Supplement for Change Point Analysis of Correlation in Non-stationary Time Series

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A Proofs of main results

In this section we provide proofs of the main results, where some of the technical details are deferred to the Appendix [see Section B]. In the following discussion we will also make frequent use of the projection operator $\mathcal{P}_j(\cdot) = \mathbb{E}(\cdot|\mathcal{F}_j) - \mathbb{E}(\cdot|\mathcal{F}_{j-1})$. Throughout this section the symbol $\Rightarrow$ denotes weak convergence of a stochastic process in $\mathcal{C}(0,1)$ with the uniform topology. The moments of K and K^2 are denoted by $\mu_l = \int_{\mathbb{R}} x^l K(x)dx$ and $\phi_l = \int_{\mathbb{R}} x^l K^2(x)dx$, respectively for $l \in \mathbb{Z}$. For series a_n and b_n, denotes $a_n \asymp b_n$ if $a_n = O(b_n)$ and $b_n = O(a_n)$.

A.1 Proof of Theorem 3.1, 3.2 and Lemma 3.1

We will start with an auxiliary Lemma, which will be used in the proof of Theorem 3.1.

Lemma A.1. *Under conditions of Theorem 3.1, we have that for any fix lag-k,*

$$\max_{1\le i\le n} |S_i^\diamond - \hat{S}_i^\diamond| = O_p(\sqrt{nb_n} + nb_n^3 + b_n^{-1}), \tag{A.1}$$

where $S_i^\diamond = \sum_{s=1}^i e_i e_{i+k}$, $\hat{S}_i^\diamond = \sum_{s=1}^i \hat{e}_i \hat{e}_{i+k}$.

Proof. First note that

$$S_i^\diamond - \hat{S}_i^\diamond = A_{n,i} + B_{n,i} + C_{n,i}, \tag{A.2}$$

where the quantities $A_{n,i}$, $B_{n,i}$ and $C_{n,i}$ are defined by

$$A_{n,i} = 2\sum_{j=1}^{i} e_{j+k}\big(\hat{\mu}_{b_n}(t_j) - \mu(t_j)\big), C_{n,i} = 2\sum_{j=1}^{i} e_j\big(\hat{\mu}_{b_n}(t_{j+k}) - \mu(t_{j+k})\big),$$

$$B_{n,i} = \sum_{j=1}^{i} \big(\mu(t_j) - \hat{\mu}_{b_n}(t_j)\big)\big(\mu(t_{j+k}) - \hat{\mu}_{b_n}(t_{j+k})\big).$$

Observing the estimate (B.4) in Section B.1, we have that

$$\max_{1\le i\le n} B_{n,i} = O_p(b_n^{-1} + nb_n^4). \tag{A.3}$$

By Lemma B.1 (which is proved in Section B.1) it follows that

$$\max_{\lfloor nb_n\rfloor \le i \le n-\lfloor nb_n\rfloor} \Big| A_{n,i} - 2\sum_{j=\lfloor nb_n\rfloor+1}^{i} a_{n,j} - 2\sum_{j=1}^{\lfloor nb_n\rfloor} e_{j+k}(\hat{\mu}_{b_n}(t_j)-\mu(t_j))\Big| = O_p(n\chi_n),$$

$$\max_{n-\lfloor nb_n\rfloor+1 \le i \le n} \Big| A_{n,i} - 2\sum_{j=\lfloor nb_n\rfloor}^{n-\lfloor nb_n\rfloor} a_{n,j} - 2\sum_{j=1}^{\lfloor nb_n\rfloor-1} e_{j+k}(\hat{\mu}_{b_n}(t_j)-\mu(t_j))$$
$$- 2\sum_{j=n-\lfloor nb_n\rfloor+1}^{i} e_{j+k}(\hat{\mu}_{b_n}(t_j)-\mu(t_j))\Big| = O_p(n\chi_n),$$

where $\chi_n = b_n^3 + \frac{1}{nb_n}$, and

$$a_{n,j} = \frac{e_{j+k}}{nb_n}\sum_{s=1}^{n} K_{b_n}\big(\frac{s-j}{n}\big)e_s \qquad (j=1,\dots,n). \tag{A.4}$$

A further application of the estimate (B.4) in Section B.1 and the Cauchy-Schwarz inequality gives

$$\Big\| \max_{1\le j\le\lfloor nb_n\rfloor} \Big| \sum_{i=1}^{j} e_{j+k}\big(\mu(t_j)-\hat{\mu}_{b_n}(t_j)\big)\Big|\Big\|_2 \le \sum_{i=1}^{\lfloor nb_n\rfloor} \|e_{j+k}\|_4 \|\mu(t_j)-\hat{\mu}_{b_n}(t_j)\|_4$$
$$= O(\sqrt{nb_n} + nb_n^3 + \frac{1}{nb_n}),$$

$$\Big\| \max_{n-\lfloor nb_n\rfloor+1\le j\le n} \Big| \sum_{i=n-\lfloor nb_n\rfloor+1}^{j} e_{j+k}\big(\mu(t_j)-\hat{\mu}_{b_n}(t_j)\big)\Big|\Big\|_2 = O(\sqrt{nb_n} + nb_n^3 + \frac{1}{nb_n}).$$

This implies that

$$\max_{1\le i\le n} |A_{n,i}| \le \max_{\lfloor nb_n\rfloor \le i\le n-\lfloor nb_n\rfloor} |\bar{A}_{n,i}| + O_p(\sqrt{nb_n} + nb_n^3 + \frac{1}{nb_n}), \tag{A.5}$$

where $\bar{A}_{n,i} = 2\sum_{j=\lfloor nb_n\rfloor}^{i} a_{n,j}$ and $a_{n,j}$ is defined in (A.4).
In the following we derive an estimate for the first term on the right-hand side of (A.5). For this purpose we consider the random variables $\tilde{e}_{s,m} = \mathbb{E}(e_s|\varepsilon_s, ..., \varepsilon_{s-m})$ and note that the sequence $(\tilde{e}_{s,m})_{s=1}^n$ is m-dependent. Now define $a_{n,j}^{(m)} = e_{j+k}\sum_{s=1}^n K_{b_n}\big(\frac{s-j}{n}\big)\tilde{e}_{s,m}/(nb_n)$ and

$$\bar{A}_{n,i}^{(m)} = 2\sum_{j=\lfloor nb_n\rfloor}^{i} a_{n,j}^{(m)},$$

then a similar argument as given in the proof of Theorem 1 of Zhou (2014) shows that

$$\max_{1\le j\le n} \Big\| \sum_{s=1}^{n} K_{b_n}\big(\frac{s-j}{n}\big)(\tilde{e}_{s,m}-e_s)\Big\|_4 \le C\sqrt{nb_n}m\chi^m$$

for some constant $\chi \in (0,1)$. By the Cauchy-Schwartz inequality it follows that

$$\Big\| \max_{\lfloor nb_n \rfloor \le i \le n-\lfloor nb_n \rfloor} |\bar{A}_{n,i} - \bar{A}_{n,i}^{(m)}| \Big\|_2 \le \Big\| \frac{2}{nb_n} \sum_{j=\lfloor nb_n \rfloor}^{n-\lfloor nb_n \rfloor} |e_{j+k}| \sum_{s=1}^{n} K_{b_n}\big(\frac{s-j}{n}\big)(\tilde{e}_{s,m} - e_s)| \Big\|_2 \qquad (A.6)$$
$$= O(\sqrt{n} m \chi^m b_n^{-1/2}).$$

Write $\tilde{a}_{n,j}^{(m)} = \tilde{e}_{j+k,m} \sum_{s=1}^{n} K_{b_n}(\frac{s-j}{n}) \tilde{e}_{s,m}/(nb_n)$ and $\tilde{A}_{n,i}^{(m)} = 2\sum_{j=\lfloor nb_n \rfloor}^{i} \tilde{a}_{n,j}^{(m)}$ it is easy to see that

$$\Big\| \max_{\lfloor nb_n \rfloor \le i \le n-\lfloor nb_n \rfloor} |\tilde{A}_{n,i}^{(m)} - \bar{A}_{n,i}^{(m)}| \Big\|_2 \le \frac{2}{nb_n} \sum_{j=1}^{n} \|e_{j+k} - \tilde{e}_{j+k,m}\|_4 \Big\| \sum_{s=1}^{n} K_{b_n}(\frac{s-j}{n}) \tilde{e}_{s,m} \Big\|_4. \qquad (A.7)$$

Now an elementary calculation via Burkholder's inequality shows

$$\max_{1 \le j \le n} \Big\| \frac{1}{nb_n} \sum_{s=1}^{n} K_{b_n}\Big(\frac{s-j}{n}\Big) \tilde{e}_{s,m} \Big\|_4 = O\Big(\frac{1}{\sqrt{nb_n}}\Big),$$

and by a similar argument as given in the proof of Theorem 1 of Zhou (2014) we have for some constant $\chi \in (0,1)$ the estimate $\max_{1 \le j \le n} \|\tilde{e}_{j,m} - e_j\|_4 = O(\chi^m)$. This gives for the left-hand side of (A.7)

$$\Big\| \max_{\lfloor nb_n \rfloor \le i \le n-\lfloor nb_n \rfloor} |\bar{A}_{n,i}^{(m)} - \tilde{A}_{n,i}^{(m)}| \Big\|_2 = O(\sqrt{n/b_n} \chi^m),$$

and an application of (A.6) yields

$$\Big\| \max_{\lfloor nb_n \rfloor \le i \le n-\lfloor nb_n \rfloor} |\bar{A}_{n,i} - \tilde{A}_{n,i}^{(m)}| \Big\|_2 = O(\sqrt{n/b_n} m \chi^m). \qquad (A.8)$$

A tedious but straightforward calculation shows that $\mathcal{P}_{j-l}(\tilde{e}_{j,m}\tilde{e}_{i,m}) = 0$ for $l > 2m$. For example, if $i \ge j-m$, then by definition, $\tilde{e}_{j,m}\tilde{e}_{i,m}$ is $\sigma(\varepsilon_{j-2m}, \varepsilon_{j-2m+1}, ..., \varepsilon_{i\vee j})$ measurable. Consequently,

$$\mathbb{E}(\tilde{e}_{j,m}\tilde{e}_{i,m}|\mathcal{F}_{j-l}) = \mathbb{E}(\tilde{e}_{j,m}\tilde{e}_{i,m}|\mathcal{F}_{j-l-1}) = \mathbb{E}(\tilde{e}_{j,m}\tilde{e}_{i,m})$$

if $l > 2m$, which gives $\mathcal{P}_{j-l}(\tilde{e}_{j,m}\tilde{e}_{i,m}) = 0$. The other cases $i \le j-l-1$ and $j-l \le i \le j-m-1$ are treated similarly, and details are omitted for the sake of brevity. Observing $\mathcal{P}_{j-l}(\tilde{e}_{j,m}\tilde{e}_{i,m}) = 0$ for $l > 2m$ we obtain

$$\Big\| \max_{\lfloor nb_n \rfloor \le i \le n-\lfloor nb_n \rfloor} |\tilde{A}_{n,i}^{(m)} - \mathbb{E}\tilde{A}_{n,i}^{(m)}| \Big\|_2 \le 2\sum_{l=0}^{2m} \Big\| \max_{\lfloor nb_n \rfloor \le i \le n-\lfloor nb_n \rfloor} |\sum_{j=\lfloor nb_n \rfloor}^{i} \mathcal{P}_{j-l}\tilde{a}_{n,j}^{(m)}| \Big\|_2. \qquad (A.9)$$

Similar arguments as given in the proof of Theorem 1 in Wu (2005) show

$$\|\mathcal{P}_{j-l}\tilde{a}_{n,j}^{(m)}\|_2 \le \frac{M}{n} \Big\| \tilde{e}_{j+k,m} \sum_{s=1}^{n} \tilde{e}_{s,m} K_{b_n}\Big(\frac{s-j}{n}\Big) - \tilde{e}_{j+k,m}^{(j-l)} \sum_{s=1}^{n} \tilde{e}_{s,m}^{(j-l)} K_{b_n}\Big(\frac{s-j}{n}\Big) \Big\|_2,$$

and by the triangle inequality it follows that

$$\|\mathcal{P}_{j-l}\tilde{a}_{n,j}^{(m)}\|_2 \leq M(Z_{1,j} + Z_{2,j}),$$

where the terms $Z_{1,j}$ and $Z_{2,j}$ are defined by

$$Z_{1,j} = \frac{1}{nb_n}\Big\|\tilde{e}_{j+k,m}\sum_{s=1}^{n}K_{b_n}\Big(\frac{s-j}{n}\Big)\big[\tilde{e}_{s,m}^{(j-l)} - \tilde{e}_{s,m}\big]\Big\|_2,$$
$$Z_{2,j} = \frac{1}{nb_n}\Big\|\big[\tilde{e}_{j+k,m}^{(j-l)} - \tilde{e}_{j+k,m}\big]\sum_{s=1}^{n}K_{b_n}\Big(\frac{s-j}{n}\Big)\tilde{e}_{s,m}^{(j-l)}\Big\|_2,$$

$\tilde{e}_{s,m}^{(j)} = \mathbb{E}(e_s^{(j)}|\varepsilon_{s-m},\ldots,\varepsilon_j',\ldots,\varepsilon_s)$ for $s-m \leq j \leq s$, $e_s^{(j)} = G_l(t_s,\mathcal{F}_s^{(j)})$ for $b_l < t_s \leq b_{l+1}$ and we use the convention $\tilde{e}_{s,m}^{(j)} = \tilde{e}_{s,m}$ for $j < s-m$ or $j > s$. Elementary calculations show that for $l \geq 0$

$$\Big\|\sum_{s=1}^{n}K_{b_n}(\frac{s-j}{n})\tilde{e}_{s,m}^{(j-l)}\Big\|_4 = O(\sqrt{nb_n}), \qquad 1 \leq j \leq n,$$

while by definition $\|\tilde{e}_{j,m}^{(j-l)} - \tilde{e}_{j,m}\|_4 = 0$ for $l > m$. On the other hand, if $1 \leq j \leq n$, $0 \leq l \leq m$, we have by Assumption (A4)

$$\|\tilde{e}_{j+k,m}^{(j-l)} - \tilde{e}_{j+k,m}\|_4 \leq M\chi^{l+k},$$

which gives $Z_{2,j} = O(\frac{\chi^l}{\sqrt{nb_n}})$. Observing that $\tilde{e}_{s,m}^{(j-l)} - \tilde{e}_{s,m} = 0$ if $s \geq j-l+m+1$ or $s \leq j-l-1$, it is easy to see that $Z_{1,j} = O(\frac{m}{nb_n})$. It now follows from Doob's inequality

$$\Big\|\max_{\lfloor nb_n\rfloor \leq i \leq n-\lfloor nb_n\rfloor}|\sum_{j=\lfloor nb_n\rfloor}^{i}\mathcal{P}_{j-l}\tilde{a}_{n,j}^{(m)}|\Big\|_2 = O\Big(\sqrt{n}\Big(\frac{\chi^l}{\sqrt{nb_n}} + \frac{m}{nb_n}\Big)\Big),$$

and we obtain from (A.9) that

$$\Big\|\max_{\lfloor nb_n\rfloor \leq i \leq n-\lfloor nb_n\rfloor}|\tilde{A}_{n,i}^{(m)} - \mathbb{E}\tilde{A}_{n,i}^{(m)}|\Big\|_2 = O\Big(\frac{m^2}{n^{1/2}b_n} + (b_n)^{-1/2}\Big). \qquad \text{(A.10)}$$

Finally, similar arguments as given in the proof of Lemma 5 in Zhou and Wu (2010) show

$$\max_{\lfloor nb_n\rfloor \leq i \leq n-\lfloor nb_n\rfloor}\mathbb{E}[\tilde{A}_{n,i}^{(m)}] = O\Big(\sum_{i=1}^{n}\sum_{j=1}^{n}\chi^{|i-j|}/(nb_n)\Big) = O(b_n^{-1}).$$

Observing (A.5), (A.8) and (A.10) and taking $m = M\log n$ for a sufficiently large constant $M > 0$ yields $\max_{1\leq i\leq n}|A_{n,i}| = O_p(\sqrt{nb_n} + nb_n^3 + b_n^{-1})$. Similarly $\max_{1\leq i\leq n}|C_{n,i}| = O_p(\sqrt{nb_n} + nb_n^3 + b_n^{-1})$. Consequently, the assertion (A.1) follows from (A.2), (A.3) and these two estimates.

