 2a_1^*;$$

$$\boldsymbol{ARL}_\mu(T_C(c)) > \boldsymbol{ARL}_\mu(T_E(r)), \quad \textit{for } \ \tilde{c} > \delta,\ \frac{\delta}{\mu} < 2a_2^*;$$

$$\boldsymbol{ARL}_\mu(T_C(c)) > \boldsymbol{ARL}_\mu(T_E(r)), \quad \textit{for } \ \tilde{c} > \delta,\ \frac{\delta}{\mu} > 2a_2^*,$$

*where $a_1^*$ and $a_2^*$ are defined by $f_a$ with $a = 2\tilde{c}/\delta$.*

**Proof.** Since $\mathbf{ARL}_0(T_C(c)) = \mathbf{ARL}_0(T_E(r))$ for large $c$ and small $r$ implies that $1/r = (1+o(1))c/\tilde{c}^2$ for large $c$ and small $r$, it follows from (2.11), (2.12) and (2.4) that

$$\mathbf{ARL}_\mu(T_E(r)) = (1+o(1))\frac{c}{\tilde{c}^2}\log\frac{\mu}{\mu-\tilde{c}} < \frac{1}{bc}\exp\{\frac{c(1-2\mu/\delta)}{\sigma^2 A^2}\} \le \mathbf{ARL}_\mu(T_C(c))$$

for $\tilde{c} < \mu < \delta/2$ and large $c$. This proves (i). Similarly, we can prove (ii). For (iii), it follows from (2.10) and (2.12) that

$$\log \mathbf{ARL}_\mu(T_E(r)) = (1+o(1))\frac{c(\tilde{c}-\mu)^2}{\tilde{c}^2\sigma^2 A^2},$$
$$\log \mathbf{ARL}_\mu(T_C(c)) = (1+o(1))\frac{c(1-2\mu/\delta)}{\sigma^2 A^2}$$

for large $c$. Thus,

$$\begin{aligned}\log \mathbf{ARL}_\mu(T_E(r)) - \log \mathbf{ARL}_\mu(T_C(c)) &= (1+o(1))\frac{c}{\tilde{c}^2\sigma^2 A^2}[(\tilde{c}-\mu)^2 - \tilde{c}^2(1-\frac{2\mu}{\delta})]\\ &= (1+o(1))\frac{c\mu}{\tilde{c}^2\delta\sigma^2 A^2}[2\tilde{c}^2 - 2\tilde{c}\delta + \frac{\mu}{\delta}].\end{aligned}$$

Note that $\mu > \delta(1-\sqrt{1-2\mu/\delta})/2$ and $2\tilde{c}^2 - 2\tilde{c}\delta + \mu\delta > 0$ for $\delta < \tilde{c}$, or $\tilde{c} < \delta(1-\sqrt{1-2\mu/\delta})/2$, or $\tilde{c} > \delta/2(1+\sqrt{1-2\mu/\delta})$. Hence,

$$\begin{aligned}&\log \mathbf{ARL}_\mu(T_E(r)) - \log \mathbf{ARL}_\mu(T_C(c))\\ &\quad= (1+o(1))\frac{c\mu}{\tilde{c}^2\delta\sigma^2 A^2}[2\tilde{c}^2 - 2\tilde{c}\delta + \mu\delta] < 0 \ (> 0)\end{aligned}$$

for $\tilde{c} < \min\{\delta,\ \delta(1+\sqrt{1-2\mu/\delta})/2\}$ ($\tilde{c} > \max\{\delta,\ \delta(1+\sqrt{1-2\mu/\delta})/2\}$). This proves (iii).

For (iv), let $a = 2\tilde{c}/\delta$ and $x = \delta/(2\mu)$. By (2.11), we have

$$\begin{aligned}&\mathbf{ARL}_\mu(T_E(r)) - \mathbf{ARL}_\mu(T_C(c))\\ &\quad= (1+o(1))[\frac{c}{\tilde{c}^2}\log\Big(\frac{\mu}{\mu-\tilde{c}}\Big) - \frac{c}{\delta(\mu-\delta/2)}]\\ &\quad= (1+o(1))\frac{c}{\tilde{c}^2}[\log\Big(\frac{1}{1-ax}\Big) - \frac{a^2x}{2(1-x)}] = (1+o(1))\frac{c}{\tilde{c}^2}f_a(x).\end{aligned}$$

Thus, (iv) follows from the properties of $f_a$.

