# PROOFS OF THEOREMS OF THE PAPER "RUN LENGTH PROPERTIES OF THE CUSUM AND EWMA SCHEMES FOR THE STATIONARY LINEAR PROCESSES"

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April 22, 2015

## 1 Tow Theorems.

To obtain the asymptotic ARL for the two control charts, we need three conditions presented in the following.

Let $h(\theta) = \mathbf{E}(e^{\theta\xi_j})$ denote the moment-generating functions of $\xi_j$. We suppose that the white noise $\{\xi_j\}$ satisfies the following two conditions:

**(I) The distribution of $\xi_1$ is not a point mass at $\mathbf{E}(\xi_1)$.**

**(II) The moment-generating function of $\xi_1$ satisfies $h(\theta) < \infty$ for some $\theta > 0$ and $\bar{h} = \sup\{h'(\theta)/h(\theta) : \theta < \bar{\theta}\} > 0$, where $\bar{\theta} = \sup\{\theta : \ h(\theta) < \infty\}$.**

Note that, from condition II, it follows that $h(\theta)$ is the analytic function for $|\theta| < \bar{\theta}$. It can be shown that many distributions, such as normal, exponential, uniform and Poisson, satisfy conditions I and II.

Another condition is about $\{a_k\}$.

**(III)** $\sum_{k=1}^{\infty} k|a_k| < \infty$.

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*AMS* 1991 *subject classification.* Primary 62L10; secondary 62N10
*Key Words and Phrases.* Change point detection, average run length, autocorrelated stationary processes

This condition implies that

$$\lim_{n\to\infty} n \sum_{k=n+1}^{\infty} |a_k| = 0 \tag{1}$$

Let $\eta_j = \delta(A\xi_j - \frac{\delta}{2})$ and $h_\eta(\theta) = \mathbf{E}(e^{\theta\eta_1})$ denote the moment-generating functions of $\eta_1$. Let $\theta(y)$ satisfy $y = h'_\eta(\theta(y))/h_\eta(\theta(y))$.

Now we present the asymptotic ARLs of the CUSUM chart.

**Theorem 1.** Suppose conditions (I), (II) and (III) hold. Let $\hat{\mu} = \delta(\mu - \delta/2)$.

(i) If $0 \le \mu < \delta/2$, then

$$\frac{1}{bc} e^{c(\theta^* + o(1))} \le \mathbf{ARL}_\mu(T_C(c)) \le \frac{2c}{u} e^{c(\theta^* + o(1))} \tag{2}$$

for a large control limit, $c$, where $\theta^* > 0$ is a unique positive root of the equation $\log h(\delta A\theta) - \delta^2\theta/2 = 0$ on $\theta > 0$, $u = \delta A h'(\delta A\theta^*)/h(\delta A\theta^*) - \delta^2/2 > 0$ and $b$ is a positive constant defined by

$$b = \inf\{x > 1/u : \theta(\frac{1}{x}) - x \log h_\eta(\theta(\frac{1}{x})) \ge 2\theta^*\}. \tag{3}$$

(ii) If $\mu > \delta/2$ , then

$$-(1 + o(1))\frac{3\sqrt{c}\log c}{(\hat{\mu})^{3/2}} + \frac{c}{\hat{\mu}} \le \mathbf{ARL}_\mu(T_C(c)) \le \frac{c}{\hat{\mu}} + \frac{2\sqrt{c}\log c}{(\hat{\mu})^{3/2}} + \frac{e^{(\delta\sigma A)^2/2}}{\hat{\mu} c^{\sqrt{2}-1}}(1 + o(1)) \tag{4}$$

for large $c$.

For the EWMA chart, we let the control limit, $\tilde{c}$, be fixed and the weight parameter, $r$, be small such that the $\text{ARL}_0$ becomes large. In the following theorem, we see that the role of the control limit, $\tilde{c}$, in the EWMA chart is the same as the reference value $\delta/2$ in the CUSUM chart, and the weight parameter, $r$, in the EWMA chart is like the control limit, $c$, in the CUSUM chart.

**Theorem 2.** Suppose that conditions (I), (II) and (III) hold.

(i) If $0 \le \mu < \tilde{c}$, then

$$e^{\frac{1}{r}(\theta^*(\tilde{c}) + o(1))} \le \mathbf{ARL}_\mu(T_E(r)) \le \frac{3\log r^{-1}}{r} e^{\frac{1}{r}(\theta^*(\tilde{c}) + o(1))} \tag{5}$$

for a small weighting parameter $r$, where $\theta^*(\tilde{c}) = \tilde{c}\theta_{\tilde{c}} - \log h_\zeta(\theta_{\tilde{c}})$, $\theta_{\tilde{c}}$ is a unique positive root of the equation $\tilde{c}\theta - \log h(A\theta) = 0$ on $\theta > 0$ and $h_\zeta(\theta)$ is defined by

$$h_\zeta(\theta) = \exp\{\int_0^\theta \frac{\log h(Ax)}{x} dx\}. \tag{6}$$

(ii) If $\mu > \tilde{c}$, then

$$(1 + o(1))(1 - \frac{1}{(\log r^{-1})^p})\frac{1}{r}\log\frac{\mu}{\mu - \tilde{c}} \le \mathbf{ARL}_\mu(T_E(r)) \le (1 + o(1))\frac{1}{r}\log\frac{\mu}{\mu - \tilde{c}} \tag{7}$$

for small $r$, where $p$ is a positive number.