A.1.1 Proof of Lemma 3.1

Define $\mathcal{N}(i) = \frac{1}{L}\big(\sum_{j=i-L+1}^{i} e_j^2 - \sum_{j=i}^{i+L-1} e_j^2\big)$ and recall the definition of $\mathcal{M}(i)$ in (3.6). By similar arguments as given in the proof of Lemma B.3 (note that $\iota > 8$) we have $\|\mathcal{M}(i) - \mathcal{N}(i)\|_4 = b_n^2 + \frac{1}{\sqrt{nb_n}}$, and Proposition B.1 yields

$$\max_{L \le i \le n-L+1} |\mathcal{M}(i) - \mathcal{N}(i)| = O_p\Big(n^{1/4} b_n^2 + \frac{1}{n^{1/4} b_n^{1/2}}\Big). \tag{A.11}$$

Consider the case that $i \in B := \{i : |t_i - \tilde{t}_v| > 2L\}$. Then by our assumption on the variance function, there exists a large constant C, such that $|\mathbb{E}\mathcal{N}(i)| \le CL/n$ for $L \le i \le n-L+1$, $i \in B$. By Lemma B.3 and Lemma B.4 it now follows $\|\mathcal{N}(i) - \mathbb{E}\mathcal{N}(i)\|_{\iota/2} \le CL^{-1/2}$ ($L \le i \le n-L+1$, $i \in B$), which gives

$$\max_{L \le i \le n-L+1, i \in B} |\mathcal{N}(i)| = O_p(L^{-1/2} n^{2/\iota} + L/n).$$

Combining this estimate with (A.11) yields

$$\max_{L \le i \le n-L+1, i \in B} |\mathcal{M}(i)| = O_p\Big(L^{-1/2} n^{2/\iota} + L/n + n^{1/4} b_n^2 + \frac{1}{n^{1/4} b_n^{1/2}}\Big).$$

Similarly, we can show that $\mathcal{M}(\lfloor n\tilde{t}_v \rfloor) = \sigma(t_v^+) - \sigma(t_v^-) + O_p\big(n^{1/4} b_n^2 + \frac{1}{n^{1/4} b_n^{1/2}} + L^{-1/2} + L/n\big)$. Let $L = \lfloor n^{4/\iota} \log\log n \rfloor$. The choice of L implies that

$$\lim_{n \to \infty} \mathbb{P}\Big(|\mathcal{M}(\lfloor n\tilde{t}_v \rfloor)| > \max_{L \le i \le n-L+1, i \in B} |\mathcal{M}(i)|\Big) = 1,$$

which completes the proof of Lemma 3.1. □

A.1.2 Proof of Theorem 3.1

We restrict ourselves to the case of a variance function with 1 abrupt change point. The situation that the variance changes smoothly with time could be shown similarly and easier, with the fact that $t_n^* \in [\zeta, 1-\zeta]$. Recall that for any fixed lag-k, $S_i^{(k)} = \sum_{j=1}^{i} W_j^{(k)}$, $\hat{S}_i^{(k)} = \sum_{j=1}^{i} \hat{W}_j^{(k)}$ where

$$W_j^{(k)} = \frac{e_i e_{i+k}}{\sigma(t_i)\sigma(t_{i+k})}, \ \hat{W}_j^{(k)} = \frac{\hat{e}_i \hat{e}_{i+k}}{\hat{\sigma}^2(t_i)}, \ T_{n,i}^{(k)} = |S_i^{(k)} - \frac{i}{n} S_n^{(k)}|,$$

We will show the estimate

$$\max_{1 \le i \le n-k} |\hat{S}_i^{(k)} - S_i^{(k)}| = O_p(nc_n^2 + nb_n^3 c_n^{-1/4} + b_n^{-1} c_n^{-1} + n^{1-\upsilon'}), \tag{A.12}$$

which implies $\max_{1\le i\le n-k}|\hat{T}^{(k)}_{n,i}-T^{(k)}_{n,i}| = O_p(nc_n^2+nb_n^3c_n^{-1/4}+b_n^{-1}c_n^{-1}+n^{1-\upsilon'})$, where υ' is a constant which satisfies $\upsilon'\in(\frac{1}{2},1-\frac{4}{\iota})$. Define

$$T_n=\max_{1\le i\le n-r_l}|(\hat{T}^{(r_1)}_{n,i},\dots,\hat{T}^{(r_l)}_{n,i})^T|,$$

then it follows from Section 5 in Zhou (2013) that $T_n/\sqrt{n}$ converges weakly to the distribution of the random variable $\mathcal{K}_1$ defined in Theorem 3.1. By our choice of the bandwidth b_n we have $nc_n^2+nb_n^3c_n^{-1/4}+b_n^{-1}c_n^{-1}+n^{1-\upsilon'}=o(\sqrt{n})$, and the assertion of Theorem 3.1 follows from (A.12). For the sake of simplicity we omit in the subscripts c_n,b_n in the variance estimator $\hat\sigma_{c_n,b_n}$ and the superscript k in the definition $\hat{S}^{(k)}_i, S^{(k)}_i$ the proof of the estimate (A.12). With the notation $\tilde{S}_i=\frac{e_ie_{i+k}}{\sigma^2(t_i)}$ we obtain

$$\max_{1\le j\le n}\left|S_j-\tilde{S}_j\right|\le\max_{1\le j\le n}\sum_{i=1}^{j}\frac{|e_ie_{i+k}|\cdot|\sigma(t_i)-\sigma(t_{i+k})|}{\sigma^2(t_i)\sigma(t_{i+k})}=O_p(1), \tag{A.13}$$

where we have used the fact that the variance function is Lipschitz continuous before and after t_v. Let $\bar{S}_j=\sum_{i=1}^{j}\frac{\hat{e}_i\hat{e}_{i+k}}{\sigma^2(t_i)}$, where the estimate $\hat\sigma^2(t_i)$ has been replaced by the "true" variance $\sigma^2(t_i)$. By Lemma A.1, it can be seen that by similar argument,

$$\max_{1\le j\le n}\left|\bar{S}_j-\tilde{S}_j\right|=O_p(\sqrt{nb_n}+nb_n^3+b_n^{-1}). \tag{A.14}$$

Define

$$\Lambda_j:=(\hat{S}_j-\bar{S}_j)=\sum_{i=1}^{j}\frac{\hat{e}_i\hat{e}_{i+k}(-\hat\sigma^2(t_i)+\sigma^2(t_i))}{\hat\sigma^2(t_i)\sigma^2(t_i)},$$

then our next goal is to estimate $\max_{1\le j\le n}|\Lambda_j|$. For this purpose we consider the random variable

$$\bar\Lambda_j:=\sum_{i=1}^{j}\frac{\hat{e}_i\hat{e}_{i+k}(-\hat\sigma^2(t_i)+\sigma^2(t_i))}{\sigma^4(t_i)}=\sum_{i=1,t_i\notin[\tilde{t}_v-n^{-\upsilon'},\tilde{t}_v+n^{-\upsilon'}]}^{j}\frac{\hat{e}_i\hat{e}_{i+k}(-\hat\sigma^2(t_i)+\sigma^2(t_i))}{\sigma^4(t_i)}+O_p(n^{1-\upsilon'})$$

(here the estimator in the denominator has been replaced by the true variance function, and the remaining order is due to Lemma B.3), and obtain

$$\max_{1\le j\le n}|\Lambda_j-\bar\Lambda_j|\le\sum_{i=1}^{n}\frac{|\hat{e}_i\hat{e}_{i+k}|(\hat\sigma^2(t_i)-\sigma^2(t_i))^2}{\hat\sigma^2(t_i)\sigma^4(t_i)}\mathbf{1}(t_i\notin[\tilde{t}_v-n^{-\upsilon'},\tilde{t}_v+n^{-\upsilon'}])+O_p(n^{1-\upsilon'}). \tag{A.15}$$

For the expectation of the right-hand side it follows

$$\begin{aligned}\mathbb{E}\Big[\sum_{i=1}^{n}\frac{|\hat{e}_i\hat{e}_{i+k}|(\hat\sigma^2(t_i)-\sigma^2(t_i))^2}{\hat\sigma^2(t_i)\sigma^4(t_i)}\mathbf{1}(t_i\notin[\tilde{t}_v-n^{-\upsilon'},\tilde{t}_v+n^{-\upsilon'}]\Big]) \qquad (A.16)\\ \le C\sum_{i=1}^{n}\|\hat{e}_i\|_4\|\hat{e}_{i+k}\|_4\|(\hat\sigma^2(t_i)-\sigma^2(t_i))^2\|_2\mathbf{1}(t_i\notin[\tilde{t}_v-n^{-\upsilon'},\tilde{t}_v+n^{-\upsilon'}]).\end{aligned}$$

By Lemma B.3 of Section B.1, we have that

$$\|\hat{\mu}_{b_n}(t) - \mu(t)\|_4 = O\Big(b_n^2 + \frac{1}{\sqrt{nb_n}}\Big), \tag{A.17}$$

which implies $\|\hat{e}_i\|_4 \leq C$. On the other hand, Corollary B.2 in Section B.1 shows

$$\max_{1\leq i\leq n} \|(\hat{\sigma}^2(t_i) - \sigma^2(t_i))^2 \mathbf{1}(t_i \notin [\tilde{t}_v - n^{-v'}, \tilde{t}_v + n^{-v'}])\|_2 = O\Big(b_n^4 + \frac{1}{nb_n} + c_n^4 + \frac{1}{nc_n}\Big),$$

and we obtain from (A.15), (A.16) and Proposition B.1 in Section B.2 the estimate

$$\begin{aligned} \max_{1\leq j\leq n} |\Lambda_j| &\leq \max_{1\leq j\leq n} |\bar{\Lambda}_j| + \max_{1\leq j\leq n} |\Lambda_j - \bar{\Lambda}_j| \qquad\qquad (A.18) \\ &= \max_{1\leq j\leq n} |\bar{\Lambda}_j| + O_p(nb_n^4 + b_n^{-1} + nc_n^4 + c_n^{-1}). \end{aligned}$$

Now the remaining problem is to derive an appropriate estimate for the quantity $\max_{1\leq j\leq n} |\bar{\Lambda}_j|$. For this purpose note that $\bar{\Lambda}_j = \bar{\lambda}_{j,1} + \bar{\lambda}_{j,2} + O_p(n^{1-v'})$, where

$$\begin{aligned} \bar{\lambda}_{j,1} &= \sum_{i=1}^{j} \frac{(\hat{e}_i\hat{e}_{i+k} - e_ie_{i+k})(\sigma^2(t_i) - \hat{\sigma}^2(t_i))}{\sigma^4(t_i)} \mathbf{1}(t_i \notin [\tilde{t}_v - n^{-v'}, \tilde{t}_v + n^{-v'}]), \\ \bar{\lambda}_{s,j,2} &= \sum_{i=s}^{j} \frac{e_ie_{i+k}(\sigma^2(t_i) - \hat{\sigma}^2(t_i))}{\sigma^4(t_i)} \mathbf{1}(t_i \notin [\tilde{t}_v - n^{-v'}, \tilde{t}_v + n^{-v'}]). \end{aligned}$$

and $\bar{\lambda}_{j,2} = \bar{\lambda}_{1,j,2}$ and $\mathcal{A}_i = \{t_i \notin [\tilde{t}_v - n^{-v'}, \tilde{t}_v + n^{-v'}]\}$ for short. By Lemma B.1, Corollary B.2 of Section B.1 and the estimate (A.17) it is easy to see that

$$\begin{aligned} \mathbb{E}\Big[\max_{1\leq j\leq n} |\bar{\lambda}_{j,1}|\Big] &\leq \sum_{i=1}^{n} \frac{\|\hat{e}_i\hat{e}_{i+k} - e_ie_{i+k}\|_2}{\sigma^4(t_i)} \|\sigma^2(t_i) - \hat{\sigma}^2(t_i)\|_2 \mathbf{1}(\mathcal{A}_i) = O(\underline{\pi}_n), \qquad (A.19) \\ \mathbb{E}\Big[\max_{1\leq j\leq \lfloor nb_n+nc_n\rfloor} |\bar{\lambda}_{j,2}|\Big] &\leq \sum_{i=1}^{\lfloor nb_n+nc_n\rfloor} \frac{\|e_ie_{i+k}\|_2}{\sigma^4(t_i)} \|\sigma^2(t_i) - \hat{\sigma}^2(t_i)\|_2 \mathbf{1}(\mathcal{A}_i) = O(\pi_n), \qquad (A.20) \end{aligned}$$

$$\begin{aligned} \max_{n-\lfloor nb_n+nc_n\rfloor\leq j\leq n} |\bar{\lambda}_{j,2}| &\leq |\bar{\lambda}_{n-\lfloor nb_n+nc_n\rfloor-1,2}| + \sum_{i=n-\lfloor nb_n+nc_n\rfloor}^{n} \frac{|e_ie_{i+k}|}{\sigma^4(t_i)} |\sigma^2(t_i) - \hat{\sigma}^2(t_i)| \\ &\leq \max_{\lfloor nb_n+nc_n\rfloor = s\leq j\leq n-\lfloor nb_n+nc_n\rfloor-1} |\bar{\lambda}_{s,j,2}| \\ &\quad + \sum_{i=1}^{\lfloor nb_n+nc_n\rfloor} \frac{|e_ie_{i+k}|}{\sigma^4(t_i)} |\sigma^2(t_i) - \hat{\sigma}^2(t_i)| + \sum_{i=n-\lfloor nb_n+nc_n\rfloor}^{n} \frac{|e_ie_{i+k}|}{\sigma^4(t_i)} |\sigma^2(t_i) - \hat{\sigma}^2(t_i)| \\ &= \max_{\lfloor nb_n+nc_n\rfloor = s\leq j\leq n-\lfloor nb_n+nc_n\rfloor-1} |\bar{\lambda}_{s,j,2}| + O_p(\pi_n). \qquad (A.21) \end{aligned}$$

where the constants $\underline{\pi}_n$ and π_n are given by $\underline{\pi}_n = nb_n^2c_n^2 + \sqrt{\frac{n}{c_n}}b_n^2 + \sqrt{\frac{n}{b_n}}c_n^2 + \frac{1}{\sqrt{b_nc_n}}$, $\pi_n = (nb_n + nc_n)(b_n^2 + c_n^2 + \frac{1}{\sqrt{nb_n}} + \frac{1}{\sqrt{nc_n}})$, respectively.