### 3.2. Numerical comparison of the ARL for the two control charts

The purpose of this section is to present some simulation results for ARLs of the upper-sided EWMA and CUSUM control charts for detecting mean shifts in the AR(1) model: $X_{i+1} = \rho X_i + \xi_{i+1}$, where $\xi_i \sim N(\bar{\xi}, 1)$.

Table 1. Comparison of ARLs for AR(1) model with $\rho = 0$ (i.i.d. normal data).

| $\mu$ | EWMA $r = 0.043615$ $\tilde{c} = 0.357665$ | EWMA $r = 0.12782$ $\tilde{c} = 0.71533$ | EWMA $r = 0.384112$ $\tilde{c} = 1.43066$ | CUSUM $c = 3.952$ $\delta = 0.5$ | CUSUM $c = 4.722$ $\delta = 1$ | CUSUM $c = 4.9732$ $\delta = 2$ |
|---|---|---|---|---|---|---|
| 0 | 699.1(682.8) | 702.4(695.7) | 698.1(696.3) | 702.3(687.4) | 701.6(695.6) | 698(695.4) |
| 0.1 | 211.4(194.8) | 280.7(273.3) | 382 (378.7) | 255.9(241.1) | 329(320.3) | 411(399.6) |
| 0.25 | 68.27(52.92) | 94.49(86.56) | 168.4(166.4) | 82.05(67.16) | 119.6(113.2) | 203.7(200.9) |
| 0.5 | 25.7(14.29) | 29.26(22.18) | 54.13(51.2) | 28.39(16.54) | 34.61(28.08) | 66.8(65.37) |
| 0.75 | 15.22(6.63) | 14.7 (8.96) | 22.9 (20.03) | 16.22(7.34) | 16 (10.61) | 27.08(24.89) |
| 1 | 10.77(3.93) | 9.44(4.74) | 12.13(9.52) | 11.26(4.21) | 9.82(5.25) | 13.21(10.74) |
| 1.25 | 8.34(2.66) | 6.94(2.99) | 7.56(5.16) | 8.65(2.81) | 7.02(3.17) | 7.91(5.67) |
| 1.5 | 6.84(1.96) | 5.47(2.05) | 5.36(3.2) | 7.04(2.06) | 5.48(2.18) | 5.46(3.4) |
| 2 | 5.06(1.24) | 3.91(1.22) | 3.31(1.55) | 5.16(1.28) | 3.83(1.26) | 3.25(1.62) |
| 3 | 3.42(0.68) | 2.56(0.63) | 1.96(0.69) | 3.44(0.69) | 2.47(0.63) | 1.84(0.7) |
| 4 | 2.63(0.53) | 2.02(0.36) | 1.4(0.51) | 2.64(0.53) | 1.94(0.4) | 1.32(0.48) |

Table 2. Comparison of ARLs for AR(1) model with $\rho = 0.5$.