**Remark 1.** It is convenient to rewrite the results of the two theorems in the following expressions. For large $c$ and small $r$ we have

$$\mathbf{ARL}_\mu(T_C(c)) = L_C e^{c(\theta^*+o(1))}, \qquad \mathbf{ARL}_\mu(T_E(r)) = L_E e^{\frac{1}{r}(\theta^*(\tilde{c})+o(1))} \tag{8}$$

for $0 \le \mu < \delta/2$ and $0 \le \mu < \tilde{c}$ respectively, and

$$\mathbf{ARL}_\mu(T_C(c)) = (1+o(1))\frac{c}{\delta(\mu-\delta/2)}, \qquad \mathbf{ARL}_\mu(T_E(r)) = (1+o(1))\frac{1}{r}\log\frac{\mu}{\mu-\tilde{c}} \tag{9}$$

for $\mu > \delta/2$ and $\mu > \tilde{c}$, respectively, where $c$ and $\tilde{c}$ are the control limits of the CUSUM and EWMA, respectively, and $L_C$ and $L_E$ satisfy $1/(bc) \le L_C \le 2c/u$ and $1 \le L_E \le 3\log r^{-1}/r$, respectively.

## 2 Proofs of Theorem 1

We first present two lemmas. Here, lemma 1 in the following is a slight generalization of the lemma given in Durrett (2005, P.73) and lemma 2 is the same as Lemma 2 in Han and Tsung (2006). We omit the proofs of lemma 2.

**Lemma 1**. Let $Z_k, 1 \le k \le n$, be independent with distributions $F_k(x)$ and the moment-generating functions $h_k(\lambda)$, and let $Z_k^\lambda, 1 \le k \le n$, be independent with the distributions $F_k^\lambda(y)$ and the moment-generating functions $h_k^\lambda(\theta)$, where $h_k(\lambda) < \infty, 1 \le k \le n$, for some $\lambda > 0$ and

$$F_k^\lambda(y) = \frac{1}{h_k(\lambda)}\int_{-\infty}^{y} e^{\lambda x} dF_k(x), \qquad h_k^\lambda(\theta) = \mathbf{E}_k^\lambda(e^{\theta Z_k^\lambda}) \tag{10}$$

for some $\lambda > 0$. Let $F^n$ and $F_\lambda^n$ denote the distributions of $S_n = Z_1 + ... + Z_n$ and $S_n^\lambda = Z_1^\lambda + ... + Z_n^\lambda$ respectively. Then,

$$\frac{dF^n}{dF_\lambda^n} = e^{-\lambda z} h_1(\lambda)...h_n(\lambda) \tag{11}$$

and

$$\mathbf{P}(S_n \ge ma) \ge \exp\{-m\lambda b + \sum_{k=1}^{n} \log h_k(\lambda) + \log(F_\lambda^n(mb) - F_\lambda^n(ma))\} \tag{12}$$

for $b > 0$ and $m > 0$.

**Proof.** Since

$$\begin{aligned}
F^2(z) &= \int_{-\infty}^{\infty} dF_1(x) \int_{-\infty}^{z-x} dF_2(y) \\
&= \int_{-\infty}^{\infty} e^{-\lambda x} h_1(\lambda) dF_1^\lambda(x) \int_{-\infty}^{z-x} e^{-\lambda y} h_2(\lambda) dF_2^\lambda(y) \\
&= h_1(\lambda)h_2(\lambda) \int\int_{x+y<z} e^{-\lambda(x+y)} dF_1^\lambda(x) dF_2^\lambda(y) \\
&= h_1(\lambda)h_2(\lambda) \int_{-\infty}^{z} e^{-\lambda u} dF_\lambda^2(u),
\end{aligned}$$

the result holds for $n = 1, 2$. By mathematical induction, we can similarly show that (11) holds for $n \geq 1$.

From (11), it follows that

$$\begin{aligned}
\mathbf{P}(S_n \geq ma) &= \int_{ma}^{\infty} e^{-\lambda z} h_1(\lambda)...h_n(\lambda) dF_\lambda^n \\
&\geq h_1(\lambda)...h_n(\lambda) \int_{ma}^{mb} e^{-\lambda z} dF_\lambda^n \\
&\geq h_1(\lambda)...h_n(\lambda) e^{-\lambda mb} \int_{ma}^{mb} dF_\lambda^n \\
&= h_1(\lambda)...h_n(\lambda) e^{-\lambda mb} [F_\lambda^n(mb) - F_\lambda^n(ma)] \\
&= \exp\{-m\lambda b + \sum_{k=1}^{n} \log h_k(\lambda) + \log(F_\lambda^n(mb) - F_\lambda^n(ma))\}.
\end{aligned}$$

This completes the proof.

Note that, by (10), the mean and the moment-generating function of $Z_k^\lambda$ can be, respectively, expressed as

$$\mathbf{E}_k^\lambda(Z_k^\lambda) = \frac{h_k'(\lambda)}{h_k(\lambda)}, \qquad h_k^\lambda(\theta) = \mathbf{E}_k^\lambda(e^{\theta Z_k^\lambda}) = \frac{h_k(\lambda+\theta)}{h_k(\lambda)}. \tag{13}$$

Let $\eta_j = \delta(A\xi_j - \frac{\delta}{2})$ and $h_\eta(\theta) = \mathbf{E}(e^{\theta\eta_1})$ denote the moment-generating functions of $\eta_1$. Let $\theta(y)$ satisfy $y = h_\eta'(\theta(y))/h_\eta(\theta(y))$.