In order to prove a corresponding estimate for the remaining term

$$\max_{\lfloor nb_n+nc_n\rfloor=s\le j\le n-\lfloor nb_n+nc_n\rfloor-1} |\bar{\lambda}_{s,j,2}|$$

in (A.21) we study the asymptotic behavior of the quantity $\hat{\sigma^2}(t)-\sigma^2(t)$. By similar arguments as given above, we have that

$$\begin{aligned}\max_{\lfloor nb_n+nc_n\rfloor=s\le j\le n-\lfloor nb_n+nc_n\rfloor-1} |\bar{\lambda}_{s,j,2}| &\le \max_{\lfloor nb_n+nc_n\rfloor=s\le j\le\lfloor n\tilde{t}_v-n^{1-v'}-nb_n-nc_n\rfloor} |\bar{\lambda}_{s,j,2}| \\ &+ \max_{\lfloor n\tilde{t}_v+n^{1-v'}+nb_n+nc_n\rfloor=s\le j\le n-\lfloor nb_n+nc_n\rfloor} |\bar{\lambda}_{j,2}| + O_p(\pi_n+n^{1-v'})\end{aligned}$$

and by Corollary B.1 in Section B.1 it easily follows that

$$\sup_{t\in \mathrm{T}_n}\Big|\hat{\sigma}^2(t)-\sigma^2(t)-\frac{\mu_2\ddot{\sigma}^2(t)c_n^2}{2}-\frac{1}{nc_n}\sum_{i=1}^n K_{c_n}(t_i-t)(\hat{e}_i^2-\mathbb{E}(e_i^2))\Big| = O\Big(c_n^3+\frac{1}{nc_n}\Big), \tag{A.22}$$

where $\mathrm{T_n} = [c_n, \tilde{t}_v - c_n - n^{-v'}] \cup [\tilde{t}_v + c_n + n^{-v'}, 1-c_n]$. We now consider the decomposition

$$\sum_{i=1}^n K_{c_n}(t_i-t)\big(\hat{e}_i^2-\mathbb{E}(e_i^2)-(e_i^2-\mathbb{E}(e_i^2))\big) = \sum_{i=1}^n K_{c_n}(t_i-t)Q_i,$$

where $Q_i = Q_{1,i}+Q_{2,i}$, $Q_{1,i} = 2e_i[\mu(t_i)-\hat{\mu}(t_i)]$, $Q_{2,i} = [\mu(t_i)-\hat{\mu}(t_i)]^2$. By Lemma B.1 in Section B.1 we obtain

$$\sup_{\lfloor nb_n\rfloor\le i\le n-\lfloor nb_n\rfloor}\Big|\hat{\mu}_{b_n}(t_i)-\mu(t_i)-\frac{\mu_2\ddot{\mu}(t_i)}{2}b_n^2-\frac{1}{nb_n}\sum_{j=1}^n e_jK_{b_n}(t_j-t_i)\Big| = O(b_n^3+\frac{1}{nb_n}).$$

The triangle inequality and Proposition B.1 in Section B.2 imply

$$\Big\|\sup_{t\in\mathfrak{T}_n''}\Big|\sum_{i=1}^n K_{c_n}(t_i-t)\big[Q_{1,i}-\frac{2e_i}{nb_n}\sum_{j=1}^n e_jK_{b_n}(t_i-t_j)-\mu_2\ddot{\mu}(t_i)b_n^2e_i\big]\Big|\Big\|_4 = O(nb_n^3c_n^{3/4}+b_n^{-1}c_n^{3/4}), \tag{A.23}$$

where we use the notation $\mathfrak{T}_n'' = [b_n+c_n, \tilde{t}_v-b_n-c_n-n^{-v'}]\cup[\tilde{t}_v+b_n+c_n+n^{-v'}, 1-b_n-c_n]$. Similar arguments as given in the calculation of $\max_{\lfloor nb_n\rfloor\le i\le n-\lfloor nb_n\rfloor}|A_{n,i}|$ in the proof of (A.1) and the summation by parts formula show

$$\Big\|\sup_{t\in\mathfrak{T}_n''}\frac{2}{nb_n}\Big|\sum_{i=1}^n K_{c_n}(t_i-t)e_i\sum_{j=1}^n e_jK_{b_n}(t_i-t_j)\Big|\Big\|_2 = O(b_n^{-1}),$$

$$\Big\|\sup_{t\in\mathfrak{T}_n''}\Big|\sum_{i=1}^n K_{c_n}(t_i-t)\mu_2\ddot{\mu}(t_i)b_n^2e_i\Big|\Big\|_2 = O(n^{1/2}b_n^2),$$

and (A.23) gives $\big\|\sup_{t\in\mathfrak{T}_n''}\big|\sum_{i=1}^n K_{b_n}(t_i-t)Q_{1,i}\big|\big\|_2 = O(nb_n^3c_n^{3/4}+b_n^{-1}+n^{1/2}b_n^2)$. On the other hand, note that

$$\Big\|\sup_{t\in\mathfrak{T}_n''}\Big|\sum_{i=1}^n K_{c_n}(t_i-t)Q_{2,i}\Big|\Big\|_2 \le R_{n,1}+R_{n,2}$$

where

$$\begin{aligned}
R_{n,1} &= \Big\|\sup_{t\in\mathfrak{T}_n''}\Big|\sum_{i=1}^n K_{c_n}(t_i-t)\Big(\frac{1}{nb_n}\sum_{j=1}^n e_jK_{b_n}(t_i-t_j)+\frac{\mu_2\ddot{\mu}(t_i)}{2}b_n^2\Big)^2\Big|\Big\|_2\\
R_{n,2} &= \Big\|\sup_{t\in\mathfrak{T}_n''}\Big|\sum_{i=1}^n K_{c_n}(t_i-t)\Big(\mu(t_i)-\hat{\mu}(t_i)+\frac{1}{nb_n}\sum_{j=1}^n e_jK_{b_n}(t_i-t_j)+\frac{\mu_2\ddot{\mu}(t_i)}{2}b_n^2\Big)\\
&\quad\times\Big(\mu(t_i)-\hat{\mu}(t_i)-\frac{1}{nb_n}\sum_{j=1}^n e_jK_{b_n}(t_i-t_j)-\frac{\mu_2\ddot{\mu}(t_i)}{2}b_n^2\Big)\Big|\Big\|_2.
\end{aligned}$$

Proposition B.1 in Section B.2 and similar calculations as given in the proof of (A.1) show that

$$R_{n,1} = O\Big(nc_nc_n^{-1/2}\big(\frac{1}{nb_n}+b_n^4\big)\Big) = O(c_n^{1/2}b_n^{-1}+nc_n^{1/2}b_n^4),$$

while a further application of Lemma B.1 in Section B.1 yields

$$R_{n,2} = O\Big(\frac{nb_n^3c_n}{\sqrt{nb_n}}c_n^{-1/2}+nb_n^5c_n^{1/2}\Big) = O(\sqrt{n}b_n^{5/2}c_n^{1/2}+nb_n^5c_n^{1/2}). \tag{A.24}$$

Consequently, combining the arguments in (A.22)-(A.24), it follows that

$$\Big\|\sup_{t\in\mathfrak{T}_n''}\Big|\hat{\sigma^2}(t)-\sigma^2(t)-\frac{\mu_2\ddot{\sigma^2}(t)c_n^2}{2}-\frac{1}{nc_n}\sum_{i=1}^n K_{b_n}(t_i-t)\big(e_i^2-\mathbb{E}(e_i^2)\big)\Big|\Big\|_2 = O(\bar{\pi}_n), \tag{A.25}$$

where

$$\bar{\pi}_n = c_n^3+\frac{1}{nc_n}+b_n^3c_n^{-1/4}+\frac{1}{nb_nc_n}+\frac{b_n^2}{\sqrt{n}c_n}+c_n^{-1/2}b_n^{-1}n^{-1}+c_n^{-1/2}b_n^4+b_n^{5/2}(nc_n)^{-1/2}+b_n^5c_n^{-1/2}.$$

Let

$$\begin{aligned}
\mathrm{T}_{n,1} &= [\lfloor nb_n+nc_n\rfloor, n\tilde{t}_v-n^{1-\upsilon'}-nb_n-nc_n\rfloor]\cap\mathbb{Z},\\
\mathrm{T}_{n,2} &= [n\tilde{t}_v+n^{1-\upsilon'}+nb_n+nc_n\rfloor, n-\lfloor nb_n+nc_n\rfloor]\cap\mathbb{Z},
\end{aligned}$$

defining $\tilde{W}_i=\frac{e_ie_{i+k}}{\sigma^2(t_i)}$ and $Z_i'=e_i^2-\mathbb{E}e_i^2$, then it follows from (A.25) that

$$\begin{aligned}
&\mathbb{E}\Big(\max_{\lfloor nb_n+nc_n\rfloor=s,j\in\mathrm{T}_{n,1}}\Big|\bar{\lambda}_{s,j,2}+\sum_{i=\lfloor nb_n+nc_n\rfloor}^{j}\frac{\tilde{W}_i(\sum_{j=1}^n K_{c_n}(t_j-t_i)Z_i'+\mu_2\ddot{\sigma}^2(t_i)nc_n^3/2)}{\sigma^2(t_i)nc_n}\Big|\Big)\le\\
&\sum_{i\in\mathrm{T}_{n,1}}\frac{1}{\sigma^4(t_i)}\|e_ie_{i+k}\|_2\Big\|\hat{\sigma}^2(t_i)-\sigma^2(t_i)-\frac{1}{nc_n}\sum_{j=1}^n K_{c_n}(t_j-t_i)Z_i'-\mu_2\ddot{\sigma}^2(t_i)c_n^2/2\Big\|_2 = O(n\bar{\pi}_n).
\end{aligned} \tag{A.26}$$

By the Cauchy-Schwarz inequality we obtain $\|\tilde{W}_i - \tilde{W}_i^{(m)}\|_4 = O(\chi^{|i-m|})$, $\|Z_i' - Z_i'^{(m)}\|_4 = O(\chi^{|i-m|})$, where $Z_i'^{(m)} = (e_i^{(m)})^2 - \mathbb{E}(e_i^{(m)})^2$, $\tilde{W}_i^{(m)} = \frac{e_i^{(m)} e_{i+k}^{(m)}}{\sigma(t_i)^2}$, and

$$e_i^{(m)} = G_j(t_i, \mathcal{F}_i^{(m)}), \text{ if } \; b_j < t_i \leq b_{j+1}.$$

Hence, with similar arguments as given in the proof of Lemma 5 of Zhou and Wu (2010) we get

$$\max_{j \in \mathrm{T}_{n,1}} \mathbb{E}\Big[\sum_{i=\lfloor nb_n + nc_n \rfloor}^{j} \frac{\tilde{W}_i \sum_{j=1}^{n} K_{c_n}(t_j - t_i) Z_j'}{\sigma^2(t_i) n c_n} \Big] = O(c_n^{-1}).$$

Then by a similar m-dependent approximating technique as given in the proof of (A.1) we get

$$\max_{j \in \mathrm{T}_{n,1}} \Big| \sum_{i=\lfloor nb_n + nc_n \rfloor}^{j} \frac{1}{\sigma^2(t_i) n c_n} \sum_{j=1}^{n} \tilde{W}_i K_{c_n}(t_j - t_i) Z_j' - \mathbb{E}[\tilde{W}_i K_{c_n}(t_j - t_i) Z_j'] \Big| = O_p(c_n^{-1}).$$

Similarly, and more easily one obtains

$$\max_{j \in \mathrm{T}_{n,1}} \Big| \sum_{i=1}^{j} e_i e_{i+k} \mu_2 \ddot{\sigma}^2(t_i) c_n^2 / (2\sigma^4(t_i)) \Big| = O_p(nc_n^2). \tag{A.27}$$

Hence, it follows from (A.26) and (A.27) that

$$\max_{j \in \mathrm{T}_{n,1}} |\bar{\lambda}_{j,2}| = O_p(n\bar{\pi}_n + nc_n^2).$$

Similarly,

$$\max_{j \in \mathrm{T}_{n,2}} |\bar{\lambda}_{j,2}| = O_p(n\bar{\pi}_n + nc_n^2),$$

which implies, observing (A.19) - (A.21),

$$\max_{1 \leq j \leq n} |\bar{\Lambda}_j| = O_p(\pi_n + \underline{\pi}_n + n\bar{\pi}_n + nc_n^2).$$

Combining this result with the estimates (A.13), (A.14) and (A.18), and by our choice of the bandwidths, we have that

$$\max_{1 \leq j \leq n} \Big| \sum_{i=1}^{j} (\hat{W}_i - W_i) \Big| = O_p(nc_n^2 + nb_n^3 c_n^{-1/4} + b_n^{-1} c_n^{-1} + n^{1-\upsilon'}),$$

which establishes the estimate (A.12) and completes the proof of Theorem 3.1. $\square$

A.1.3 Proof of Theorem 3.2

We proof the theorem when there is an abrupt change of variance at $\tilde{t}_v$. The case that there is no abrupt change in variance is similar and easier. Recall that $\hat{W}_i^{(k)} = \frac{\hat{e}_i\hat{e}_{i+k}}{\hat{\sigma}^2(t_i)}$ and $\tilde{W}_i^{(k)} = \frac{e_ie_{i+k}}{\sigma(t_i)\sigma(t_{i+k})}$ and $W_i^{(k)} = \frac{e_ie_{i+k}}{\sigma^2(t_i)}$. We consider the corresponding partial sums $S_{j,m}^{(k)} = \sum_{r=j}^{j+m-1} W_r^{(k)}$, $\tilde{S}_{j,m}^{(k)} = \sum_{r=j}^{j+m-1} \tilde{W}_r^{(k)}$ and $\hat{S}_{j,m}^{(k)} = \sum_{r=j}^{j+m-1} \hat{W}_r^{(k)}$ and define $S_n^{(k)} = \sum_{r=1}^n W_r^{(k)}$, $\tilde{S}_n^{(k)} = \sum_{r=1}^n \tilde{W}_r^{(k)}$, $\hat{S}_n^{(k)} = \sum_{r=1}^n \hat{W}_r^{(k)}$. Recall the definition of $\hat{\mathbf{\Phi}}_{i,m}$ in (3.14) and $\hat{\mathbf{S}}_{j,m} = (S_{j,m}^{(r_1)}, ..., S_{j,m}^{(r_l)})^T$, $\hat{\mathbf{S}}_n = \hat{\mathbf{S}}_{1,n}$. Similarly, we define $\mathbf{S}_{j,m}$, $\mathbf{S}_n$, $\tilde{\mathbf{S}}_{j,m}$ and $\tilde{\mathbf{S}}_n$ and the linear interpolation on the interval $[0,1]$ by

$$\hat{\mathbf{\Phi}}_{m,n}(t) = \hat{\mathbf{\Phi}}_{\lfloor nt \rfloor,m} + (nt - \lfloor nt \rfloor)(\hat{\mathbf{\Phi}}_{\lfloor nt \rfloor+1,m} - \hat{\mathbf{\Phi}}_{\lfloor nt \rfloor,m}). \tag{A.28}$$

The assertion follows from the continuous mapping theorem if the weak convergence

$$\{\hat{\mathbf{\Phi}}_{m,n}(t)\}_{t\in[0,1]} \Rightarrow \{\mathbf{U}(t)\}_{t\in[0,1]}$$

conditional on $\mathcal{F}_n$ can be established. For a proof of this statement define $(\mathbf{\Phi}_{i,m},\ \mathbf{\Phi}_{m,n}(t))$ and $(\tilde{\mathbf{\Phi}}_{i,m},\ \tilde{\mathbf{\Phi}}_{m,n}(t))$ by replacing $(\hat{\mathbf{S}}_{j,m}, \hat{\mathbf{S}}_n)$ in the definition of $\hat{\Phi}_{i,m}$ and $\hat{\Phi}_{m,n}(t)$ with $(\mathbf{S}_{j,m}, \mathbf{S}_n)$ and $(\tilde{\mathbf{S}}_{j,m}, \tilde{\mathbf{S}}_n)$, respectively. Note that similar arguments as given in the proof of Theorem 3 in Zhou (2013) show that $\{\tilde{\mathbf{\Phi}}_{m,n}(t)\}_{t\in[0,1]} \Rightarrow \{\mathbf{U}(t)\}_{t\in[0,1]}$. The assertion of Theorem 3.2 then follows from the estimate

$$\sup_{t\in[0,1]} \left|\tilde{\mathbf{\Phi}}_{m,n}(t) - \hat{\mathbf{\Phi}}_{m,n}(t)\right| = O_p\Big(\frac{m}{\sqrt{n}} + \frac{\sqrt{m}}{n^{\upsilon'/2}} + \sqrt{m}\delta_n\Big), \tag{A.29}$$

where $\delta_n = \big(c_n^2 + (\frac{1}{\sqrt{nc_n}} + b_n^2 + \frac{1}{\sqrt{nb_n}})c_n^{-1/4}\big)\log n$. In order to prove (A.29) let C denote a sufficiently large constant, which may vary from line to line in the following calculations, and consider the event

$$\begin{aligned} A_n &= \Big\{\sup_{t\in[0,1]} |\hat{\mu}_{b_n}(t) - \mu(t)| \le C\frac{\log n}{\sqrt{nb_n}b_n^{1/4}} + Cb_n^2\log n\Big\}, \\ B_n &= \Big\{\sup_{t\in[0,t_v-n^{-\upsilon'}]\cup[t_v+n^{-\upsilon'},1]} |\hat{\sigma}^2(t) - \sigma^2(t)| \le C\Big(c_n^2 + (\frac{1}{\sqrt{nc_n}} + b_n^2 + \frac{1}{\sqrt{nb_n}})c_n^{-1/4}\Big)\log n\Big\}, \end{aligned}$$

where $\upsilon' \in (\frac{1}{2}, 1-\frac{4}{\iota})$. By Lemma B.3 and Corollary B.1 of Section B.1, we have that $\lim_{n\to\infty}\mathbb{P}(A_n\cap B_n) = 1$. Then it is easy to see that

$$\|\mathbf{\Phi}_{n,m} - \tilde{\mathbf{\Phi}}_{n,m}\| = O(\frac{m^{1/2}}{n} + \frac{1}{\sqrt{n}}) = O(\frac{m}{\sqrt{n}} + \frac{\sqrt{m}}{n^{\upsilon'/2}} + \sqrt{m}\delta_n).$$