| $\mu$ | EWMA $r = 0.01798$ $\tilde{c} = 0.357665$ | EWMA $r = 0.04996$ $\tilde{c} = 0.71533$ | EWMA $r = 0.1652$ $\tilde{c} = 1.43066$ | CUSUM $c = 10.6122$ $\delta = 0.5$ | CUSUM $c = 13.152$ $\delta = 1$ | CUSUM $c = 13.352$ $\delta = 2$ |
|---|---|---|---|---|---|---|
| 0 | 702.1(684.9) | 698(694.2) | 702.5(698.8) | 697.5(673.5) | 701.7(687.9) | 701.1(694.3) |
| 0.1 | 319.6(293.7) | 363.9(349.3) | 442.7(436.8) | 368.3(336.1) | 403.5(391.5) | 466.5(465.2) |
| 0.25 | 141.8(115) | 171(154.9) | 239.3(232.3) | 170(137.9) | 202.2(185.9) | 262.8(258) |
| 0.5 | 62.39(39.79) | 69.36(54.48) | 99.86(92.62) | 73.41(47.54) | 80.74(66.46) | 112.45(105) |
| 0.75 | 38.26(20.06) | 38.35(25.57) | 50.77(44.07) | 43.08(22.18) | 43.25(29.68) | 57.52(52.41) |
| 1 | 27.29(12.15) | 25.75(14.71) | 30.24(23.75) | 30.84(13.72) | 27.68(16.17) | 32.67(26.46) |
| 1.25 | 21.28(8.27) | 18.97(9.22) | 20.23(14.29) | 23.74(9.01) | 20.26(10.49) | 21.46(15.83) |
| 1.5 | 17.41(6.05) | 15.17(6.51) | 14.78(9.38) | 19.26(6.47) | 15.7(7.09) | 15.42(10.16) |
| 2 | 12.94(3.72) | 10.82(3.79) | 9.43(4.74) | 14.26(4) | 11.08(4.01) | 9.52(4.91) |
| 3 | 8.76(1.89) | 7.11(1.8) | 5.59(1.88) | 9.57(2.03) | 7.2(1.85) | 5.57(2) |
| 4 | 6.8(1.19) | 5.51(1.11) | 4.16(1.06) | 7.36(1.25) | 5.52(1.16) | 4.1(1.13) |

The numerical results were obtained based on a 1,000,000-repetition experiment. Tables 1−5 compare the simulation results for various values of the mean shift $\mu$ with change point $\tau = 1$ and $\text{ARL}_0 \approx 700$. The first column in the five tables lists the mean shifts ($\mu$). In the five tables, the $\tilde{c}$ and $c$ are the control limits, respectively, for EWMA and CUSUM charts, and $r$ and $\delta$ are, respectively, the weight parameters of the EWMA and the reference values of the CUSUM. The values in the parentheses in every column of Tables 1−5 are the standard deviations of the simulation results.

By (3.1), we know that the EWMA chart has the best performance in detecting the mean shift of i.i.d. processes if the control limits are chosen to be

Table 3. Comparison of ARLs for AR(1) model with $\rho = 0.9$.

| $\mu$ | EWMA $r = 0.003578$ $\tilde{c} = 0.357665$ | EWMA $r = 0.00731$ $\tilde{c} = 0.71533$ | EWMA $r = 0.0176$ $\tilde{c} = 1.43066$ | CUSUM $c = 77.732$ $\delta = 0.5$ | CUSUM $c = 122.121$ $\delta = 1$ | CUSUM $c = 161.35$ $\delta = 2$ |
|---|---|---|---|---|---|---|
| 0 | 699.6(775.4) | 700.4(704.4) | 701.5(688.8) | 698.8(612.5) | 701(644.4) | 699.5(662.5) |
| 0.1 | 554.1(606.8) | 567.6(571.2) | 601.7(581.8) | 587(498.9) | 595.4(540.6) | 598(559.5) |
| 0.25 | 399.1(417.3) | 430.2(416.6) | 470.5(453.7) | 469.5(391.6) | 478.3(407.5) | 483.8(448.3) |
| 0.5 | 263.6(255.8) | 289(262.3) | 326.3(301.7) | 334.2(258.3) | 335.8(279.3) | 346.9(309.3) |
| 0.75 | 188.7(161.7) | 209.6(175.1) | 239.6(207.7) | 252.2(179.5) | 252.5(198.6) | 257.1(219.1) |
| 1 | 144.3(114.1) | 161(128) | 179.7(151.4) | 199.2(129.1) | 196(141.8) | 202.9(168.1) |
| 1.25 | 117.1(84.25) | 129(94.43) | 145.4(113.3) | 160.3(97.61) | 158.1(109.1) | 157.9(124.5) |
| 1.5 | 98.32(65) | 107.1(72.27) | 118.1(88.98) | 137.8(76.68) | 132.3(84.78) | 130.8(96.76) |
| 2 | 73.67(41.78) | 79.47(46.81) | 83.97(55.44) | 105.1(50.83) | 98.09(53.88) | 92.92(61.18) |
| 3 | 49.58(22.54) | 52.29(24.4) | 52.93(27.32) | 70.33(27.08) | 64.02(27.96) | 57.35(30.08) |
| 4 | 38.31(14.15) | 39.59(14.86) | 38.66(16.71) | 53.91(17.1) | 48.4(17.29) | 41.62(17.68) |