**Lemma 2**. Suppose that the two conditions, (I) and (II), hold. Let $\mu < \delta/2$; that is, $\mathbf{E}(\eta_j) = \delta(\mu - \delta/2) < 0$. Then, there exists at most one $\theta^* \in (\theta(0), \bar{\theta})$ such that $h_\eta(\theta^*) = 1$; that is, $\log h(\delta A\theta^*) - \delta^2\theta^*/2 = 0$, where $\theta(0) > 0$ satisfies $0 = h_\eta'(\theta(0))/h_\eta(\theta(0))$. Moreover, $u = h_\eta'(\theta^*) > 0$, $\log h_\eta(\theta(x)) < 0$ for $x < u$ and $\log h_\eta(\theta(x)) > 0$ for $x > u$, and

$$\theta(\frac{1}{x}) - x \log h_\eta(\theta(\frac{1}{x})) \geq \theta^* \tag{14}$$

for $x > 0$ and

$$\theta(\frac{1}{x}) - x \log h_\eta(\theta(\frac{1}{x})) \geq 2\theta^* \tag{15}$$

for $x \geq b$, where the number $b$ is defined by

$$b = \inf\{x > 1/u : \theta(\frac{1}{x}) - x \log h_\eta(\theta(\frac{1}{x})) \geq 2\theta^*\}. \tag{16}$$

**Proof of Theorem 1.** (i). We first prove the upward inequality of (2). Without loss of generality, the number $x$ is considered to be the same as $[x]$ when $x$ is large, where the number $[x]$ denotes the smallest integer greater than or equal to $x$. Let $A_k = \sum_{j=1}^{k} a_{j-1}$. It follows that

$$\lim_{n\to\infty} A_n = A, \qquad \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} A_k = A, \qquad \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} |A_{n+k} - A_k| = 0, \tag{17}$$

and

$$\lim_{n\to\infty}\sum_{k=n+1}^{\infty}|A_{n+k}-A_k| \le \lim_{n\to\infty}\sum_{k=1}^{n}\sum_{j=n}^{\infty}|a_{k+j}| \le \lim_{n\to\infty} n\sum_{k=n+1}^{\infty}|a_k| = 0. \tag{18}$$

Here, the last limit follows from (1). For $n \le m$, we have

$$\sum_{k=m-n+1}^{m}\delta\Big(X_k-\frac{\delta}{2}\Big) = Y_{m,n}+Z_{m,n}+U_{m,n}$$

where

$$Y_{m,n}=\sum_{k=1}^{n}\delta\Big(A_k\xi_{m+1-k}-\frac{\delta}{2}\Big), \qquad Z_{m,n}=\delta\sum_{k=1}^{n}(A_{n+k}-A_k)\xi_{m+1-n-k}$$

and

$$U_{m,n}=\delta\sum_{k=n+1}^{\infty}(A_{n+k}-A_k)\xi_{m+1-n-k}.$$

Since $Y_{(2k-1)n,n}+Z_{(2k-1)n,n}, k\ge 1$, are mutually independent and identically distributed, and

$$\begin{aligned}
\mathbf{P}_\mu(T_C>m) &= \mathbf{P}_\mu\Big(\sum_{i=n-k+1}^{n}\delta\Big(X_i-\frac{\delta}{2}\Big)<c,\quad 1\le k\le n, 1\le n\le m\Big)\\
&= \mathbf{P}_\mu\Big(Y_{n,k}+Z_{n,k}+U_{n,k}<c,\quad 1\le k\le n, 1\le n\le m\Big)\\
&\le \mathbf{P}_\mu\Big(Y_{(2k-1)n,n}+Z_{(2k-1)n,n}+U_{(2k-1)n,n}<c,\quad 1\le k\le K\Big)
\end{aligned}$$

for large $m$, where $K$ is a natural number such that $K=\max\{k:(2k-1)n\le m\}$, it follows that

$$\begin{aligned}
&\mathbf{P}_\mu(T_C>m)\\
\le\ & \mathbf{P}_\mu\Big(Y_{(2k-1)n,n}+Z_{(2k-1)n,n}<c+\epsilon,\quad 1\le k\le K,\Big)+\mathbf{P}_\mu\Big(\max_{1\le k\le K}|U_{(2k-1)n,n}|\ge\epsilon\Big)\\
\le\ & [\mathbf{P}_\mu\Big(Y_{n,n}+Z_{n,n}<c+\epsilon\Big)]^K+\sum_{k=1}^{K}[\mathbf{P}_\mu\Big(U_{(2k-1)n,n}\ge\epsilon\Big)+\mathbf{P}_\mu\Big(-U_{(2k-1)n,n}\ge\epsilon\Big)]
\end{aligned}$$

for any small positive number, $\epsilon$.

Next, we estimate $\mathbf{P}_\mu\Big(Y_{n,n}+Z_{n,n}<c+\epsilon\Big)$. Let

$$\begin{aligned}
Y_j(n) &= \delta\Big(A_j\xi_{n+1-j}-\frac{\delta}{2}\Big),\quad 1\le j\le n\\
Z_j(n) &= (A_{n+j}-A_j)\xi_{1-j},\quad 1\le j\le n
\end{aligned}$$

Then, $Y_{n,n}+Z_{n,n}=\sum_{j=1}^{n}Y_j(n)+\sum_{j=1}^{n}Z_j(n)$. Let $F_j(x)$ and $G_j(x)$ denote, respectively, the distributions of $Y_j(n)$ and $Z_j(n)$, and let $h_j(\lambda)$ and $I_j(\lambda)$ be, respectively, the moment-generating functions of $Y_j(n)$ and $Z_j(n)$ for some $\lambda > \theta^* = \theta(u)$, where $\theta^*$ and $u$ are defined in Lemma 2. Let $Y_j^\lambda(n), Z_j^\lambda(n), 1 \le j \le n$, be independent variables with the distributions $F_j^\lambda(y)$ and $G_j^\lambda(y)$, respectively, where $F_j^\lambda(y)$, $G_j^\lambda(y)$ and the corresponding moment-generating functions $h_j^\lambda(\theta)$ and $I_j^\lambda(\theta)$ are defined in (15). Denote by $F^{2n}$ and $F_\lambda^{2n}$ the distributions of $S_{2n} = \sum_{j=1}^n Y_j(n) + \sum_{j=1}^n Z_j(n)$ and $S_{2n}^\lambda = \sum_{j=1}^n Y_j^\lambda(n) + \sum_{j=1}^n Z_j^\lambda(n)$ respectively.