Write $E_n = A_n \cap B_n$. On the other hand, for $1 \le j \le n-m+1$ and any fix lag-k_i, $1 \le i \le l$, the estimate (omit the supscript for short)

$$\mathbb{E}[(S_{j,m} - \hat{S}_{j,m})^2\mathbf{1}(E_n)] = \mathbb{E}\Big\{\sum_{r=j}^{j+m-1}\Big(\frac{\hat{e}_i\hat{e}_{i+k}}{\hat{\sigma}^2(t_i)} - \frac{e_ie_{i+k}}{\sigma^2(t_i)}\Big)\mathbf{1}(E_n)\Big\}^2 = O(m^2\delta_n^2)$$

for $j \notin [\lfloor n\tilde{t}_v - n^{1-v'} \rfloor - m - 1, \lfloor n\tilde{t}_v + n^{1-v'} \rfloor + m + 1]$, and

$$\mathbb{E}[(S_{j,m} - \hat{S}_{j,m})^2 \mathbf{1}(E_n)] \leq Cm^2, \text{ for } j \in [\lfloor n\tilde{t}_v - n^{1-v'} \rfloor - m - 1, \lfloor n\tilde{t}_v + n^{1-v'} \rfloor + m + 1]$$

Similarly,

$$\mathbb{E}[(S_n - \hat{S}_n)^2 \mathbf{1}(E_n)] = O(m^2 + n^{2-2v'} + n^2 \delta_n^2).$$

Note that

$$\begin{aligned} \|\mathbf{\Phi}_{n,m} - \hat{\mathbf{\Phi}}_{n,m}\|_2^2 &= \frac{1}{m(n-m+1)} \sum_{s=1}^{l} \sum_{i=1}^{n-m+1} \Big(S_{j,m}^{(r_s)} - \hat{S}_{j,m}^{(r_s)} - \frac{m}{n}(S_n^{(r_s)} - \hat{S}_n^{(r_s)}) \Big)^2 \\ &= O(m\delta_n^2 + \frac{m^2}{n} + \frac{m}{n^{v'}}). \end{aligned}$$

An application of Doob's inequality and Proposition B.3 in Section B.2 finally yields

$$\max_{1 \leq i \leq n-m+1} |\mathbf{\Phi}_{i,m} - \hat{\mathbf{\Phi}}_{i,m}| = O_p\Big(\frac{m}{\sqrt{n}} + \frac{\sqrt{m}}{n^{v'/2}} + \sqrt{m}\delta_n \Big),$$
$$\max_{1 \leq i \leq n-m+1} |\tilde{\mathbf{\Phi}}_{i,m} - \mathbf{\Phi}_{i,m}| = O_p\Big(\frac{m}{\sqrt{n}} + \frac{\sqrt{m}}{n^{v'/2}} + \sqrt{m}\delta_n \Big).$$

The estimate (A.29) now follows from this result and definition (A.28) and an application of triangle inequality, which completes the proof of Theorem 3.2. □

A.2 Proof of Lemma 4.1 - 4.2

In order to simplify the notation define $G_n^{(k)}(m) = S_m^{(k)} - \frac{m}{n} S_n^{(k)}$, $\hat{G}_n^{(k)}(m) = \hat{S}_m^{(k)} - \frac{m}{n} \hat{S}_n^{(k)}$, where as before, $S_m^{(k)} = \sum_{i=1}^{m} \frac{e_i e_{i+k}}{\sigma(t_i)\sigma(t_{i+k})}$, $\hat{S}_m^{(k)} = \sum_{i=1}^{m} \frac{\hat{e}_i \hat{e}_{i+k}}{\hat{\sigma}^2(t_i)}$, Then it is easy to see that the estimator $\hat{t}_n^{(k)}$ of the change point in the correlation function defined in (4.4) can be represented as $\hat{t}_n^{(k)} = \frac{1}{n} \operatorname{argmax}_{1 \leq m \leq n} (\hat{G}_n^{(k)}(m))^2$.

A.2.1 Proof of Lemma 4.1

We fix a lag-r_s for some $1 \leq s \leq l$. Recall that under the null hypothesis (4.1), we have $\rho_{u,r_s} = \rho_1^{r_s}$ for $u \leq \lfloor nt_{r_s} \rfloor$ and $\rho_{u,r_s} = \rho_1^{r_s} + \Delta_{r_s} = \rho_2^{(r_s)}$ for $u > \lfloor nt_{r_s} \rfloor$, where Δ_{r_s} is an unknown (without loss of generality) positive constant. We omit the superscript and subscript r_s in this proof. A simple calculation shows that

$$f_n(m) := \mathbb{E} G_n(m) = n(m(n)t(n) - m(n) \wedge t(n))\Delta, \tag{A.30}$$

where we used the notation $m(n) = m/n$ and $t(n) = \lfloor nt \rfloor / n$. By Proposition 5 of Zhou (2013), on a possibly richer probability space, there exist $i.i.d$ standard normal variables, say $\{V_i\}_{i\in\mathbb{Z}}$, such that

$$\max_{1\le i\le n} \Big| S_i - \mathbb{E}(S_i) - \sum_{j=1}^{i} \kappa_s(t_j) V_j \Big| = o_p(n^{1/4}\log n), \tag{A.31}$$

where κ_s is the s_{th} diagonal element of κ, which is defined in assumption (A5). Define $\Xi_j = \sum_{i=1}^{j} \kappa_s(t_i) V_i$. By the arguments given in the proof of Theorem 3.1, we have

$$\max_{1\le m\le n} |G_n(m) - \hat{G}_n(m)| = O_p(\varrho_n), \tag{A.32}$$

where $\varrho_n = nc_n^2 + nb_n^3 c_n^{-1/4} + b_n^{-1}c_n^{-1} + n^{1-\upsilon'}$ and $\upsilon' \in (\frac{1}{2}, 1 - \frac{4}{\iota})$. Now a similar reasoning as given in the proof of Lemma 5 of Zhou and Wu (2010) and assumptions (A3) (A4) and (A5) yield that there exists a constant C such that $\kappa_s^2(t) \le C$ for all $t \in [0,1]$. Then it is easy to see that $\|\Xi_n\|_2^2 = O(n)$. By Doob's inequality, we have that

$$\max_{1\le j\le n} |\Xi_j| = O_p(\sqrt{n}), \tag{A.33}$$

and observing (A.31) we obtain

$$\max_{1\le m\le n} \left| G_n^2(m) - \hat{G}_n^2(m) \right| = \max_{1\le m\le n} |G_n(m) + \hat{G}_n(m)||G_n(m) - \hat{G}_n(m)| = O_p(n\varrho_n).$$

Define $\hat{V}_n(m) = \hat{G}_n^2(m) - \hat{G}_n^2(\lfloor nt \rfloor)$, note that $\hat{V}_n(\lfloor nt \rfloor) = 0$ and consider a constant $\beta \in (\frac{1}{2}, \frac{2}{3})$, such that $n^{1-\beta}/\varrho_n \to \infty$. By the choices of b_n and c_n, there exists qualified β. Observing the definition (A.30) and the estimate (A.31), it follows that

$$\max_{1\le m\le n} \left| G_n^2(m) - \Big(f_n(m) + \Xi_m - \frac{m}{n}\Xi_n \Big)^2 \right| = O_p(n^{5/4}\log n). \tag{A.34}$$

By (A.33), we have $\max_{1\le m\le n}(\Xi_m - \frac{m}{n}\Xi_n)^2 = O_p(n)$, and together with (A.32) and (A.34) this yields

$$\begin{aligned} \max_{m\in\mathcal{M}_n} \hat{V}_n(m) = \max_{m\in\mathcal{M}_n} \left[G_n^2(m) - G_n^2(\lfloor nt \rfloor) \right] + O_p(n\rho_n) = \max_{m\in\mathcal{M}_n} \Big\{ & f_n^2(m) - f_n^2(\lfloor nt \rfloor) \\ & + 2(f_n(m) - f_n(\lfloor nt \rfloor))\Xi_m + 2f_n(\lfloor nt \rfloor)(\Xi_m - \Xi_{\lfloor nt \rfloor}) \\ & - 2\frac{m}{n} f_n(m)\Xi_n + 2\frac{\lfloor nt \rfloor}{n} f_n(\lfloor nt \rfloor)\Xi_n \Big\} + O_p(n\varrho_n + n^{5/4}\log n), \end{aligned} \tag{A.35}$$

where the maxima are taken over the set

$$\mathcal{M}_n = \{ m \mid \lfloor nt \rfloor - \lfloor n^{1-\beta/2} \rfloor \le m \le \lfloor nt \rfloor - \lfloor n^{1-\beta} \rfloor | \}.$$

Observing the definition of $f_n(m)$ in (A.30) we have for some positive constant C,

$$\max_{m\in\mathcal{M}_n} (f_n^2(m) - f_n^2(\lfloor nt\rfloor)) \le -Cn^{2-\beta},$$

and (A.33) implies

$$\begin{aligned}\max_{m\in\mathcal{M}_n} (f_n(m) - f_n(\lfloor nt\rfloor))\Xi_m &= O_p(n^{3/2-\beta/2}\log n),\\ \max_{m\in\mathcal{M}_n} \Big(\frac{m}{n}f_n(m) - \frac{\lfloor nt\rfloor}{n}f_n(\lfloor nt\rfloor)\Big)\Xi_n &= O_p(n^{3/2-\beta/2}\log n).\end{aligned}$$

Using the representation $\Xi_m - \Xi_{\lfloor nt\rfloor} = \sum_{i=m+1}^{\lfloor nt\rfloor}\sigma(t_i)V_i$ and similar arguments as in the derivation of (A.33) yields

$$\max_{m\in\mathcal{M}_n} (\Xi_m - \Xi_{\lfloor nt\rfloor}) = O_p(n^{1/2(1-\beta/2)}\log n).$$

Consequently,

$$\max_{m\in\mathcal{M}_n} f_n(\lfloor nt\rfloor)[\Xi_m - \Xi_{\lfloor nt\rfloor}] = O_p(n^{3/2-\beta/4}\log n). \tag{A.36}$$

By our choice of β, it now follows from (A.35) - (A.36) that

$$\mathbb{P}\Big(\limsup_{n\to\infty}\max_{m\in\mathcal{M}(n)}\hat{V}_n(m) = -\infty\Big) = 1. \tag{A.37}$$

On the other hand, similar arguments give the estimates

$$\begin{aligned}\max_{1\le m\le\lfloor nt\rfloor-\lfloor n^{1-\beta/2}\rfloor}[f_n^2(m) - f_n^2(\lfloor nt\rfloor)] &\le -Cn^{2-\beta/2},\\ \max_{1\le m\le\lfloor nt\rfloor-\lfloor n^{1-\beta/2}\rfloor} f_n(\lfloor nt\rfloor)[\Xi_m - \Xi_{\lfloor nt\rfloor}] &= O_p(n^{3/2}\log n),\\ \max_{1\le m\le\lfloor nt\rfloor-\lfloor n^{1-\beta/2}\rfloor}[\frac{m}{n}f_n(m) - \frac{\lfloor nt\rfloor}{n}f_n(\lfloor nt\rfloor)]\Xi_n &= O_p(n^{3/2}\log n),\\ \max_{1\le m\le\lfloor nt\rfloor-\lfloor n^{1-\beta/2}\rfloor}[f_n(m) - f_n(\lfloor nt\rfloor)]\Xi_m &= O_p(n^{3/2}\log n),\end{aligned}$$

and by our choice of β we obtain

$$\mathbb{P}(\limsup_{n\to\infty}\max_{1\le m\le\lfloor nt\rfloor-\lfloor n^{1-\beta/2}\rfloor}\hat{V}_n(m) = -\infty) = 1.$$

Combined with (A.37) this gives

$$\mathbb{P}(\limsup_{n\to\infty}\max_{1\le m\le\lfloor nt\rfloor-\lfloor n^{1-\beta}\rfloor}\hat{V}_n(m) = -\infty) = 1,$$

and it can be shown by similar arguments that

$$\mathbb{P}(\limsup_{n\to\infty} \max_{\lfloor nt\rfloor+\lfloor n^{1-\beta}\rfloor\le m\le n} \hat{V}_n(m) = -\infty) = 1.$$

Consequently, it follows that

$$\lim_{n\to\infty} \mathbb{P}(|n\hat{t}_n - \lfloor nt\rfloor| \le n^{1-\beta}) = 1,$$

which proves (4.6) of Lemma 4.1. In the case where the variance has no jump at time t, the result (4.5) follows from the fact that for any lag-r_s, $\hat{G}_n^{(r_s)}(m)/\sqrt{n}$ converges weakly to some Gaussian process $\{\mathbf{U}^{(s)}(u) - u\mathbf{U}^{(s)}(1)\}_{u\in[0,1]}$, which implies $\hat{t}_n^{(r_s)} \xrightarrow{\mathcal{D}} \tilde{T}^{(r_s)} = \mathrm{argmax}_{u\in(0,1)} |\mathbf{U}^{(s)}(u) - u\mathbf{U}^{(s)}(1)|$, where the Gaussian process $\{\mathbf{U}^{(s)}(u)\}_{u\in[0,1]}$ is the s_{th} entry of the vector Gaussian process $\{\mathbf{U}(t)\}_{t\in[0,1]}$ which is defined in Theorem 3.1.