Table 4. Comparison of ARLs for AR(1) model with $\rho = -0.5$.

| $\mu$ | EWMA $r = 0.0703995$ $\tilde{c} = 0.357665$ | EWMA $r = 0.19198$ $\tilde{c} = 0.71533$ | EWMA $r = 0.46241$ $\tilde{c} = 1.43066$ | CUSUM $c = 2.4$ $\delta = 0.5$ | CUSUM $c = 3.2685$ $\delta = 1$ | CUSUM $c = 4.9125$ $\delta = 2$ |
|---|---|---|---|---|---|---|
| 0 | 701.3(687.2) | 698.9(703.5) | 702.7(696.8) | 701.9(708.5) | 701.8(703.5) | 698.9(703.7) |
| 0.1 | 158.7(146.1) | 230.4(226.4) | 366(362.3) | 199.9(186.2) | 336.6(332.7) | 493.8(492.5) |
| 0.25 | 42.09(31.38) | 65.64(59.98) | 152.2(150.4) | 52.21(42.21) | 108(104.5) | 308.8(305.5) |
| 0.5 | 14.69(7.38) | 17.27(12.69) | 42.1(39.77) | 16.24(8.68) | 24.85(20.14) | 116.4(113.1) |
| 0.75 | 8.7(3.36) | 8.43(4.77) | 16.33(14.45) | 9.23(3.73) | 10.13(6.19) | 37.91(35.07) |
| 1 | 6.17(2.04) | 5.48(2.56) | 8.14(6.28) | 6.44(2.22) | 6.08(2.96) | 13.62(11.09) |
| 1.25 | 4.79(1.45) | 4(1.63) | 4.85(3.35) | 4.94(1.53) | 4.31(1.8) | 6.76(4.54) |
| 1.5 | 3.96(1.07) | 3.17(1.2) | 3.28(1.99) | 3.98(1.11) | 3.33(1.3) | 4.17(2.44) |
| 2 | 2.94(0.73) | 2.19(0.87) | 1.93(1.06) | 2.95(0.74) | 2.21(0.86) | 2.26(1.17) |
| 3 | 1.82(0.59) | 1.23(0.44) | 1.09(0.3) | 1.79(0.58) | 1.24(0.45) | 1.16(0.39) |
| 4 | 1.18(0.39) | 1.01(0.11) | 1(0.04) | 1.17(0.38) | 1.01(0.11) | 1.01(0.07) |

equal to $\tilde{c} = 0.71533\mu$, where $\mu$ is the real mean shift. Thus, like CUSUM charts (with reference values $\delta = 0.5, \delta = 1$ and $\delta = 2$), three control limits for the EWMA chart, $\tilde{c} = 0.71533 \times 0.5 = 0.357665$, $\tilde{c} = 0.71533 \times 1 = 0.71533$ and $\tilde{c} = 0.71533 \times 2 = 1.43066$, are considered in the numerical simulations.

By comparing the simulation results of the ARLs illustrated in Tables 1−5, we can make the following remarks.

1. The two charts for detecting the mean shift in the i.i.d. process have better performance on the whole than for the process with positive auto-covariance , $\rho > 0$, while they have worse performance than for the process with negative auto-covariance, $\rho < 0$.

Table 5. Comparison of ARLs for AR(1) model with $\rho = -0.9$.