Taking $n = (c+\epsilon)/u$ and $v > u$, it follows from Lemma 1 that

$$\mathbf{P}_\mu\Big(Y_{n,n} + Z_{n,n} \ge c + \epsilon\Big) = \mathbf{P}_\mu\Big(S_{2n} \ge un\Big)$$
$$\ge \exp\{-n\lambda v + \sum_{j=1}^n \log h_j(\lambda) + \sum_{j=1}^n \log I_j(\lambda) + \log(F_\lambda^{2n}(nv) - F_\lambda^{2n}(nu))\}. \tag{19}$$

We now prove

$$F_\lambda^{2n}(nv) - F_\lambda^{2n}(nu) \to 1 \tag{20}$$

or equality

$$\mathbf{P}\Big(\{S_{2n}^\lambda > nv\} \cup \{S_{2n}^\lambda < nu\}\Big) \to 0$$

for $u < h'_\eta(\lambda)/h_\eta(\lambda) < v$ as $n \to \infty$, where $h_\eta(\lambda)$ is the moment-generating function of $\delta(A\xi_1 - \delta/2)$.

It follows from (13) and (17) that

$$\lim_{j\to\infty} \log h_j(\lambda) = \lim_{j\to\infty} \log h(\delta A_j \lambda) - \frac{\delta^2\lambda}{2} = \log h_\eta(\lambda), \qquad \lim_{j\to\infty} \frac{h'_j(\lambda)}{h_j(\lambda)} = \frac{h'_\eta(\lambda)}{h_\eta(\lambda)}$$

and

$$\begin{aligned}\frac{(h_j^\lambda(0))'}{h_j^\lambda(0)} &= \lim_{\theta\searrow 0} \frac{1}{\theta} \log h_j^\lambda(\theta) = \lim_{\theta\searrow 0} \frac{1}{\theta} \log \frac{h_j(\lambda+\theta)}{h_j(\lambda)} \\ &= \lim_{\theta\searrow 0} \frac{1}{\theta} \log[1 + \frac{h_j(\lambda+\theta) - h_j(\lambda)}{h_j(\lambda)}] = \lim_{\theta\searrow 0} \frac{1}{\theta} \frac{h_j(\lambda+\theta) - h_j(\lambda)}{h_j(\lambda)} = \frac{h'_j(\lambda)}{h_j(\lambda)}.\end{aligned}$$

Similarly,

$$\lim_{j\to\infty} I_j(\lambda) = \lim_{j\to\infty} h(\delta(A_{n+j} - A_j)\lambda)) = h(0) = 1, \qquad \frac{(I_j^\lambda(0))'}{I_j^\lambda(0)} = \frac{I'_j(\lambda)}{I_j(\lambda)}$$

and

$$\lim_{j\to\infty} \frac{I'_j(\lambda)}{I_j(\lambda)} = \lim_{j\to\infty} \frac{\delta(A_{n+j} - A_j) h'(\delta(A_{n+j} - A_j)\lambda))}{h(\delta(A_{n+j} - A_j)\lambda))} = 0.$$

Hence

$$\log h_j^\lambda(\theta) = \frac{(h_j^\lambda(0))'}{h_j^\lambda(0)}\theta + o(\theta) = \frac{h'_j(\lambda)}{h_j(\lambda)}\theta + o(\theta), \qquad \log I_j^\lambda(\theta) = \frac{I'_j(\lambda)}{I_j(\lambda)}\theta + o(\theta). \tag{21}$$

By Chebyshev's inequality, we have

$$\begin{aligned}
\mathbf{P}\Big(S_{2n}^{\lambda} > nv\Big) &\leq \exp\{-n\theta\Big(v - \frac{1}{n\theta}\sum_{j=1}^{n}\log h_j^{\lambda}(\theta) + \frac{1}{n\theta}\sum_{j=1}^{n}\log I_j^{\lambda}(\theta)\Big)\} \\
&= \exp\{-n\theta\Big(v - \frac{1}{n}\sum_{j=1}^{n}\frac{h_j'(\lambda)}{h_j(\lambda)} + \frac{1}{n}\sum_{j=1}^{n}\frac{I_j'(\lambda)}{I_j(\lambda)} + o(1)\Big)\} \\
&= \exp\{-n\theta\Big(v - \frac{h_\eta'(\lambda)}{h_\eta(\lambda)} + o(1)\Big)\} \to 0
\end{aligned}$$

as $n \to \infty$ for small $\theta$. Similarly, we have

$$\begin{aligned}
\mathbf{P}\Big(-S_{2n}^{\lambda} > -nu\Big) &\leq \exp\{-n\theta\Big(-u - \frac{1}{n}\sum_{j=1}^{n}\log h_j^{\lambda}(-\theta) + \frac{1}{n}\sum_{j=1}^{n}\log I_j^{\lambda}(-\theta) + o(1)\Big)\} \\
&= \exp\{-n\theta\Big(-u + \frac{h_\eta'(\lambda)}{h_\eta(\lambda)} + o(1)\Big)\} \to 0
\end{aligned}$$

as $n \to \infty$ for small $\theta$. This proves (20).

Note that $\log h_j(\lambda) \to \log h_\eta(\lambda)$ and $\log I_j(\lambda) \to 0$ as $j \to \infty$. It follows from (19) that

$$\begin{aligned}
&\mathbf{P}_\mu\Big(Y_{n,n} + Z_{n,n} \geq c + \epsilon\Big) \\
\geq\ &\exp\{-n\Big(\lambda v - \frac{1}{n}\sum_{j=1}^{n}\log h_j(\lambda) - \frac{1}{n}\sum_{j=1}^{n}\log I_j(\lambda) - \frac{1}{n}\log(F_\lambda^{2n}(nv) - F_\lambda^{2n}(nu))\Big)\} \\
=\ &\exp\{-(c+\epsilon)\Big(\frac{1}{u}\lambda v - \frac{1}{u}\log h_\eta(\lambda) + o(1)\Big)\}
\end{aligned}$$

for large $c$, where $n = (c+\epsilon)/u$. Since $\lambda, v$ ($\lambda > \theta^*, v > h_\eta'(\lambda)/h_\eta(\lambda)$) are arbitrary and $h_\eta(\theta^*) = 1$, $h_\eta'(\theta^*)/h_\eta(\theta^*) = u$. Taking $\lambda \searrow \theta^*$ and $v \searrow h'(\lambda)/h(\lambda)$, we have

$$\mathbf{P}_\mu\Big(Y_{n,n} + Z_{n,n} \geq c + \epsilon\Big) \geq e^{-(c+\epsilon)(\theta^* + o(1))} \tag{22}$$

for large $c$.