A.2.2 Proof of Lemma 4.2

We fix a lag-r_s for some $1 \le s \le l$ and then omit the superscript/subscript r_s. Recall the definition of (3.8), the notation $W_j = W_j^{(r_s)} = \frac{e_j e_{j+r_s}}{\sigma(t_j)\sigma(t_{j+r_s})}$, and denote the change point by $t = t_{r_s}$. Finally define

$$\Delta_{n,1} = \sum_{j=1}^{\lfloor nt\rfloor} W_j\ , \quad \Delta_{n,2} = \frac{1}{n-\lfloor nt\rfloor} \sum_{j=\lfloor nt\rfloor+1}^{n} W_j \tag{A.38}$$

We first consider the situation of (4.6) in the main article, that is $|\Delta| > 0$. From the proof of Theorem 3.1 we have that

$$\Delta_{n,1} - \mathbb{E}[\Delta_{n,1}] = O_p\Big(\frac{1}{\sqrt{n}}\Big), \quad \Delta_{n,2} - \mathbb{E}[\Delta_{n,2}] = O_p\Big(\frac{1}{\sqrt{n}}\Big).$$

Since $\Delta = \rho_2 - \rho_1 = \mathbb{E}[\Delta_{n,2}] - \mathbb{E}[\Delta_{n,1}]$ we have $\Delta_n := \Delta_{n,2} - \Delta_{n,1} = \Delta + O_p(1/\sqrt{n})$.
In order to prove this estimate we introduce the notation $\mathcal{A}_n = \{|\hat{t}_n - t| \le \frac{C}{\sqrt{n}}\}$. Then by Lemma 4.1, we have that $\lim_{n\to\infty} \mathbb{P}(\mathcal{A}_n) = 1$. This yields

$$(\Delta_{n,1} - \hat{\Delta}_{n,1}) I(\mathcal{A}_n) = I(\mathcal{A}_n)(A_n + B_n + C_n),$$

where

$$A_n = \sum_{j=1}^{\lfloor nt\rfloor} \frac{W_j}{\lfloor nt\rfloor} - \sum_{j=1}^{\lfloor n\hat{t}_n\rfloor} \frac{W_j}{\lfloor nt\rfloor}\ ,\ B_n = \sum_{j=1}^{\lfloor n\hat{t}_n\rfloor} \Big(\frac{W_j}{\lfloor nt\rfloor} - \frac{\hat{W}_j}{\lfloor nt\rfloor}\Big)\ , \tag{A.39}$$

$$C_n = \sum_{j=1}^{\lfloor n\hat{t}_n\rfloor} \Big(\frac{\hat{W}_j}{\lfloor nt\rfloor} - \frac{\hat{W}_j}{\lfloor n\hat{t}_n\rfloor}\Big).$$

It is easy to see that $I(\mathcal{A}_n)A_n = O_p(\frac{1}{\sqrt{n}})$. Using the same arguments as in the proof of Theorem 3.1, we obtain $I(\mathcal{A}_n)B_n = o_p(\sqrt{n}/n) = o_p(1/\sqrt{n})$ and

$$
\begin{aligned}
I(\mathcal{A}_n)\cdot C_n &= I(\mathcal{A}_n)\sum_{j=1}^{\lfloor n\hat{t}_n\rfloor}\hat{W}_j\frac{\lfloor n\hat{t}_n\rfloor-\lfloor nt\rfloor}{\lfloor nt\rfloor\lfloor n\hat{t}_n\rfloor} \le C\cdot I(\mathcal{A}_n)\sum_{j=1}^{\lfloor n\hat{t}_n\rfloor}\hat{W}_j\frac{1}{n\sqrt{n}} \\
&= C\cdot I(\mathcal{A}_n)\Big(\sum_{j=1}^{\lfloor n\hat{t}_n\rfloor}W_j + o_p(\sqrt{n})\Big)\frac{1}{n\sqrt{n}} \\
&= CI(\mathcal{A}_n)\cdot\Big(\sum_{j=1}^{\lfloor nt\rfloor}W_j + \sum_{j=\lfloor nt\rfloor+1}^{\lfloor n\hat{t}_n\rfloor}W_jI(t\le\hat{t}_n) - \sum_{j=\lfloor n\hat{t}_n\rfloor+1}^{\lfloor nt\rfloor}W_jI(t>\hat{t}_n) + o_p(\sqrt{n})\Big)\frac{1}{n\sqrt{n}} \\
&= O_p\Big(\frac{1}{\sqrt{n}}\Big). \qquad (A.40)
\end{aligned}
$$

Combining (A.39) - (A.40) and using Proposition B.3 in Section B.2 shows $\Delta_{n,1}-\hat{\Delta}_{n,1} = O_p\big(\frac{1}{\sqrt{n}}\big)$. Similarly, we have $\Delta_{n,2}-\hat{\Delta}_{n,2} = O_p(\frac{1}{\sqrt{n}})$, and the assertion of the lemma follows when $|\Delta| > 0$. For the case that $|\Delta| = 0$, define the following two functions of u, $0 \le u \le 1$,

$$
\hat{\Delta}_{n,1}(u) = \frac{1}{\lfloor nu\rfloor}\sum_{j=1}^{\lfloor nu\rfloor}\frac{\hat{e}_j\hat{e}_{j+k}}{\hat{\sigma}_n^2(t_j)}\ ,\quad \hat{\Delta}_{n,2}(u) = \frac{1}{n-\lfloor nu\rfloor}\sum_{j=\lfloor nu\rfloor+1}^{n}\frac{\hat{e}_j\hat{e}_{j+k}}{\hat{\sigma}_n^2(t_j)}.
$$

Then using Doob's inequality and similar arguments to that in Theorem 3.1, we have that

$$
\max_{\frac{1}{\log n}\le u\le 1-\frac{1}{\log n}}\Big|\hat{\Delta}_{n,1}(u)\Big| = O_p\Big(\frac{\log n}{\sqrt{n}}\Big),\ \max_{\frac{1}{\log n}\le u\le 1-\frac{1}{\log n}}\Big|\hat{\Delta}_{n,2}(u)\Big| = O_p\Big(\frac{\log n}{\sqrt{n}}\Big). \qquad (A.41)
$$

Recall the definition of $\tilde{T} = \mathrm{argmax}_{x\in(0,1)}|U(x)-xU(1)|$ in the proof of Lemma 4.1, where $U(x) := \mathbf{U}^{(r_s)}(x)$ for short. Write $\tilde{U}(x) = U(x)-xU(1)$, $\mathcal{W}_n = [0,\frac{1}{\log n}]\cup[1-\frac{1}{\log n},1]$, $\bar{\mathcal{W}}_n = [\frac{1}{\log n}, 1-\frac{1}{\log n}]$. Then by observing the variance structure, we can see that

$$
\lim_{n\to\infty}\mathbb{P}\left\{\max_{x\in\mathcal{W}_n}\tilde{U}(x)\ge\max_{x\in\bar{\mathcal{W}}_n}\tilde{U}(x)\right\} = 0,
$$

which shows that the event $\tilde{\mathcal{W}}_n := \{\hat{t}_n\in\bar{\mathcal{W}}_n\}$ satisfies that $\lim_{n\to\infty}\mathbb{P}(\tilde{\mathcal{W}}_n) = 1$. By (A.41) and Proposition B.3, we have that

$$
\hat{\Delta}_{n,1} = O_p\Big(\frac{\log n}{\sqrt{n}}\Big)\ ,\quad \hat{\Delta}_{n,2} = O_p\Big(\frac{\log n}{\sqrt{n}}\Big), \qquad (A.42)
$$

which finishes the proof.

A.3 Proof of Theorem 4.1 and 4.2

A.3.1 Proof of Theorem 4.1

We consider the non-observable analogue

$$T_n^{(k),r} = \frac{3}{{t_k}^2(1-t_k)^2}\int_0^1 (U_n^{(k)})^2(s)ds.$$

of the statistic $\hat{T}_n^{(k),r}$ defined in (4.7), where the process $U_n^{(k)}$ is given by

$$U_n^{(k)}(s) = \frac{1}{n}\sum_{j=1}^{\lfloor ns\rfloor}\frac{e_j e_{j+k}}{\sigma(t_j)\sigma(t_{j+k})} - \frac{s}{n}\sum_{j=1}^{n}\frac{e_j e_{j+k}}{\sigma(t_j)\sigma(t_{j+k})}.$$

It follows from the proof of Theorem 3.1 and Lemma 4.1 that, for any fixed lag-r_k ($1 \le k \le l$), we have that

$$\{\sqrt{n}(U_n^{(r_k)}(u) + (u\wedge t - ut)\Delta_{r_k})\}_{u\in[0,1],k\in[1,l]\cap\mathbb{Z}} \Rightarrow \{\mathbf{U}^{(k)}(u) - u\mathbf{U}^{(k)}(1)\}_{u\in[0,1],k\in[1,l]\cap\mathbb{Z}}, \quad \text{(A.43)}$$

whenever $\Delta_{r_k} \neq 0$. The continuous mapping theorem, elementary calculations, and the identity $3\int_0^1 [st - s\wedge t]^2 ds = t^2(1-t)^2$ imply $\{\sqrt{n}(T_n^{(r_k),r} - \Delta_{r_k}^2)\}_{k\in[1,l]\cap\mathbb{Z}} \xrightarrow{\mathcal{D}} \{\mathcal{Z}^{(r_k)}(\Delta_{r_s})\}_{k\in[1,l]\cap\mathbb{Z}}$, where the random variable $\mathcal{Z}^{(r_k)}$ is defined in Theorem 4.1. By the proof of Theorem 3.1, we have that for $1 \le k \le l$, and constant υ' satisfies $\upsilon' \in (\frac{1}{2}, 1-\frac{4}{\iota})$

$$\sup_{0\le s\le 1} n|U_n^{(r_k)}(s) - \hat{\mathcal{V}}_n^{(r_k)}(s)| = O_p(nc_n^2 + nb_n^3 c_n^{-1/4} + b_n^{-1}c_n^{-1} + n^{1-\upsilon'}),$$

From (A.43) it follows that $\int_0^1 |U_n^{(r_k)}(s)|ds = O_p(1)$. Consequently, we have that for $1 \le k \le l$,

$$\begin{aligned}
&n^{1/2}\int_0^1 [(U^{(r_k)})_n^2(s) - (\hat{\mathcal{V}}^{(r_k)})_n^2(s)]ds \le \\
&\sup_{0\le s\le 1} n^{1/2}|U_n^{(r_k)}(s) - \hat{\mathcal{V}}_n^{(r_k)}(s)|\int_0^1 |U_n^{(r_k)}(s) + \hat{U}_n^{(r_k)}(s)|ds \\
&\le 2n^{1/2}\sup_{0\le s\le 1}|U_n^{(r_k)}(s) - \hat{\mathcal{V}}_n^{(r_k)}(s)|\int_0^1 |U_n^{(r_k)}(s)|ds + n^{1/2}\sup_{0\le s\le 1}|U_n^{(r_k)}(s) - \hat{\mathcal{V}}_n^{(r_k)}(s)|^2 \\
&= O_p(n^{1/2}c_n^2 + n^{1/2}b_n^3 c_n^{-1/4} + n^{-1/2}b_n^{-1}c_n^{-1} + n^{1/2-\upsilon'}),
\end{aligned}$$

which completes the proof. □

A.3.2 Proof of Theorem 4.2

For lag-k, recall the definition of $\hat{\Delta}_n^{(k)}$, $\hat{A}_j^{(k)}$, $\hat{\Phi}_{i,m}^{A,(k)}$ in (4.11), (4.12), (4.13) and define

$$A_j^{(k)} = \frac{e_j e_{j+k}}{\sigma(t_j)\sigma(t_{j+k})} - \Delta_k \mathbf{1}(j \geq \lfloor nt_k \rfloor),$$
$$\Phi_{i,m}^{A,(k)} = \frac{1}{\sqrt{m(n-m+1)}} \sum_{j=1}^{n-m+1} (S_{j,m}^{A,(k)} - \frac{m}{n} S_n^{A,(k)}) R_j,$$

where $S_{j,m}^{A,(k)} = \sum_{r=j}^{j+m-1} A_r^{(k)}$, $S_n^{A,(k)} = \sum_{r=1}^{n} A_r^{(k)}$. We introduce the processes

$$\Phi_{m,n}^{A,(k)}(s) = \Phi_{\lfloor ns \rfloor,m}^{A,(k)} + (ns - \lfloor ns \rfloor)(\Phi_{\lfloor ns \rfloor+1,m}^{A,(k)} - \Phi_{\lfloor ns \rfloor,m}^{A,(k)}),$$
$$\hat{\Phi}_{m,n}^{A,(k)}(s) = \hat{\Phi}_{\lfloor ns \rfloor,m}^{A,(k)} + (ns - \lfloor ns \rfloor)(\hat{\Phi}_{\lfloor ns \rfloor+1,m}^{A,(k)} - \hat{\Phi}_{\lfloor ns \rfloor,m}^{A,(k)}).$$

and note that by Zhou (2013), $\{\mathbf{\Phi}_{m,n}^{A}(s)\}_{s\in[0,1]} \Rightarrow \{\mathbf{U}(s)\}_{s\in[0,1]}$ conditional on $\mathcal{F}_n$, where $\mathbf{\Phi}_{m,n}^{A}(s) = (\Phi_{m,n}^{A,(r_1)}(s), ..., \Phi_{m,n}^{A,(r_l)}(s))^T$. The assertion of Theorem 4.2 is therefore a consequence of the estimate

$$\max_{1\leq u\leq l} \sup_{s\in(0,1)} |\Phi_{m,n}^{A,(r_u)}(s) - \hat{\Phi}_{m,n}^{A,(r_u)}(s)| = O_p\Big(\frac{m}{\sqrt{n}} + \Big(\frac{m \log n}{\sqrt{n}}\Big)^{1/2} + \sqrt{m}\delta_n\Big), \tag{A.44}$$

To see this, note that for any fixed lag-r_u $(1 \leq u \leq l)$

$$\frac{1}{m(n-m+1)} \sum_{j=1}^{n-m+1} (\hat{S}_{j,m}^{A,(r_u)} - S_{j,m}^{A,(r_u)})^2 \leq 2(I + II),$$

where

$$I = \frac{1}{m(n-m+1)} \sum_{j=1}^{n-m+1} \Big(\sum_{r=j}^{j+m-1} \frac{e_r e_{r+r_u}}{\sigma(t_r)\sigma(t_{r+r_u})} - \frac{\hat{e}_r \hat{e}_{r+r_u}}{\hat{\sigma}^2(t_r)} \Big)^2,$$
$$II = \frac{1}{m(n-m+1)} \sum_{j=1}^{n-m+1} \Big(\sum_{r=j}^{j+m-1} \Big(\Delta_{r_u} \mathbf{1}\left(r \geq \lfloor nt_{r_u} \rfloor\right) - \hat{\Delta}_n^{(r_u)} \mathbf{1}\left(r \geq \lfloor n\hat{t}_n^{(r_u)} \rfloor\right)\Big)\Big)^2$$
$$\leq 2(II_1 + II_2),$$
$$II_1 = \frac{1}{m(n-m+1)} \sum_{j=1}^{n-m+1} \Big(\sum_{r=j}^{j+m-1} \Big(\Delta_{r_u} - \hat{\Delta}_n^{(r_u)}\Big) \mathbf{1}\left(r \geq \lfloor n\hat{t}_n^{(r_u)} \rfloor\right) \Big)^2,$$
$$II_2 = \frac{1}{m(n-m+1)} \sum_{j=1}^{n-m+1} \Big(\sum_{r=j}^{j+m-1} \Delta_{r_u} \left(\mathbf{1}\left(r \geq \lfloor nt_{r_u} \rfloor\right) - \mathbf{1}\left(r \geq \lfloor n\hat{t}_n^{(r_u)} \rfloor\right)\right) \Big)^2.$$

By the proof of Theorem 3.2, we have $I = O_p(m\delta_n^2 + \frac{m^2}{n} + \frac{m}{n^{v'}})$, where v' and δ_n is defined in the proof of Theorem 3.2. First consider the case that $\Delta_{r_u} > 0$. Similarly by Lemma 4.2 and

Proposition B.3, $II_1 = O_p(\frac{m \log^2 n}{n})$. Let $\mathcal{W}_n = \{|\hat{t}_n^{(r_u)} - t_{r_u}| \le n^{-\alpha'}\}$ for some $\alpha' \in (1/2, \upsilon')$. Then $\lim_{n\to\infty} \mathbb{P}(\mathcal{W}_n) = 1$ and an application of Proposition B.3 shows that $II_2 = O_p(\frac{m \log n}{\sqrt{n}})$. So

$$\frac{1}{m(n-m+1)} \sum_{j=1}^{n-m+1} (\hat{S}_{j,m}^{A,(r_u)} - S_{j,m}^{A,(r_u)})^2 = O_p\Big(m\delta_n^2 + \frac{m^2}{n} + \frac{m \log n}{\sqrt{n}}\Big),$$

Similarly

$$\frac{m}{n^2(n-m+1)} \sum_{j=1}^{n-m+1} (\hat{S}_n^{A,(r_u)} - S_n^{A,(r_u)})^2 = O_p\Big(m\delta_n^2 + \frac{m^2}{n} + \frac{m \log n}{\sqrt{n}}\Big),$$

By a similar argument as given in the proof of Theorem 3.2 and an application of Doob's inequality we can show

$$\sup_{s\in(0,1)} |\Phi_{m,n}^{A,(r_u)}(s) - \hat{\Phi}_{m,n}^{A,(r_u)}(s)| = O_p\Big(\frac{m}{\sqrt{n}} + \Big(\frac{m \log n}{\sqrt{n}}\Big)^{1/2} + \sqrt{m}\delta_n\Big), \tag{A.45}$$

When $\Delta_{r_u} = 0$ it follows from (A.42) that $II = O_p(\frac{m \log^2 n}{n})$. Similarly (A.45) holds. Thus (A.44) holds, which finishes the proof. □

A.4 Proof of Algorithm 4.1

Proof. For any lag-r_s, if $\Delta_{r_s} = 0$, then the type 1 error is protected since $\hat{T}_n^{(r_s),r} = O_P(1/n)$ and $M_n^{r,(r_1)}$ is symmetric. Otherwise, the algorithm is valid in view of Lemma 4.2 and Proposition B.3. □

B More technical details

B.1 Uniform bounds for nonparametric estimates

The following two lemmas provide uniform bounds for the estimate $\hat{\mu}_{b_n}$ in the interior $\mathfrak{T}_n = [b_n, 1-b_n]$ and at the boundary $\mathfrak{T}_n' = [0, b_n) \cup (1-b_n, 1]$ of the interval $[0,1]$.