| $\mu$ | EWMA $r = 0.069315$ $\tilde{c} = 0.357665$ | EWMA $r = 0.15822$ $\tilde{c} = 0.71533$ | EWMA $r = 0.32385$ $\tilde{c} = 1.43066$ | CUSUM $c = 3.2284$ $\delta = 0.5$ | CUSUM $c = 6.015$ $\delta = 1$ | CUSUM $c = 10.9558$ $\delta = 2$ |
|---|---|---|---|---|---|---|
| 0 | 701.9(690.2) | 700(704.6) | 697.6(690.9) | 698.4(689.4) | 698.6(680.5) | 698.5(688.8) |
| 0.1 | 135.2(120.9) | 241(235.6) | 396.6(391.6) | 377.7(363.5) | 591.9(593.8) | 616.2(607.1) |
| 0.25 | 34.64(23.16) | 68.59(60.95) | 179.2(173.5) | 78.36(60.49) | 426.5(424.3) | 516(509) |
| 0.5 | 12.89(5.59) | 18.04(11.58) | 60.32(54.71) | 20.23(8.42) | 65.63(49.78) | 375.9(371.1) |
| 0.75 | 7.86(2.69) | 9.01(4.26) | 24.29(20.31) | 11.4(3.5) | 18.53(8.08) | 237.5(227.9) |
| 1 | 5.66(1.7) | 5.9(2.31) | 11.58(8.12) | 7.94(2.08) | 10.39(3.4) | 50.64(39.57) |
| 1.25 | 4.41(1.22) | 4.38(1.48) | 6.66(3.76) | 6.07(1.45) | 7.25(2.01) | 15.89(7.46) |
| 1.5 | 3.63(0.91) | 3.46(1.06) | 4.51(2.13) | 4.91(1.13) | 5.53(1.44) | 8.95(3.22) |
| 2 | 2.8(0.69) | 2.45(0.91) | 2.63(1.18) | 3.49(0.73) | 3.69(0.87) | 4.7(1.38) |
| 3 | 1.37(0.61) | 1.13(0.38) | 1.15(0.46) | 2.48(0.75) | 2.42(0.81) | 2.51(0.84) |
| 4 | 1.01(0.09) | 1(0.04) | 1(0.03) | 1.2(0.45) | 1.16(0.42) | 1.17(0.45) |

2. The optimality of the CUSUM chart for $\mu = \delta$, and the EWMA chart for $\tilde{c} = 0.71533\mu$, in detecting the mean shift of the i.i.d. process does not hold in the autocorrelated process. For instance, the CUSUM with $\delta = 1$ and the EWMA with $\tilde{c} = 0.71533$ in Table 1 ($\rho = 0$, i.i.d. case) have the minimal values $\mathrm{ARL}_\mu(T_C) = 9.82$ and $\mathrm{ARL}_\mu(T_E) = 9.44$ respectively, but in Table 5 ($\rho = -0.9$, negatively autocorrelated case) the ARLs of the CUSUM with $\delta = 0.5$ and the EWMA with $\tilde{c} = 0.357665$ are the minimums.

3. The two charts have their own merits and weaknesses. The EWMA chart has better performance on the whole than the CUSUM in detecting the mean shift of the positively autocorrelated process (see Tables 2 and 3), but the CUSUM chart is more efficient than the EWMA in detecting the mean shift of the negatively autocorrelated process (see Tables 4 and 5).

4. The simulation results support the results of Corollary 1. Since the $\mathrm{ARL}_0$ is not large enough in the numerical simulation, the simulation results do not always coincide with the theoretical comparisons given in Corollary 2, the condition that $\mathrm{ARL}_0$ is large is a necessary one.

In Tables 6−8 we give some comparisons between the numerical results and the estimations in (2.10) and (2.11) for ARL. Let $ARL_E(\rho, \mu, \tilde{c}, r)$ and $ARL_C(\rho, \mu, \delta, c)$ denote the numerical $\mathrm{ARL}_\mu$, respectively, for the EWMA and CUSUM control charts. It follows from (2.12) that

$$\theta^* = (1 - \frac{2\mu}{\delta})(1-\rho)^2, \ \ (\mu < \frac{\delta}{2}), \qquad \theta^*(\tilde{c}) = (\tilde{c} - \mu)^2(1-\rho)^2, \quad (\mu < \tilde{c}),$$

Table 6. $\{L_E\}$ and $\{L_C\}$ with $\text{ARL}_0 \approx 700$ and $\rho = 0$ (i.i.d. normal data).