Let $m = t(c+\epsilon)(2e^{(c+\epsilon)(\theta^*+o(1))} - 1)/u$ for $t > 0$ and large $c$. Then, $K = te^{(c+\epsilon)(\theta^*+o(1))}$. It follows from (22) that

$$[\mathbf{P}_\mu\Big(Y_{n,n} + Z_{n,n} < c + \epsilon\Big)]^K \leq \Big(1 - e^{-(c+\epsilon)(\theta^*+o(1))}\Big)^K \to e^{-t} \tag{23}$$

as $c \to \infty$. On the other hand, by Chebyshev's inequality we have

$$\begin{aligned}
\mathbf{P}_\mu\Big(U_{n,n} \geq \epsilon\Big) &\leq \exp\{-\theta\epsilon + \sum_{k=n+1}^{\infty}\log h(\delta(A_{n+k} - A_k)\theta)\} \\
&= \exp\{-\theta\epsilon + \delta\sum_{k=n+1}^{\infty}(1 + o(1))h'(0)(A_{n+k} - A_k)\theta\}
\end{aligned}$$

for large $n$. Note that $n=(c+\epsilon)/u$. Taking $\theta=(c+\epsilon)(\theta^*+a)/\epsilon$, where $a$ is a positive constant, by (18), we have

$$\begin{aligned}\mathbf{P}_\mu\Big(U_{n,n}\ge\epsilon\Big) &\le \exp\{-(c+\epsilon)\Big(\theta^*+a-\frac{\theta^*+a}{\epsilon}|h'(0)|\delta\sum_{k=n+1}^{\infty}(1+o(1))|(A_{n+k}-A_k)|\Big)\}\\ &= \exp\{-(c+\epsilon)(\theta^*+a-o(1))\}\end{aligned}$$

for large $c$. Since $U_{(2k-1)n,n}, k\ge 1$ are identically distributed, it follows that

$$\sum_{k=1}^{K}\mathbf{P}_\mu\Big(U_{(2k-1)n,n}\ge\epsilon\Big)=K\mathbf{P}_\mu\Big(U_{n,n}\ge\epsilon\Big)\le K\exp\{-(c+\epsilon)(\theta^*+a+o(1))\}\to 0 \quad (24)$$

as $c\to\infty$. Similarly, we can prove that

$$\sum_{k=1}^{K}\mathbf{P}_\mu\Big(-U_{(2k-1)n,n}\ge\epsilon\Big)\le K\exp\{-(c+\epsilon)(\theta^*+a-o(1))\}\to 0 \quad (25)$$

as $c\to\infty$. From (23) (24) and (25) it follows that $\mathbf{P}_\mu(T_C>m)\le e^{-t}(1+o(1))$ for large $c$. Thus, by the properties of exponential distribution, we have

$$\mathbf{E}_\mu(T_C)\le(1+o(1))(c+\epsilon)(2e^{(c+\epsilon)(\theta^*+o(1))}-1)/u$$

for large $c$. Since $\epsilon$ is arbitrary, the upward inequality of (2) is true.

To prove the downward inequality of (2), let

$$V_m = \{\sum_{i=n-k+1}^{n}\delta\Big(X_i-\frac{\delta}{2}\Big)<c,\ \ 1\le k\le\min\{n,bc-1\},\ 1\le n\le m\}$$

and

$$W_m=\{\sum_{i=n-k+1}^{n}\delta\Big(X_i-\frac{\delta}{2}\Big)<c,\ \ bc\le k\le n,\ bc\le n\le m\}$$

for large $c$, where $b$ is defined in (16). Then $\{T_C>m\}=V_mW_m$. Since $\{X_i\}$ is the linear combination of the i.i.d. $\{\xi_j\}$, it follows from Theorem 5.1 in Esary, Proschan and Walkup (1967) that $\mathbf{P}_\mu(T_C>m)\ge\mathbf{P}_\mu(W_m)\mathbf{P}_\mu(V_m)$,

$$\mathbf{P}_\mu(V_m) \ge \prod_{n=1}^{m}\prod_{k=1}^{\min\{n,bc\}}\mathbf{P}_\mu\Big(\sum_{i=n-k+1}^{n}\delta(X_i-\frac{\delta}{2})<c\Big)$$

and

$$\mathbf{P}_\mu(W_m) \ge \prod_{n=bc}^{m}\prod_{k=bc}^{n}\mathbf{P}_\mu\Big(\sum_{i=n-k+1}^{n}\delta(X_i-\frac{\delta}{2})<c\Big).$$