Lemma B.1. *If assumptions (A1)-(A3) are satisfied and $b_n \to 0$, $nb_n \to \infty$, we have*

$$\sup_{t\in\mathfrak{T}_n} \Big| \hat{\mu}_{b_n}(t) - \mu(t) - \frac{\mu_2 \ddot{\mu}(t)}{2} b_n^2 - \frac{1}{nb_n} \sum_{i=1}^n e_i K_{b_n}(t_1 - t) \Big| = O(b_n^3 + \frac{1}{nb_n}),$$

where $\mathfrak{T}_n = [b_n, 1-b_n]$.

Proof. With the notations

$$S_{n,l}(t) = \frac{1}{nb_n}\sum_{i=1}^{n}\big(\frac{t_i-t}{b_n}\big)^l K_{b_n}(t_i-t),$$
$$R_{n,l}(t) = \frac{1}{nb_n}\sum_{i=1}^{n}Y_i\big(\frac{t_i-t}{b_n}\big)^l K_{b_n}(t_i-t),$$

$(l = 0, 1, ...)$ we obtain the representation

$$\begin{bmatrix}\hat{\mu}_{b_n}(t)\\ b_n\hat{\dot{\mu}}_{b_n}(t)\end{bmatrix} = \begin{bmatrix}S_{n,0}(t) & S_{n,1}(t)\\ S_{n,1}(t) & S_{n,2}(t)\end{bmatrix}^{-1}\begin{bmatrix}R_{n,0}(t)\\ R_{n,1}(t)\end{bmatrix} =: S_n^{-1}(t)R_n(t), \tag{B.1}$$

for the local linear estimate $\tilde{\mu}_{b_n}$, where the last identity defines the 2×2 matrix $S_n(t)$ and the vector $R_n(t)$ in an obvious manner. By elementary calculation and a Taylor expansion we have

$$S_n(t)\begin{bmatrix}\hat{\mu}_{b_n}(t)-\mu(t)\\ b_n(\hat{\dot{\mu}}_{b_n}(t)-\dot{\mu}(t))\end{bmatrix} = \begin{bmatrix}\frac{1}{nb_n}\sum_{i=1}^n e_iK_{b_n}(t_i-t)+\frac{1}{2}\ddot{\mu}(t)\mu_2b_n^2\\ \frac{1}{nb_n}\sum_{i=1}^n e_iK_{b_n}(t_i-t)(\frac{t_i-t}{b_n})\end{bmatrix} + O(b_n^3+\frac{1}{nb_n})$$

uniformly with respect to $t\in\mathfrak{T}_n$. Note that $S_{n,0}(t) = 1+O(\frac{1}{nb_n})$ and $S_{n,1}(t) = O(\frac{1}{nb_n})$, uniformly with respect to $t\in\mathfrak{T}_n$, which yields

$$\sup_{t\in\mathfrak{T}_n}\Big|\hat{\mu}_{b_n}(t)-\mu(t)-\frac{\mu_2\ddot{\mu}(t)}{2}b_n^2-\frac{1}{nb_n}\sum_{i=1}^n e_iK_{b_n}(t_i-t)\Big| = O\big(b_n^3+\frac{1}{nb_n}\big).$$

Therefore the lemma follows from the definition of the estimate $\hat{\mu}_{b_n}$ in (3.4). $\square$

Lemma B.2. *Assume that the conditions of Lemma B.1 hold, then*

$$\sup_{t\in\mathfrak{T}'_n}\Big|c(t)(\hat{\mu}_{b_n}(t)-\mu(t))-\frac{1}{nb_n}\sum_{i=1}^n\Big[\nu_{2,b_n}(t)-\nu_{1,b_n}(t)\Big(\frac{t_i-t}{b_n}\Big)\Big]e_iK_{b_n}(t_i-t)+$$
$$\frac{b_n^2}{2}\ddot{\mu}(t)(\nu_{2,b_n}^2(t)-\nu_{1,b_n}(t)\nu_{3,b_n}(t))\Big| = O(b_n^3+\frac{1}{nb_n}),$$

where $\mathfrak{T}'_n = [0,b_n)\cup(1-b_n,1]$, $\nu_{j,b_n}(t) = \int_{-t/b_n}^{(1-t)/b_n}x^jK(x)dx$ *and* $c(t) = \nu_{0,b_n}(t)\nu_{2,b_n}(t)-\nu_{1,b_n}^2(t)$.

Proof. For any $t\in[0,b_n)\cup(1-b_n,1]$, using $(B.1)$, we obtain

$$S_n(t)\begin{bmatrix}\hat{\mu}_{b_n}(t)-\mu(t)\\ b_n(\hat{\dot{\mu}}_{b_n}(t)-\dot{\mu}(t))\end{bmatrix} = \begin{bmatrix}\frac{1}{nb_n}\sum_{i=1}^n[Y_i-\mu(t)-\dot{\mu}(t)(t_i-t)]K_{b_n}(t_i-t)\\ \frac{1}{nb_n}\sum_{i=1}^n[Y_i-\mu(t)-\dot{\mu}(t)(t_i-t)]K_{b_n}(t_i-t)(\frac{t_i-t}{b_n})\end{bmatrix} + O(\frac{1}{nb_n}),$$

and a Taylor expansion yields

$$S_n(t)\begin{bmatrix}\hat{\mu}_{b_n}(t)-\mu(t)\\ b_n(\hat{\dot{\mu}}_{b_n}(t)-\dot{\mu}(t))\end{bmatrix}=\begin{bmatrix}\frac{1}{nb_n}\sum_{i=1}^n e_iK_{b_n}(t_i-t)+\frac{b_n^2}{2}\nu_{2,b_n}(t)\ddot{\mu}(t)\\ \frac{1}{nb_n}\sum_{i=1}^n e_iK_{b_n}(t_i-t)(\frac{t_i-t}{b_n})+\frac{b_n^2}{2}\nu_{3,b_n}(t)\ddot{\mu}(t)\end{bmatrix}+O(b_n^3+\frac{1}{nb_n}) \tag{B.2}$$

uniformly with respect to $t\in[0,b_n)\cup(1-b_n,1]$. On the other hand, uniformly with respect to $t\in[0,b_n)\cup(1-b_n,1]$, we have that

$$S_n(t)=\begin{bmatrix}\nu_{0,b_n}(t) & \nu_{1,b_n}(t)\\ \nu_{1,b_n}(t) & \nu_{2,b_n}(t)\end{bmatrix}+O(\frac{1}{nb_n}). \tag{B.3}$$

Therefore, combining (B.2) and (B.3), it follows that

$$\begin{aligned}c(t)(\hat{\mu}_{b_n}(t)-\mu(t))=\frac{1}{nb_n}&\sum_{i=1}^n\big[\nu_{2,b_n}(t)-\nu_{1,b_n}(t)\big(\frac{t_i-t}{b_n}\big)\big]e_iK_{b_n}(t_i-t)+\\ &\frac{b_n^2}{2}\ddot{\mu}(t)\big(\nu_{2,b_n}^2(t)-\nu_{1,b_n}(t)\nu_{3,b_n}(t)\big)+O\Big(b_n^3+\frac{1}{nb_n}\Big)\end{aligned}$$

uniformly with respect to $t\in[0,b_n)\cup(1-b_n,1]$. □

The next lemma concerns the order of deviations of $\hat{\mu}_{b_n}$ from μ in the $\|\cdot\|_4$-norm.

Lemma B.3. *Assume that assumptions (A1)-(A4) are satisfied and that $nb_n^3\to\infty$, $nb_n^6\to 0$, then*

$$\sup_{t\in[0,1]}\|\hat{\mu}_{b_n}(t)-\mu(t)\|_4=O(b_n^2+(nb_n)^{-1/2}), \tag{B.4}$$

$$\Big\|\sup_{t\in[0,1]}|\hat{\mu}_{b_n}(t)-\mu(t)|\Big\|_4=O(b_n^2+(nb_n)^{-1/2}b_n^{-1/4}). \tag{B.5}$$

Proof. Observing the stochastic expansion in Lemma B.1 we first evaluate $\|\sum_{i=1}^n e_iK_{b_n}(t_i-t)\|_4$ and $\|\frac{\partial}{\partial t}\sum_{i=1}^n e_iK_{b_n}(t_i-t)\|_4$. Recalling the definition of projection operator $\mathcal{P}_i$ we note that

$$\Big\|\sum_{i=1}^n e_iK_{b_n}(t_i-t)\Big\|_4\le\sum_{k=0}^\infty\Big\|\sum_{i=1}^n\mathcal{P}_{i-k}e_iK_{b_n}(t_i-t)\Big\|_4.$$

Since for each k, $\mathcal{P}_{i-k}e_iK_{b_n}(t_i-t)$, $1\le i\le n$ is a martingale difference sequence, it follows from Burkholder's inequality

$$\Big\|\sum_{i=1}^n\mathcal{P}_{i-k}e_iK_{b_n}(t_i-t)\Big\|_4^2\le C\sum_{i=1}^n\|(\mathcal{P}_{i-k}e_iK_{b_n}(t_i-t))\|_4^2,$$

and condition (A4) implies $\| \sum_{i=1}^{n} \mathcal{P}_{i-k} e_i K_{b_n}(t_i - t)\|_4 = O(\sqrt{nb_n}\chi^k)$, uniformly with respect to $t \in [0,1]$. This yields

$$\sup_{t\in[0,1]} \Big\| \sum_{i=1}^{n} e_i K_{b_n}(t_i - t) \Big\|_4 = O(\sqrt{nb_n}). \tag{B.6}$$

Similar arguments show $\sup_{t\in[0,1]} \|\frac{\partial}{\partial t} \sum_{i=1}^{n} e_i K_{b_n}(t_i - t)\|_4 = O(\sqrt{nb_n} b_n^{-1})$. By Proposition B.1 in Section B.2 it follows that

$$\| \sup_{t\in[0,1]} |\sum_{i=1}^{n} e_i K_{b_n}(t_i - t)/(nb_n)| \|_4 = O((nb_n)^{-1/2} b_n^{-1/4}), \tag{B.7}$$

and by Lemma B.1 we obtain

$$\Big\| \sup_{t\in\mathfrak{T}_n} \Big| (\hat{\mu}_{b_n}(t) - \mu(t))^2 - \big(\frac{1}{nb_n} \sum_{i=1}^{n} e_i K_{b_n}(t_i - t) + \frac{\mu_2 \ddot{\mu}(t)}{2} b_n^2\big)^2 \Big| \Big\|_2 = O\Big(\frac{\chi_n}{\sqrt{nb_n} b_n^{1/4}} + \chi_n^2 \Big), \tag{B.8}$$

where $\chi_n = b_n^3 + \frac{1}{nb_n}$. Hence

$$\| \sup_{t\in\mathfrak{T}_n} (\hat{\mu}_{b_n}(t) - \mu(t))^2 \|_2 = O(\frac{1}{nb_n^{3/2}} + b_n^4).$$

By similar arguments and Lemma B.2 it follows that

$$\| \sup_{t\in\mathfrak{T}'_n} (\hat{\mu}_{b_n}(t) - \mu(t))^2 \|_2 = O(\frac{1}{nb_n^{3/2}} + b_n^4),$$

and a combination of the last two estimates gives (B.5). On the other hand, Lemma B.1, (B.6) and similar but easier arguments as given in the derivation of (B.8) show that

$$\sup_{t\in\mathfrak{T}'_n} \|(\hat{\mu}_{b_n}(t) - \mu(t))^2\|_2 = O((\frac{1}{\sqrt{nb_n}} + b_n^2)^2),$$

which proves the remaining estimate (B.4). □

The following results give a uniform bound for the p-mean of $\hat{\sigma}^2(t) - \sigma^2(t)$, where $\hat{\sigma}^2(\cdot)$ is the variance estimator defined above (3.7) in the main article.

Lemma B.4. *Suppose that Assumptions (A1)-(A4) are satisfied, $c_n \to 0$, $nc_n \to \infty$, and i) The variance function σ^2 is strictly positive, twice differentiable with a Lipschitz continuous second derivative $\ddot{\sigma}^2$. Then the estimate $\hat{\sigma}^2(t,k) = \hat{\sigma}^2_{c_n,b_n}(t,k)$ defined above (3.7) in the main artilce satisfies*

$$\max_{k\in[\lfloor n\zeta\rfloor, n-\lfloor n\zeta\rfloor]} \sup_{t\in\mathfrak{T}_{k,n}} \Big| \hat{\sigma}^2(t,k) - \sigma^2(t) - \frac{\mu_2 \ddot{\sigma}^2(t) c_n^2}{2} - \frac{1}{nc_n} \sum_{i=1}^{n} (\hat{e}_i^2 - \mathbb{E}e_i^2) K_{c_n}(t_i - t) \Big| = O\Big(c_n^3 + \frac{1}{nc_n} \Big), \tag{B.9}$$

$$\max_{k\in[\lfloor n\zeta\rfloor, n-\lfloor n\zeta\rfloor]}\sup_{t\in\mathfrak{T}'_{k-,n}}\Big|c(t,k-)(\hat\sigma^2(t,k)-\sigma^2(t)-\frac{1}{nc_n}\sum_{i=1}^n\Big[\nu_{2,c_n}(t,k-)-\nu_{1,c_n}(t,k-)\Big(\frac{t_i-t}{c_n}\Big)\Big]$$
$$\times[\hat e_i^2-\mathbb{E}(e_i^2)]K_{c_n}(t_i-t)+\frac{c_n^2}{2}\ddot\sigma^2(t)(\nu^2_{2,c_n}(t,k-)-\nu_{1,c_n}(t,k-)\nu_{3,c_n}(t,k-))\Big|=O\Big(c_n^3+\frac{1}{nc_n}\Big), \tag{B.10}$$

$$\max_{k\in[\lfloor n\zeta\rfloor, n-\lfloor n\zeta\rfloor]}\sup_{t\in\mathfrak{T}'_{k+,n}}\Big|c(t,k+)(\hat\sigma^2(t,k)-\sigma^2(t)-\frac{1}{nc_n}\sum_{i=1}^n\Big[\nu_{2,c_n}(t,k+)-\nu_{1,c_n}(t,k+)\Big(\frac{t_i-t}{c_n}\Big)\Big]$$
$$\times[\hat e_i^2-\mathbb{E}(e_i^2)]K_{c_n}(t_i-t)+\frac{c_n^2}{2}\ddot\sigma^2(t)(\nu^2_{2,c_n}(t,k+)-\nu_{1,c_n}(t,k+)\nu_{3,c_n}(t,k+))\Big|=O\Big(c_n^3+\frac{1}{nc_n}\Big), \tag{B.11}$$

where

$$\begin{aligned}
\mathfrak{T}_{k,n}&=[c_n,t_k-c_n]\cup[t_k+c_n,1-c_n],\ \mathfrak{T}'_{k-,n}=[0,c_n)\cup(t_k-c_n,t_k),\\
\mathfrak{T}'_{k+,n}&=[t_k,t_k+c_n)\cup(1-c_n,1],\\
\nu_{j,c_n}(t,k-)&=\int_{-t/c_n}^{(t_k-t)/c_n}x^jK(x)dx,\\
\nu_{j,c_n}(t,k+)&=\int_{(t_k-t)/c_n}^{(1-t)/c_n}x^jK(x)dx,\ c(t,k+)=\nu_{0,c_n}(t,k+)\nu_{2,c_n}(t,k+)-\nu^2_{1,c_n}(t,k+),\\
c(t,k-)&=\nu_{0,c_n}(t,k-)\nu_{2,c_n}(t,k-)-\nu^2_{1,c_n}(t,k-).
\end{aligned}$$