| $\mu$ | EWMA<br>$r = 0.043615$<br>$\tilde{c} = 0.357665$<br>$L_E$ | EWMA<br>$r = 0.12782$<br>$\tilde{c} = 0.71533$<br>$L_E$ | EWMA<br>$r = 0.384112$<br>$\tilde{c} = 1.43066$<br>$L_E$ | CUSUM<br>$c = 3.952$<br>$\delta = 0.5$<br>$L_C$ | CUSUM<br>$c = 4.722$<br>$\delta = 1$<br>$L_C$ | CUSUM<br>$c = 4.9732$<br>$\delta = 2$<br>$L_C$ |
|---|---|---|---|---|---|---|
| 0 | 37.22(40.80) | 12.83(13.81) | 3.39(3.56) | 13.50(13.71) | 6.24(6.26) | 4.83(4.84) |
| 0.1 | 46.13(52.86) | 14.51(15.46) | 3.80(4.12) | 23.89(25.32) | 7.53(7.64) | 4.68(4.83) |
| 0.25 | 52.34(65.98) | 17.37(19.28) | 4.47(4.73) | | 11.28(11.99) | 4.89(4.83) |
| 0.5 | 2.05(2.07) | 20.36(23.53) | 5.68(6.27) | 0.90(0.91) | | 5.56(5.72) |
| 0.75 | 2.37(2.34) | 1.41(1.43) | 6.86(7.69) | 1.03(1.02) | 0.85(0.86) | 7.81(8.52) |
| 1 | 2.44(2.43) | 2.21(2.22) | 7.48(8.47) | 1.07(1.06) | 1.04(1.03) | |
| 1.25 | 2.48(2.45) | 2.41(2.38) | 6.94(7.98) | 1.09(1.08) | 1.11(1.10) | 0.80 (0.80) |
| 1.5 | 2.52(2.49) | 2.48(2.45) | 1.54(1.56) | 1.11(1.01) | 1.16(1.14) | 1.10(1.06) |
| 2 | 2.58(2.53) | 2.60(2.56) | 2.33(2.32) | 1.14(1.12) | 1.22(1.19) | 1.31(1.28) |
| 3 | 2.71(2.62) | 2.77(2.71) | 2.68(2.65) | 1.20(1.17) | 1.31(1.26) | 1.48(1.44) |
| 4 | 2.82(2.75) | 3.02(2.85) | 2.80(2.81) | 1.25(1.22) | 1.41(1.36) | 1.60(1.53) |

Table 7. $\{L_E\}$ and $\{L_C\}$ with $\text{ARL}_0 \approx 700$ and $\rho = 0.5$.

| $\mu$ | EWMA<br>$r = 0.01798$<br>$\tilde{c} = 0.357665$<br>$L_E$ | EWMA<br>$r = 0.04996$<br>$\tilde{c} = 0.71533$<br>$L_E$ | EWMA<br>$r = 0.1652$<br>$\tilde{c} = 1.43066$<br>$L_E$ | CUSUM<br>$c = 10.6122$<br>$\delta = 0.5$<br>$L_C$ | CUSUM<br>$c = 13.152$<br>$\delta = 1$<br>$L_C$ | CUSUM<br>$c = 13.352$<br>$\delta = 2$<br>$L_C$ |
|---|---|---|---|---|---|---|
| 0 | 118.6(140.1) | 53.93(60.35) | 31.73(34.70) | 49.13(52.78) | 26.20(26.36) | 24.90(25.29) |
| 0.1 | 127.0(158.7) | 54.72(63.04) | 30.36(34.06) | 74.96(87.28) | 29.08(30.67) | 23.13(23.70) |
| 0.25 | 120.7(162.9) | 57.85(67.57) | 29.03(31.34) | | 39.06(42.35) | 21.50(21.96) |
| 0.5 | 2.06(2.07) | 54.00(68.24) | 26.92(30.88) | 1.73(1.76) | | 21.19(22.64) |
| 0.75 | 2.45(2.41) | 1.44(1.47) | 25.18(28.92) | 1.52(1.54) | 2.47(2.49) | 24.97(28.68) |
| 1 | 2.56(2.49) | 2.36(2.34) | 22.84(26.52) | 1.45(1.43) | 2.10(2.07) | |
| 1.25 | 2.61(2.55) | 2.57(2.54) | 19.26(22.98) | 1.40(1.38) | 1.93(1.87) | 4.02(4.01) |
| 1.5 | 2.65(2.57) | 2.69(2.62) | 1.83(1.81) | 1.36(1.34) | 1.79(1.73) | 3.46(3.33) |
| 2 | 2.72(2.63) | 2.81(2.73) | 2.85(2.76) | 1.34(1.30) | 1.68(1.63) | 2.85(2.74) |
| 3 | 2.86(2.73) | 3.00(2.88) | 3.28(3.12) | 1.35(1.30) | 1.64(1.57) | 2.50(2.35) |
| 4 | 3.01(2.84) | 3.22(3.05) | 3.57(3.38) | 1.39(1.32) | 1.68(1.59) | 2.46(2.29) |