Note that $\sum_{i=n-k+1}^{n} \delta(X_i - \delta/2)$ can be rewritten as

$$\sum_{k=n-k+1}^{n} \delta\Big(X_k - \frac{\delta}{2}\Big) = Y_{n,k} + Z_{n,k,c} + U_{n,k,c}$$

where

$$Y_{n,k} = \sum_{j=1}^{k} \delta\Big(A_j \xi_{n+1-j} - \frac{\delta}{2}\Big), \qquad Z_{n,k,c} = \sum_{j=1}^{c} \delta(A_{k+j} - A_j)\xi_{n+1-k-j}$$

and

$$U_{n,k,c} = \sum_{j=c+1}^{\infty} \delta(A_{k+j} - A_j)\xi_{n+1-k-j}.$$

Let $f_k(\theta)$, $g_{k,c}(\theta)$ and $h_{k,c}(\theta)$ be the moment-generating functions of $Y_{n,k}$, $Z_{n,k,c}$ and $U_{n,k,c}$, respectively. It follows from (17) and (18) that

$$\lim_{k\to\infty} \frac{\log f_k(\theta)}{k} = \lim_{k\to\infty} \frac{1}{k} \sum_{j=1}^{k} [\log h(\delta A_j \theta) - \frac{\delta^2 \theta}{2}] = h_\eta(\theta), \tag{26}$$

$$\lim_{c\to\infty} \frac{\log g_{k,c}(\theta)}{c} = \lim_{c\to\infty} \frac{1}{c} \sum_{j=1}^{c} \log h(\delta(A_{k+j} - A_j)\theta) = 0 \quad (\text{ uniformly for } k \geq 1) \tag{27}$$

and

$$\begin{aligned} \lim_{c\to\infty} \log h_{k,c}(\theta) &= \lim_{c\to\infty} \sum_{j=c+1}^{\infty} \log h(\delta(A_{k+j} - A_j)\theta) \\ &= \lim_{c\to\infty} \delta \sum_{j=c+1}^{\infty} (1 + o(1))h'(0)(A_{k+j} - A_j)\theta \\ &\leq \lim_{c\to\infty} \delta \sum_{j=c+1}^{\infty} (1 + o(1))|h'(0)||A_{k+j} - A_j||\theta| \\ &\leq (1 + o(1)) \lim_{c\to\infty} \frac{k}{c} c\delta \sum_{j=c+1}^{\infty} |a_j| = 0 \end{aligned} \tag{28}$$

uniformly for $k \leq Mc$, where $M > 0$ is a constant.

For $k \geq 1$, let $x = k/c$. By Chebyshev's inequality, we have

$$\begin{aligned} &\mathbf{P}_\mu\Big( \sum_{i=n-k+1}^{n} \delta(X_i - \frac{\delta}{2}) \geq c\Big) \\ \leq\ & \exp\{-c\Big(\theta - \frac{1}{c}\log f_k(\theta) - \frac{1}{c}\log g_{k,c}(\theta) - \frac{1}{c}\log h_{k,c}(\theta)\Big)\}. \end{aligned}$$

If $x = k/c \to 0$ as $c \to \infty$, taking $\theta \geq \theta^*$ it follows from (26), (27) and (28) that

$$\begin{aligned}
&\mathbf{P}_\mu\Big(\sum_{i=n-k+1}^{n} \delta(X_i - \frac{\delta}{2}) \geq c\Big) \\
\leq\ & \exp\{-c\Big(\theta - \frac{1}{c}\log f_k(\theta) - \frac{1}{c}\log g_{k,c}(\theta) - \frac{1}{c}\log h_{k,c}(\theta)\Big)\} \leq e^{-c(\theta^* - o(1))}
\end{aligned}$$

for large $c$. If $b > x = k/c \geq a > 0$, where $a$ is a small positive constant, taking $\theta(1/x))$ such that $1/x = h'(\theta(1/x))/h(\theta(1/x))$, we have

$$\begin{aligned}
&\mathbf{P}_\mu\Big(\sum_{i=n-k+1}^{n} \delta(X_i - \frac{\delta}{2}) \geq c\Big) = \mathbf{P}_\mu\Big(\sum_{i=n-k+1}^{n} \delta(X_i - \frac{\delta}{2}) \geq k/x\Big) \\
\leq\ & \exp\{-k\Big(\theta(1/x)/x - \frac{1}{k}\log f_k(\theta(1/x)) - \frac{1}{k}\log g_{k,c}(\theta(1/x)) - \frac{1}{k}\log h_{k,c}(\theta(1/x))\Big)\} \\
=\ & \exp\{-c\Big(\theta(1/x) - x\frac{1}{k}\log f_k(\theta(1/x)) - \frac{1}{c}\log g_{k,c}(\theta(1/x)) - \frac{1}{c}\log h_{k,c}(\theta(1/x))\Big)\} \\
=\ & \exp\{-c\Big(\theta(1/x) - x\log h_\eta(\theta(1/x)) + o(1)\Big)\} \leq e^{-c(\theta^* + o(1))}
\end{aligned}$$

for large $c$, where the last equality follows from (14). Thus, taking $m = te^{c(\theta^*+o(1))}/bc$ for $t > 0$, we have

$$\begin{aligned}
\mathbf{P}_\mu(V_m) \ &\geq \prod_{n=1}^{m}\prod_{k=1}^{\min\{n,bc\}} \mathbf{P}_\mu\Big(\sum_{i=n-k+1}^{n} \delta(X_i - \frac{\delta}{2}) < c\Big) \\
&= \prod_{n=1}^{m}\prod_{k=1}^{\min\{n,bc\}} [1 - \mathbf{P}_\mu\Big(\sum_{i=n-k+1}^{n} \delta(X_i - \frac{\delta}{2}) \geq c\Big)] \\
&\geq [1 - e^{-c(\theta^*+o(1))}]^{bcm} \to e^{-t},
\end{aligned}$$

as $c \to +\infty$.

Similarly, for $x \geq b$, that is, $k \geq bc$, we have

$$\begin{aligned}
\mathbf{P}_\mu(W_m) \ &\geq \prod_{n=bc}^{m}\prod_{k=bc}^{n} \mathbf{P}_\mu\Big(\sum_{i=n-k+1}^{n} \delta(X_i - \frac{\delta}{2}) < c\Big) \\
&\geq \prod_{n=bc}^{m}\prod_{k=bc}^{n} \Big(1 - \exp\{-c[\theta(1/x) - x\log h_\eta(\theta(1/x)) + o(1)]\}\Big) \\
&\geq [1 - e^{-2c(\theta^*+o(1))}]^{(m-bc)^2} \to 1,
\end{aligned}$$

as $c \to +\infty$, where the last equality follows from (15). Hence, $P(T > m) \geq P(U_m)P(V_m) \to e^{-t}$ as $c \to +\infty$. This implies the downward inequality of (2).