(ii) If there is an abrupt change of variance happened at time t_v, then a similar result of (B.9) *holds as follows:*

$$\max_{k\in[\lfloor n\zeta\rfloor,\lfloor n\tilde t_v\rfloor]}\sup_{t\in\mathfrak{T}^{k,-}_{\tilde t_v,n}}\Big|\hat\sigma^2(t,k)-\sigma^2(t)-\frac{\mu_2\ddot\sigma^2(t)c_n^2}{2}-\frac{1}{nc_n}\sum_{i=1}^n(\hat e_i^2-\mathbb{E}e_i^2)K_{c_n}(t_i-t)\Big|=O\Big(c_n^3+\frac{1}{nc_n}\Big),$$
$$\max_{k\in[\lfloor n\tilde t_v+1\rfloor,\lfloor n-n\zeta\rfloor]}\sup_{t\in\mathfrak{T}^{k,+}_{\tilde t_v,n}}\Big|\hat\sigma^2(t,k)-\sigma^2(t)-\frac{\mu_2\ddot\sigma^2(t)c_n^2}{2}-\frac{1}{nc_n}\sum_{i=1}^n(\hat e_i^2-\mathbb{E}e_i^2)K_{c_n}(t_i-t)\Big|=O\Big(c_n^3+\frac{1}{nc_n}\Big),$$

where $\mathfrak{T}^{k,-}_{\tilde t_v,n}=[c_n,t_k-c_n]\cup[\tilde t_v+c_n,1-c_n]$, $\mathfrak{T}^{k,+}_{\tilde t_v,n}=[c_n,\tilde t_v-c_n]\cup[t_k+c_n,1-c_n]$. *The similar results hold for* (B.10) *and* (B.11), *with* $t\in[0,c_n]\cup[t_k-c_n,t_k]$ *uniformly for* $t_k\le\tilde t_v$ *and* $t\in[t_k,t_k+c_n]\cup[1-c_n,1]$ *uniformly for* $t_k\ge\tilde t_v$,*respectively.*

Proof. For a proof of part (i) we only show (B.11). The remaining results can be proved similarly. For any $k\in[\lfloor n\zeta\rfloor,n-\lfloor n\zeta\rfloor]$ and $t\in\mathfrak{T}'_{k+,n}$, following the argument given in the proof of Lemma B.1, we have that

$$S_n(t,k+)\begin{bmatrix}(\hat\sigma^2(t,k+)-\sigma^2(t))\\ c_n(\hat{\dot\sigma}^2(t,k+)-\dot\sigma^2(t))\end{bmatrix}=\begin{bmatrix}\frac{1}{nc_n}\sum_{i=1}^n(\hat e_i^2-\sigma^2(t)-\dot\sigma^2(t)(t_i-t))K_{c_n}(t_i-t)\\ \frac{1}{nc_n}\sum_{i=1}^n(\hat e_i^2-\sigma^2(t)-\dot\sigma^2(t)(t_i-t))(\frac{t_i-t}{c_n})K_{c_n}(t_i-t)\end{bmatrix},$$

where $S_n(t, k+)$ is defined as follows:

$$S_n(t, k+) = \begin{bmatrix} \nu_{0,c_n}(t, k+) & \nu_{1,c_n}(t, k+) \\ \nu_{1,c_n}(t, k+) & \nu_{2,c_n}(t, k+) \end{bmatrix} + O(\frac{1}{nc_n}).$$

The lemma now follows by the same arguments as given in the proof of Lemma B.1 and Lemma B.2, and the fact that the remaining order in the notations $O(\cdot)$ is independent of k.
Part (ii) can be shown similarly to (i) observing the definition of $\hat{\sigma}^2(t)$, the conditions on the bandwidths c_n and b_n and the fact that $\sigma^2(t)$ varies smoothly before and after $\tilde{t}_v$. □

Corollary B.1. *Suppose that the conditions of Lemma B.4 hold. Let $\upsilon' \in (\frac{1}{2}, 1 - \frac{4}{\iota})$ where ι is defined in condition (A3) and condition (A4). Recall that $\hat{\sigma}^2(t) = \hat{\sigma}^2(t, \lfloor nt_n^* \rfloor)$. Let $k^* = \lfloor nt_n^* \rfloor$, $\tilde{k} = \lfloor n\tilde{t}_v \rfloor$*

(i) If the variance changes smoothly in the interval $(0, 1)$, then

$$\sup_{t \in [c_n, t_n^* - c_n] \cup [t_n^* + c_n, 1 - c_n]} \Big| \hat{\sigma}^2(t) - \sigma^2(t) - \frac{\mu_2 \ddot{\sigma}^2(t) c_n^2}{2} - \frac{1}{nc_n} \sum_{i=1}^{n} (\hat{e}_i^2 - \mathbb{E}e_i^2) K_{c_n}(t_i - t) \Big| = O\Big(c_n^3 + \frac{1}{nc_n}\Big), \tag{B.12}$$

$$\sup_{t \in [0, c_n) \cup (t_n^* - c_n, t_n^*]} \Big| c(t, k^*-)(\hat{\sigma}^2(t) - \sigma^2(t) - \frac{1}{nc_n} \sum_{i=1}^{n} \Big[\nu_{2,c_n}(t, k^*-) - \nu_{1,c_n}(t, k^*-)\Big(\frac{t_i - t}{c_n}\Big)\Big]$$
$$\times [\hat{e}_i^2 - \mathbb{E}(e_i^2)] K_{c_n}(t_i - t) + \frac{c_n^2}{2} \ddot{\sigma}^2(t)(\nu_{2,c_n}^2(t, k^*-) - \nu_{1,c_n}(t, k^*-)\nu_{3,c_n}(t, k^*-)) \Big| = O\Big(c_n^3 + \frac{1}{nc_n}\Big), \tag{B.13}$$

$$\sup_{t \in (t_n^*, t_n + c_n) \cup (1 - c_n, 1]} \Big| c(t, k^*+)(\hat{\sigma}^2(t) - \sigma^2(t) - \frac{1}{nc_n} \sum_{i=1}^{n} \Big[\nu_{2,c_n}(t, k^*+) - \nu_{1,c_n}(t, k^*+)\Big(\frac{t_i - t}{c_n}\Big)\Big]$$
$$\times [\hat{e}_i^2 - \mathbb{E}(e_i^2)] K_{c_n}(t_i - t) + \frac{c_n^2}{2} \ddot{\sigma}^2(t)(\nu_{2,c_n}^2(t, k^*+) - \nu_{1,c_n}(t, k^*+)\nu_{3,c_n}(t, k^*+)) \Big| = O\Big(c_n^3 + \frac{1}{nc_n}\Big). \tag{B.14}$$

(ii) If the variance has an abrupt change point, then

$$\sup_{t\in[c_n,\tilde{t}_v-c_n-n^{-v'}]\cup[\tilde{t}_v+c_n+n^{-v'},1-c_n]}\Big|\hat{\sigma}^2(t)-\sigma^2(t)-\frac{\mu_2\ddot{\sigma}^2(t)c_n^2}{2}-\frac{1}{nc_n}\sum_{i=1}^n(\hat{e}_i^2-\mathbb{E}e_i^2)K_{c_n}(t_i-t)\Big|$$
$$=O\Big(c_n^3+\frac{1}{nc_n}\Big),$$

$$\sup_{t\in[0,c_n)\cup(t_v-c_n-n^{-v'},t_v-n^{-v'}]}\Big|c(t,k^*-)(\hat{\sigma}^2(t)-\sigma^2(t)-\frac{1}{nc_n}\sum_{i=1}^n\Big[\nu_{2,c_n}(t,k^*-)-\nu_{1,c_n}(t,k^*-)\Big(\frac{t_i-t}{c_n}\Big)\Big]$$
$$\times[\hat{e}_i^2-\mathbb{E}(e_i^2)]K_{c_n}(t_i-t)+\frac{c_n^2}{2}\ddot{\sigma}^2(t)(\nu_{2,c_n}^2(t,k^*-)-\nu_{1,c_n}(t,k^*-)\nu_{3,c_n}(t,k^*-))\Big|=O\Big(c_n^3+\frac{1}{nc_n}\Big).$$

$$\sup_{t\in[\tilde{t}_v+n^{-v'},\tilde{t}_v+c_n+n^{-v'})\cup(1-c_n,1]}\Big|c(t,k^*+)(\hat{\sigma}^2(t)-\sigma^2(t)-\frac{1}{nc_n}\sum_{i=1}^n\Big[\nu_{2,c_n}(t,k^*+)-\nu_{1,c_n}(t,k^*+)\Big(\frac{t_i-t}{c_n}\Big)\Big]$$
$$\times[\hat{e}_i^2-\mathbb{E}(e_i^2)]K_{c_n}(t_i-t)+\frac{c_n^2}{2}\ddot{\sigma}^2(t)(\nu_{2,c_n}^2(t,k^*+)-\nu_{1,c_n}(t,k^*+)\nu_{3,c_n}(t,k^*+))\Big|=O\Big(c_n^3+\frac{1}{nc_n}\Big).$$

$$\sup_{t\in(\tilde{t}_v-n^{-v'},\tilde{t}_v+n^{-v'})}|\hat{\sigma}^2(t)-\sigma^2(t)|=O_p(1).$$

Proof. Part (i) follows directly from Lemma B.4. Part (ii) follows from Lemma B.4 and the fact that $\tilde{t}_v-t_n^*=o_p(n^{-v'})$. □

Corollary B.2. *Suppose that the conditions of Lemma B.4 hold with $\iota\geq 8$.*

(i) If there is no abrupt change in variance, then

$$\sup_{t\in[0,1]}\|\hat{\sigma}^2(t)-\sigma^2(t)\|_4=O\Big(c_n^2+\frac{1}{\sqrt{nc_n}}+b_n^2+\frac{1}{\sqrt{nb_n}}\Big),\tag{B.15}$$
$$\Big\|\sup_{t\in(0,1)}|\hat{\sigma}^2(t)-\sigma^2(t)|\Big\|_4=O\Big(c_n^2+\Big(\frac{1}{\sqrt{nc_n}}+b_n^2+\frac{1}{\sqrt{nb_n}}\Big)c_n^{-1/4}\Big).\tag{B.16}$$

(ii) If there is an abrupt change in variance, then

$$\sup_{t\in[0,t_v-n^{-v'}]\cup[t_v+n^{-v'},1]}\|\hat{\sigma}^2(t)-\sigma^2(t)\|_4=O\Big(c_n^2+\frac{1}{\sqrt{nc_n}}+b_n^2+\frac{1}{\sqrt{nb_n}}\Big),$$
$$\Big\|\sup_{t\in[0,t_v-n^{-v'}]\cup[t_v+n^{-v'},1]}|\hat{\sigma}^2(t)-\sigma^2(t)|\Big\|_4=O\Big(c_n^2+\Big(\frac{1}{\sqrt{nc_n}}+b_n^2+\frac{1}{\sqrt{nb_n}}\Big)c_n^{-1/4}\Big),$$

where $v'\in(\frac{1}{2},1-\frac{4}{\iota})$ for ι defined in (A3) and (A4).

Proof. By Corollary B.1 (i), when there is no abrupt changes in variance, we have that

$$\sup_{t\in[0,1]}\|\hat{\sigma}^2(t)-\sigma^2(t)\|_4\leq\sup_{t\in[0,1]}\|(g_1(t,\mathcal{F}_i))\|_4+\sup_{t\in[0,c_n]\cup[t_n^*-c_n,t_n^*]}\|(g_2(t,\mathcal{F}_i))\|_4\tag{B.17}$$
$$+\sup_{t\in[t_n^*,t_n^*+c_n]\cup[1-c_n,1]}\|(g_3(t,\mathcal{F}_i))\|_4$$

$$\sup_{t\in[0,1]}|\hat{\sigma}^2(t)-\sigma^2(t)| \leq \sup_{t\in[0,1]}(g_1(t,\mathcal{F}_i)) + \sup_{t\in[0,c_n]\cup[t_n^*-c_n,t_n^*]}(g_2(t,\mathcal{F}_i)) + \sup_{t\in[t_n^*,t_n^*+c_n]\cup[1-c_n,1]}(g_3(t,\mathcal{F}_i)) \tag{B.18}$$

where $g_i(t,\mathcal{F}_i)$, $i=1,2,3$ is the approximation terms of (B.12)–(B.14), respectively. Noting that

$$\sup_{t\in[0,1]}\Big\|\frac{1}{nc_n}\sum_{i=1}^n(e_i^2-\mathbb{E}e_i^2)K_{c_n}(t_i-t)\Big\|_4 = O\Big(\frac{1}{\sqrt{nc_n}}\Big).$$

By the proof of Lemma B.3, we obtain (note that $\iota\geq 8$)

$$\sup_{t\in[0,1]}\|\hat{\mu}_{b_n}(t)-\mu(t)\|_8 = O\Big(b_n^2+\frac{1}{\sqrt{nb_n}}\Big),$$

which yields (note that $\hat{e}_i = e_i+\mu(t_i)-\hat{\mu}_{b_n}(t_i)$)

$$\sup_{t\in[0,1]}\Big\|\frac{1}{nc_n}\sum_{i=1}^n(e_i^2-\hat{e}_i^2)K_{c_n}(t_i-t)\Big\|_4 = O\Big(b_n^2+\frac{1}{\sqrt{nb_n}}\Big).$$

Combining with (B.17), we have shown (B.15).
Recalling for $j=0,1,2,3$, $\nu_{j,c_n}(t,k+) = \int_{(t_k-t)/c_n}^{(1-t)/c_n} x^jK(x)dx := \tilde{\nu}_{j,c_n}(t,t_k)$, then elementary calculations shows that

$$\frac{\partial}{\partial t}\tilde{\nu}_{j,c_n}(t,t_k) = O(c_n^{-1}),\ \frac{\partial}{\partial t_k}\tilde{\nu}_{j,c_n}(t,t_k) = O(c_n^{-1}),\ \frac{\partial^2}{\partial t\partial t_k}\tilde{\nu}_{j,c_n}(t,t_k) = O(c_n^{-2}).$$

Similar results hold for $\nu_{j,c_n}(t,k-)$.Then (B.16) follows from (B.18) and Proposition B.2.
ii) follows from similarly arguments and the assertion ii) of Corollary B.1. □

Corollary B.3. *Suppose the conditions of Lemma B.4 hold, with $\iota\geq 8$. Then*

$$\Big\|\sup_{t\in(0,1)}|\hat{\sigma}^2(t)-\sigma^2(t)|\Big\|_4 = O\Big(c_n^2+\Big(\frac{1}{\sqrt{nc_n}}+b_n^2+\frac{1}{\sqrt{nb_n}}\Big)c_n^{-1/4}\Big).$$

Proof: The lemma follows from Proposition B.1 in Section B.2, the triangle inequality and simple calculations. Note that the first assumption of Proposition B.1 is satisfied by the arguments in Corollary B.2. The second assumption regarding the derivative can be shown by similar arguments as given in (B.6) and (B.7). □

B.2 Three additional technical results

Proposition B.1. *Let $\{\Upsilon_n(t)\}_{t\in[0,1]}$ be a sequence of stochastic processes with differentiable paths. Assume that for some $p\geq 1$ and any $t\in[0,1]$, $\|\Upsilon_n(t)\|_p = O(m_n)$, $\|\dot{\Upsilon}_n(t)\|_p = O(l_n)$, where m_n, l_n are sequences of real numbers, $m_n = O(l_n)$, then*

$$\Big\|\sup_{t\in[0,1]}|\Upsilon_n(t)|\Big\|_p = O\Big(m_n\Big(\frac{m_n}{l_n}\Big)^{-\frac{1}{p}}\Big).$$

In particular, if $p=2$, we have $\|\sup_{t\in[0,1]}|\Upsilon_n(t)|\|_2 = O(\sqrt{m_n l_n})$.