respectively, for the CUSUM and EWMA charts. Let

$$L_E = L_E(\rho,\mu,\tilde{c},r) = \begin{cases} \frac{ARL_E(\rho,\mu,\tilde{c},r)}{\exp\{\frac{1}{r}\theta^*(\tilde{c})\}}, & \text{if } \mu < \tilde{c} \\ \frac{ARL_E(\rho,\mu,\tilde{c},r)}{\frac{1}{r}\log\frac{\mu}{\mu-\tilde{c}}}, & \text{if } \mu > \tilde{c}, \end{cases}$$

$$L_C = L_C(\rho,\mu,\delta,c) = \begin{cases} \frac{ARL_C(\rho,\mu,\delta,c)}{\exp\{c\theta^*\}}, & \text{if } \mu < \frac{\delta}{2} \\ \frac{ARL_C(\rho,\mu,\delta,c)}{\frac{c}{\delta(\mu-\frac{\delta}{2})}}, & \text{if } \mu > \frac{\delta}{2}. \end{cases}$$

Table 8. $\{L_E\}$ and $\{L_C\}$ with $\mathrm{ARL}_0 \approx 700$ and $\rho = -0.5$.

| $\mu$ | EWMA $r = 0.0703995$ $\tilde{c} = 0.357665$ $L_E$ | EWMA $r = 0.19198$ $\tilde{c} = 0.71533$ $L_E$ | EWMA $r = 0.46241$ $\tilde{c} = 1.43066$ $L_E$ | CUSUM $c = 2.4$ $\delta = 0.5$ $L_C$ | CUSUM $c = 3.2685$ $\delta = 1$ $L_C$ | CUSUM $c = 4.9125$ $\delta = 2$ $L_C$ |
|---|---|---|---|---|---|---|
| 0 | 11.76(12.37) | 1.74(1.78) | 0.03(0.03) | 3.17(3.22) | 0.45(0.47) | 0.01 (0.01) |
| 0.1 | 19.01(20.52) | 2.72(2.86) | 0.07(0.06) | 7.83(8.41) | 0.94(0.97) | 0.02(0.02) |
| 0.25 | 29.06(33.69) | 5.19(5.36) | 0.17(0.16) | | 2.76(2.89) | 0.08(0.07) |
| 0.5 | 1.90(1.94) | 10.03(11.54) | 0.62(0.63) | 0.85(0.87) | | 0.46(0.45) |
| 0.75 | 2.18(2.20) | 1.21(1.26) | 1.71(1.81) | 0.96(0.97) | 0.77(0.79) | 2.39(2.48) |
| 1 | 2.26(2.27) | 1.93(1.94) | 3.30(3.56) | 1.01(1.01) | 0.93(0.94) | |
| 1.25 | 2.30(2.32) | 2.08(2.12) | 4.14(4.66) | 1.03(1.03) | 0.99(0.99) | 0.69(0.71) |
| 1.5 | 2.36(2.34) | 2.16(2.19) | 1.14(1.19) | 1.04(1.04) | 1.02(1.02) | 0.85(0.86) |
| 2 | 2.49(2.39) | 2.19(2.23) | 1.64(1.69) | 1.07(1.06) | 1.01(1.03) | 0.92(0.93) |
| 3 | 2.32(2.43) | 1.99(2.03) | 1.79(1.73) | 1.03(1.07) | 0.95(0.95) | 0.94(0.96) |
| 4 | 2.04(2.17) | 2.27(2.08) | 2.41(2.24) | 0.91(0.95) | 1.08(1.01) | 1.23(1.12) |

It follows from (2.10) and (2.11) that $L_C$ for $\mu < \delta/2$ and $L_E$ for $\mu < \tilde{c}$ satisfy $1/(bc) \le L_C \le 2c/u$ and $1 \le L_E \le 3\log r^{-1}/r$, respectively, for large $c$ and small $r$, and both $L_C$ for $\mu > \delta/2$ and $L_E$ for $\mu > \tilde{c}$ go to 1 as $c \to \infty$ and $r \to 0$, respectively, where $u = \delta(\delta/2 - \mu)$ and $b = (4 + \sqrt{15})/u$.