(ii) Let $\hat{\mu} = \delta(\mu - \delta/2)$. Then

$$
\begin{aligned}
\{T_C > m\} &= \Big\{ \sum_{i=n-k+1}^{n} \delta\Big(X_i - \frac{\delta}{2}\Big) < c, \quad 1 \le k \le n, 1 \le n \le m \Big\} \\
&\subset \Big\{ \sum_{i=1}^{m} \delta\Big(X_i - \frac{\delta}{2}\Big) < c \Big\} = \Big\{ \sum_{i=1}^{m} \delta(X_i - \mu) < c - m\hat{\mu} \Big\} \\
&= \Big\{ Y_{m,m}(\mu) + Z_{m,m} + U_{m,m} < c - m\hat{\mu} \Big\}
\end{aligned}
$$

where

$$
Y_{m,m}(\mu) = \sum_{i=1}^{m} \delta(A_i \xi_{m+1-i} - \mu).
$$

Let $f_{Y,m}(\theta)$, $f_{Z,m}(\theta)$ and $f_{U,m}(\theta)$ denote the moment-generating functions of $Y_{m,m}(\mu)$, $Z_{m,m}$ and $U_{m,m}$, respectively. Note that $\hat{\mu} > 0$. Let $N = c/\hat{\mu} + 2\sqrt{c}\log c/(\hat{\mu})^{3/2}$. We have

$$
\begin{aligned}
\mathbf{E}_\mu(T_C) &= \sum_{m=1}^{N} \mathbf{P}_\mu(T_C > m) + \sum_{m=N+1}^{\infty} \mathbf{P}_\mu(T_C > m) \\
&\le N + \sum_{m=N+1}^{\infty} \mathbf{P}_\mu\Big( Y_{m,m}(\mu) + Z_{m,m} + U_{m,m} < c - m\hat{\mu} \Big) \\
&= N + \sum_{k=1}^{\infty} \mathbf{P}_\mu\Big( Y_{N+k,N+k}(\mu) + Z_{N+k,N+k} + U_{N+k,N+k} < -\hat{\mu}[\frac{2\sqrt{c}\log c}{(\hat{\mu})^{3/2}} + k] \Big) \\
&\le N + \sum_{k=1}^{\infty} \exp\{-\theta\hat{\mu}[\frac{2\sqrt{c}\log c}{(\hat{\mu})^{3/2}} + k] + \log f_{Y,N+k}(-\theta) + \log f_{Z,N+k}(-\theta) + \log f_{U,N+k}(-\theta)\},
\end{aligned}
$$

where the last equality follows from Chebyshev's inequality. Note that $\mu = \bar{\xi}A$,

$$
\begin{aligned}
&\frac{d}{d\theta} \log f_{Y,N+k}(-\theta)_{|\theta=0} = -\mathbf{E}(Y_{N+k,N+k}(\mu)) = -\delta\bar{\xi} \sum_{j=1}^{N+k} (A_j - A) = \delta\bar{\xi} \sum_{j=1}^{N+k} k a_k \\
&\frac{d^2}{d^2\theta} \log f_{Y,N+k}(-\theta)_{|\theta=0} = \mathbf{Var}(Y_{N+k,N+k}(\mu)) = (\delta\sigma)^2 \sum_{j=1}^{N+k} A_j^2
\end{aligned}
$$

and

$$
\log f_{Y,N+k}(-\theta) = \theta\delta\bar{\xi} \sum_{j=1}^{N+k} k a_k + \frac{\theta^2}{2} (\delta\sigma)^2 \sum_{j=1}^{N+k} A_j^2 + o(\theta^2).
$$

Taking $\theta = (\sqrt{N+k})^{-1}$, by Condition (III) and (17), we have

$$
\log f_{Y,N+k}(-\frac{1}{\sqrt{N+k}}) = \delta\bar{\xi} \frac{1}{\sqrt{N+k}} \sum_{j=1}^{N+k} k a_k + \frac{(\delta\sigma)^2}{2} \frac{1}{N+k} \sum_{j=1}^{N+k} A_j^2 \to \frac{(\delta\sigma A)^2}{2}
$$

uniformly for $k \geq 1$ as $c \to \infty$. Similarly, by (18) we can show that both $\log f_{Z,N+k}(-(\sqrt{N+k})^{-1})$ and $\log f_{U,N+k}(-(\sqrt{N+k})^{-1})$ go to 0 uniformly for $k \geq 1$ as $c \to \infty$. Thus, by taking a positive constant $\alpha$ such that $\alpha\hat{\mu} < 1$, it follows that

$$\begin{aligned}
\mathbf{E}_\mu(T_C) &\leq N + e^{(\delta\sigma A)^2/2}\sum_{k=1}^{\infty}\exp\{-\frac{\hat{\mu}}{\sqrt{N+k}}[\frac{2\sqrt{c}\log c}{(\hat{\mu})^{3/2}} + k] + o(1)\} \\
&= N + e^{(\delta\sigma A)^2/2}\sum_{k=1}^{\alpha c}\exp\{-\frac{2\sqrt{c}\log c + (\hat{\mu})^{3/2}k}{\sqrt{c + 2\sqrt{c}\log c/\sqrt{\hat{\mu}} + \hat{\mu}k}} + o(1)\} \\
&\quad + e^{(\delta\sigma A)^2/2}\sum_{k=\alpha c+1}^{\infty}\exp\{-\frac{2\sqrt{c}\log c + (\hat{\mu})^{3/2}k}{\sqrt{c + 2\sqrt{c}\log c/\sqrt{\hat{\mu}} + \hat{\mu}k}} + o(1)\} \\
&\leq N + \frac{1}{\hat{\mu}c^{\sqrt{2}-1}}e^{(\delta\sigma A)^2/2} + e^{(\delta\sigma A)^2/2}\sum_{k=\alpha c+1}^{\infty}\exp\{-\frac{(\hat{\mu})^{3/2}\sqrt{k}}{\sqrt{(\alpha\hat{\mu})^{-1}+2}} + o(1)\} \\
&\leq N + \frac{e^{(\delta\sigma A)^2/2}}{\hat{\mu}c^{\sqrt{2}-1}}(1 + o(1)) \qquad (29)
\end{aligned}$$

for large $c$. This proves the upward inequality of (4).