Proof. For a sequence b_n define $\tilde{b}_n = \lfloor {b_n}^{-1} \rfloor$ and let $\tau_i = ib_n$, $i = 1,2,...,\tilde{b}_n$ and $\tau_i = 1$ for $i = \tilde{b}_n + 1$. Then by the triangle inequality, we have

$$\sup_{t\in(0,1)} |\Upsilon_n(t)| \le \max_{0\le i\le \tilde{b}_n+1} |\Upsilon_n(\tau_i)| + \max_{1\le i\le \tilde{b}_n+1} Z_{in},$$

where $Z_{in} = \sup_{\tau_i - b_n < t < \tau_i} |\Upsilon_n(t) - \Upsilon_n(\tau_i)|$. Observing the inequalities

$$\|Z_{in}\|_p \le \Big\| \int_{\tau_i - b_n}^{\tau_i} |\dot{\Upsilon}(t)| dt \Big\|_p \le \int_{\tau_i - b_n}^{\tau_i} \|\dot{\Upsilon}_n(t)\|_p dt = O(b_n l_n)$$

and $\max_{1\le i\le \tilde{b}_n+1} Z_{in}^p \le \sum_{i=1}^{\tilde{b}_n+1} Z_{in}^p$, we have

$$\Big\| \max_{1\le i\le \tilde{b}_n+1} Z_{in} \Big\|_p = O((l_n^p b_n^{(p-1)})^{1/p}) = O(l_n b_n^{(p-1)/p}).$$

Similarly, we obtain the estimate $\|\max_{0\le i\le \tilde{b}_n+1} |\Upsilon_n(t_i)|\|_p = O_p(b_n^{-1/p} m_n)$, and picking $b_n = m_n/l_n$ proves the assertion. □

Proposition B.2. *Let $\{\Upsilon_n(x,y)\}_{x,y\in[0,1]}$ be a sequence of stochastic processes with differentiable paths. Assume that for some $p \ge 1$ and any $x, y \in [0,1]$, $\|\Upsilon_n(x,y)\|_p = O(m_n)$, $\|\frac{\partial}{\partial x}\Upsilon_n(x,y)\|_p = O(l_{1,n})$, $\|\frac{\partial}{\partial y}\Upsilon_n(x,y)\|_p = O(l_{2,n})$, $\|\frac{\partial^2}{\partial x \partial y}\Upsilon_n(x,y)\|_p = O(l_{3,n})$ where m_n, l_n are sequences of real numbers, $m_n = O(l_n)$, let $c_n \to 0$. If $l_{1,n} \asymp l_{2,n}$, then*

$$\Big\| \sup_{x\in[0,1], y\in[(x-c_n)\vee 0, (x+c_n)\wedge 1]} |\Upsilon_n(x,y)| \Big\|_p = O(m_n (c_n')^{-2/p} c_n^{1/p}),$$

where $c_n' = \frac{l_{1,n}}{l_{3,n}} + \frac{m_n}{l_{1,n}} + \left(\frac{m_n}{l_{3,n}}\right)^{1/2}$.

Proof. For a sequence b_n define $\tilde{b}_n = \lfloor {b_n}^{-1} \rfloor$ and let $\tau_i = ib_n$, $i = 1,2,...,\tilde{b}_n$ and $\tau_i = 1$ for $i = \tilde{b}_n + 1$. Then by the triangle inequality, we have

$$\begin{aligned}\sup_{t\in(0,1)} |\Upsilon_n(t)| \le & \max_{0\le i\le \tilde{b}_n+1, 0\vee(i-\lfloor c_n/b_n\rfloor)\le j\le (i+\lfloor c_n/b_n\rfloor)\wedge 1} |\Upsilon_n(\tau_i,\tau_j)| \\ & + \max_{0\le i\le \tilde{b}_n+1, 0\vee(i-\lfloor c_n/b_n\rfloor)\le j\le (i+\lfloor c_n/b_n\rfloor)\wedge 1} Z_{i,j},\end{aligned}$$

where $Z_{i,j} = \sup_{\tau_i - b_n < t_1 < \tau_i, \tau_j - b_n < t_2 < \tau_j} |\Upsilon_n(t_1,t_2) - \Upsilon_n(\tau_i,\tau_j)|$. Observing the inequalities

$$\begin{aligned}\|Z_{i,j}\|_p \le & \int_{\tau_i - b_n}^{\tau_i} \|\frac{\partial}{\partial x}\Upsilon_n(x,y)\|_p dx + \int_{\tau_i - b_n}^{\tau_i} \|\frac{\partial}{\partial y}\Upsilon_n(x,y)\|_p dy + \\ & \int_{\tau_i - b_n}^{\tau_i} \int_{\tau_i - b_n}^{\tau_i} \|\frac{\partial^2}{\partial x \partial y}\Upsilon_n(x,y)\|_p dx dy = O(b_n^2 l_{3,n} + b_n l_{1,n} + b_n l_{2,n})\end{aligned} \quad \text{(B.19)}$$

and $\max_{0\le i\le \tilde b_n+1, 0\vee(i-\lfloor c_n/b_n\rfloor)\le j\le (i+\lfloor c_n/b_n\rfloor)\wedge 1} Z^p_{ij} \le \sum_{i=1}^{\tilde b_n+1}\sum_{j=0\vee(i-\lfloor c_n/b_n\rfloor)}^{(i+\lfloor c_n/b_n\rfloor)\wedge 1} Z^p_{i,j}$, we have

$$\Big\|\max_{1\le i\le \tilde b_n+1} Z_{in}\Big\|_p = O(l_{1,n}b_n^{(p-1)/p}(\frac{c_n}{b_n})^{1/p} + l_{2,n}b_n^{(p-1)/p}(\frac{c_n}{b_n})^{1/p} + l_{3,n}b_n^{(2p-1)/p}(\frac{c_n}{b_n})^{1/p}).$$

Similarly, we obtain the estimate

$$\|\max_{0\le i\le \tilde b_n+1, 0\le j\le \tilde b_n+1}|\Upsilon_n(t_i,t_j)|\|_p = O_p(b_n^{-2/p}c_n^{1/p}m_n),$$

and picking $b_n = c_n'$ proves the assertion. □

Proposition B.3. *Suppose A_n are sets such that $\mathbb{P}(A_n)\to 0$ as $n\to\infty$, and $X_nI(\bar A_n) = O_p(1)$. Then $X_n = O_p(1)$.*

Proof. For any $\epsilon>0$, let N be a large constant such that $\mathbb{P}(A_n)\le \epsilon/2$ for $n\ge N$, and M be a large constant such that $\mathbb{P}(|X_n|I(\bar A_n)\ge M/2)\le\epsilon/2$ for $n\ge N$. Then

$$\begin{aligned}\mathbb{P}(|X_n|\ge M) &\le \mathbb{P}(|X_n|I(A_n)\ge M/2) + \mathbb{P}(|X_n|I(\bar A_n)\ge M/2)\\ &\le \mathbb{P}(A_n) + \mathbb{P}(|X_n|I(\bar A_n)\ge M/2)\le\epsilon\end{aligned}$$

for all $n\ge N$. □

C Additional Simulation Result

C.1 Simulation Results under Stationarity

We consider

(I^*) $G(t,\mathcal{F}_i) = H(t,\mathcal{F}_i)\sqrt{c(t)}/2$ for $t\le 0.5$, and $G(t,\mathcal{F}_i) = H_1(t,\mathcal{F}_i)\sqrt{c(t)}/2$ for $t>0.5$, where $c(t) = 1-(t-0.5)^2$, $H(t,\mathcal{F}_i) = 0.2H(t,\mathcal{F}_{i-1})+\varepsilon_i$ for $t\le 0.5$, $H_1(t,\mathcal{F}_i) = (0.2-\lambda)H_1(t,\mathcal{F}_{i-1})+\varepsilon_i$ for $t>0.5$ and random variables $\{\varepsilon_i, i\in\mathbb{Z}\}$ are *i.i.d.* $N(0,1)$.

We use Algorithm 3.1 in page 13 of the main article to test change points in the lag-1 correlation of model I^*, and compare the results with the following algorithm C.1 tailored to model I^* under the **stationary** assumption.

Let $B(t)$ be a standard Brownian motion and κ^2 be the long run variance of $\frac{e_ie_{i+1}}{\sigma^2}$. Note that under stationarity, the variance $\sigma^2(t)$ and long run variance $\kappa^2(t)$ are now time invariant.

Algorithm C.1.

[1] Calculate the statistic $\hat{T}_n$ defined in (3.3) of the main article with $\frac{\hat{e}_i\hat{e}_{i+1}}{\hat{\sigma}^2(t_i)}$ replaced by $\hat{L}_i := \frac{\hat{e}_i\hat{e}_{i+1}}{\hat{\sigma}^2}$, where $\hat{e}_i$ is obtained by local linear estimation with bandwidth selected by GCV, and $\hat{\sigma}^2 = \frac{\sum_{i=1}^n \hat{e}_i^2}{n}$.

[2] Estimate the long run variance by $\hat{\kappa}^2 = \sum_{i=m+1}^{n-m} \hat{Q}_i^2/((2m+1)(n-2m))$, $\hat{Q}_i = \sum_{j=-m}^{m}(\hat{L}_i - \sum_{i=1}^n \hat{L}_i/n)$ where m is selected by GCV.

[3] Reject the null hypothesis at nominal level α if

$$\hat{T}_n/\sqrt{n} > \hat{\kappa} M_{1-\alpha}, \tag{C.20}$$

where $M_{1-\alpha}$ is the $(1-\alpha)_{th}$ quantile of the $\max_{0\le t\le 1}|B(t) - tB(1)|$.

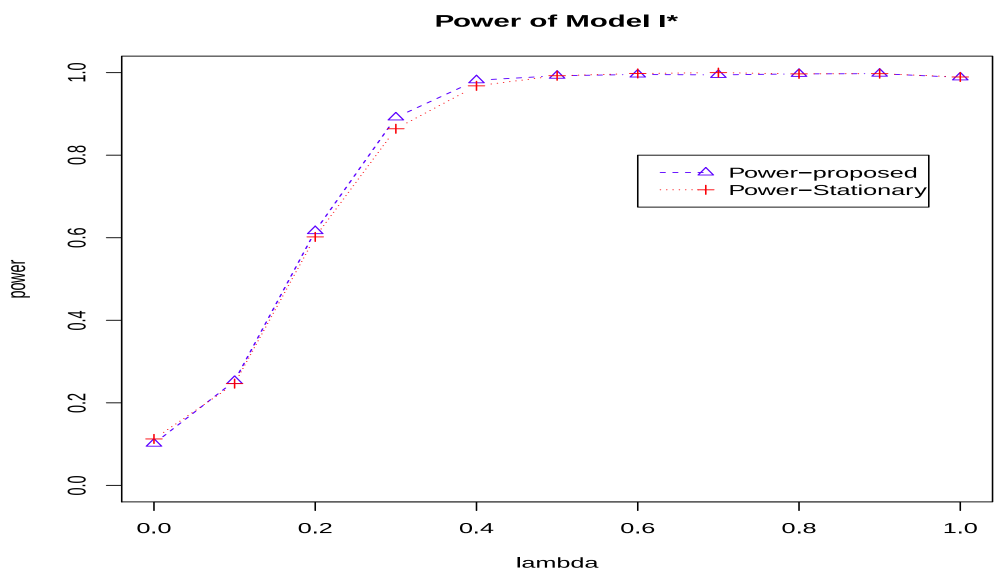

Figure C.1: *Power comparison of change point tests in the lag-1 correlation of model I^*. Blue line: simulated power of Algorithm 3.1 defined in the main article; red line: simulated power of Algorithm C.1.*

The simulated results are presented in Figure C.1. It shows that our method has decent power. It is slighter more powerful than Algorithm C.1 designed for stationary processes.

Table C.1: *Simulated type I error of the test* (3.15) *for a change point in the lag-1 correlation for model II and of the test* (4.15) *for a relevant change in the lag-1 correlation for model III using a Biweight kernel (at the boundary point of the null). The last column represents the simulated Type I error if the bandwidth is* b_n *selected by GCV.*

	b_n	0.075	0.1	0.125	0.15	0.175	0.2	0.225	GCV
II	5%	4.5	5.4	4.4	4.25	3.25	2.65	3.55	3.6
	10%	10.6	10.75	9.6	9.6	9.4	7.5	8.4	8.9
III	5%	6.1	5.85	5.95	6.2	6.3	4.95	6.05	5.5
	10%	10.6	10.2	10.45	10.55	11.65	9.1	10.5	9.4

C.2 The Impact of Different Kernel Functions

Under piecewise local stationarity, Lemma B.1 in the supplementary material and Proposition 5 of Zhou (2013) imply that for a given kernel function K, the optimal bandwidth is given by

$$b_n^{optimal} = \left(\frac{\phi_0 \int_0^1 \tilde{\kappa}^2(t)dt}{\mu_2^2 \int_0^1 |\mu''(t)|dt} \right)^{1/5} n^{1/5},$$

where $\tilde{\kappa}^2(t)$ is the long run variance of e_i, $\mu_2 = \int x^2 K(x)dx$ and $\phi_0 = \int K^2(x)dx$. Fan and Yao (2003) pointed out that the performance of procedures with bandwidth is not very sensitive with respect to the choice of different kernel functions. We have confirmed these observations in further simulations. Exemplarily we show in Table C.1 the simulated type I error of the test (3.15) for a change point in the lag-1 correlation for model II and of the test (4.15) for a relevant change in the lag-1 correlation for model III using a Biweight kernel instead of the Epanechnikov kernel (for this kernel the corresponding results can be found in Table 1 of the main document). We do not observe any significant differences between the results obtained for the two different kernels and the simulated type I error rates are quite close to their nominal levels.

C.3 Performance under Different Sample Sizes

In this section we investigate the performance of the new tests for different sample sizes in more detail. In Table C.2 and C.3 we display the simualted type I error of the test (3.15) for a change point in the lag-1 correlation for model II and of the test (4.15) for a relevant change in the lag-1

Table C.2: *Simulated Type I error of the test* (3.15) *for a change point in the lag-1 correlation for model II and the sample sizes* 300 *and* 800*. The last column represents the simulated Type I error if the bandwidth is* b_n *selected by GCV.*

	b_n	0.075	0.1	0.125	0.15	0.175	0.2	0.225	GCV
300	5%	4.95	4.5	4.4	4.45	3.8	4.1	3.6	5.25
	10%	11.25	10.65	10.15	10.5	8.6	9.5	8.8	10.3
800	5%	3.9	3.25	4.5	3.8	4.4	3.6	3.4	4.6
	10%	9.3	9.3	9.6	9.2	9.1	8.95	7.55	9.95

Table C.3: *Simulated Type I error of the test* (4.15) *for a relevant change in the lag-1 correlation (at the boundary point of the null) for model III and the sample sizes* 300 *and* 800*. The last column represents the simulated Type I error if the bandwidth is* b_n *selected by GCV.*

	b_n	0.075	0.1	0.125	0.15	0.175	0.2	0.225	GCV
300	5%	5.2	6.4	6.25	5.35	5.7	4.55	5.05	4.95
	10%	11.2	11.3	11.4	9.9	11.05	9.05	9.4	9.65
800	5%	5.7	5.7	6.75	5.6	6.7	4.9	4.9	5.45
	10%	10.8	10.35	10.8	9.8	10.85	8.45	8.7	10.65

correlation (at the boundary point of the null) for model III and the sample sizes 300 and 800, respectively. Again we observe a reasonable approximation of the nominal level.

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