The numbers $\{L_E\}$ and $\{L_C\}$ with $\rho = 0, 0.5, -0.5$ and $\mathrm{ARL}_0 \approx 700$ are given in Tables 6, 7 and 8. The numbers in the parentheses in every column of Tables 6–8 are $\{L_E\}$ and $\{L_C\}$ with $\mathrm{ARL}_0 \approx 1,400$.

As can be seen, both $\{L_E\}$ for $\mu > \tilde{c}$ and $\{L_C\}$ for $\mu > \delta/2$ do not depend on the number $\rho$, and they are close to 1 from $\mathrm{ARL}_0 \approx 700$ to $\mathrm{ARL}_0 \approx 1,400$, with $\{L_C\}$ approaching 1 more quickly. Conversely, when $\mu < \tilde{c}$ and $\mu < \delta/2$, $\{L_E\}$ and $\{L_C\}$ depend heavily on the number $\rho$. For $\mu < \tilde{c}$ and $\mu < \delta/2$, we may conjecture that

$$L_E \sim \frac{\sqrt{\pi}}{\sqrt{r}(\tilde{c} - \mu)(1 - \rho)}, \qquad L_C \sim \frac{2\exp\{1.166(\delta - 2\mu)(1 - \rho)\}}{(\delta - 2\mu)^2(1 - \rho)^2}$$

as $r \to 0$ and $c \to \infty$, respectively. In fact, the above conjecture is true when $\rho = 0$ (see Srivastava and Wu (1993, 1997)).

## 4 Conclusion and Discussion

The asymptotic expressions of the ARLs for detecting the mean shifts of the stationary linear processes extensively generalize known results for i.i.d. processes. Our asymptotic expressions for the ARLs in stationary linear processes demonstrate that the auto-covariance of the processes play an important role on the detection performance of the CUSUM and EWMA charts, and they point out the merits and weaknesses of the two charts, especially in the cases of $\mu < \delta/2$ and $\mu < \tilde{c}$.

The assumption in this paper that the change point, is $\tau = 1$ is not essential when the $\mathrm{ARL}_0$ is large enough. In fact, by the same methods used to prove our theorems, we can prove that

$$-(1+o(1))\frac{3\sqrt{c}\log c}{(\hat{\mu})^{3/2}} + \frac{c}{\hat{\mu}} + \tau - 1$$
$$\leq \mathbf{ARL}_\mu(T_C(c)) \leq \tau - 1 + \frac{c}{\hat{\mu}} + \frac{2\sqrt{c}\log c}{(\hat{\mu})^{3/2}} + \frac{e^{(\delta\sigma A)^2/2}}{\hat{\mu}c^{\sqrt{2}-1}}(1+o(1))$$

for $\mu > \delta/2$ and

$$(1+o(1))(1-\frac{1}{(\log r^{-1})^p})\frac{1}{r}\log\frac{\mu}{\mu-\tilde{c}} + \tau - 1$$
$$\leq \mathbf{ARL}_\mu(T_E(r)) \leq \tau - 1 + \frac{1}{r}\log\frac{\mu}{\mu-\tilde{c}}(1+o(1))$$

for $\mu > \tilde{c}$, when $\tau > 1$ and the $\mathrm{ALR}_0$ is large enough.

Two problems deserve further study: (1) the asymptotic expressions of the ARLs for the CUSUM and EWMA when $\mu = \delta/2$ and $\mu = \tilde{c}$; and (2) the asymptotic expressions for the ARLs for the CUSUM and EWMA in detecting a dynamic mean change of non-stationary processes.

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