To prove the downward inequality of (4), let $M = c/\hat{\mu} - 3\sqrt{c}\log c/(\hat{\mu})^{3/2}$. Then,

$$\begin{aligned}
\mathbf{E}_\mu(T_C) &\geq \sum_{m=1}^{M}\mathbf{P}_\mu(T_C > m) \\
&\geq \sum_{m=1}^{M}\prod_{n=1}^{m}\prod_{k=1}^{n}\mathbf{P}_\mu\Big(\sum_{i=n-k+1}^{n}\delta(X_i - \frac{\delta}{2}) < c\Big) \\
&= \sum_{m=1}^{M}\prod_{n=1}^{m}\prod_{k=1}^{n}\mathbf{P}_\mu\Big(Y_{n,k}(\mu) + Z_{n,k} + U_{n,k} < c - k\hat{\mu}\Big) \\
&\geq \sum_{m=1}^{M}\Big[\mathbf{P}_\mu\Big(Y_{M,M}(\mu) + Z_{M,M} + U_{M,M} < c - M\hat{\mu}\Big)\Big]^{mM} \\
&= \sum_{m=1}^{M}\Big[1 - \mathbf{P}_\mu\Big(Y_{M,M}(\mu) + Z_{M,M} + U_{M,M} \geq \frac{3\sqrt{c}\log c}{(\hat{\mu})^{1/2}}\Big)\Big]^{mM}.
\end{aligned}$$

As in (29), we can similarly check that

$$\begin{aligned}
&\mathbf{P}_\mu\Big(Y_{M,M}(\mu) + Z_{M,M} + U_{M,M} \geq \frac{3\sqrt{c}\log c}{(\hat{\mu})^{1/2}}\Big) \\
&\leq e^{(\delta\sigma A)^2/2}\exp\{-\frac{\hat{\mu}}{\sqrt{M}}\frac{3\sqrt{c}\log c}{(\hat{\mu})^{3/2}} + o(1)\} \\
&= e^{(\delta\sigma A)^2/2}\exp\{-3\log c + o(1)\} = \frac{e^{(\delta\sigma A)^2/2}}{c^3}(1 + o(1))
\end{aligned}$$

for large $c$. Note that if $x/c^3 \to 0$ for $x > 0$ as $c \to \infty$, then

$$1 - \Big(1 - \frac{e^{(\delta\sigma A)^2/2}}{c^3}\Big)^x = \frac{xe^{(\delta\sigma A)^2/2}}{c^3}(1 + o(1))$$

as $c \to \infty$. Thus, taking $x = M$ or $x = M^2$, we have

$$\begin{aligned}
\mathbf{E}_\mu(T_C) &\geq \sum_{m=1}^{M} \Big[1 - \frac{e^{(\delta\sigma A)^2/2}}{c^3}\Big]^{mM} \\
&= \frac{[1 - \frac{e^{(\delta\sigma A)^2/2}}{c^3}]^M}{1 - [1 - \frac{e^{(\delta\sigma A)^2/2}}{c^3}\Big]^M} \Big(1 - [1 - \frac{e^{(\delta\sigma A)^2/2}}{c^3}\Big]^{M^2}\Big) \to M
\end{aligned}$$

as $c \to \infty$. That is, the downward inequality of (4) holds. This completes the proof of Theorem 1.

# 3 Proof of Theorem 2

We will first prove a lemma before proving Theorem 2. In the following proofs we shall use $c$ simply to replace $\tilde{c}$ which is the control limit of EWMA chart.

**Lemma 3.** Let $Y_n = \sum_{k=0}^{n-1} C_k(r)\xi_{n-k}$ and $\zeta_n = A\sum_{k=0}^{n-1}(1-r)^k\xi_{n-k}$, where $C_k(r) = \sum_{j=0}^{k} a_{k-j}(1-r)^j, 0 < r \leq 1$. Let $h_{Y,n}(\theta)$ and $h_{\zeta,n}(\theta)$ denote the moment-generating functions of $Y_n$ and $\zeta_n$, respectively. Let $n = (ar)^{-1}$, where $a$ is a positive number. Then

$$\lim_{r\to 0} r \log h_{Y,n}(\theta) = \lim_{r\to 0} r \log h_{\zeta,n}(\theta) = \log h_{\zeta,a}(\theta), \tag{30}$$

where

$$\log h_{\zeta,a}(\theta) = \sum_{m=1}^{\infty} (1 - e^{-m/a}) \frac{A^m}{m} \frac{(\log h(0))^{(m)}}{m!} \theta^m \tag{31}$$

$(\log h(0))^{(m)}$ denotes the $m$th derivative of the function $\log h(\theta)$ at $\theta = 0$. Moreover, if $a = a(r) \leq C(-\log r)^{-1}$ for some constant $C$ and any $0 < r < 1$, then

$$\lim_{r\to 0} r \log h_{Y,n}(\theta) = \lim_{r\to 0} r \log h_{\zeta,n}(\theta) = \log h_{\zeta,0}(\theta) = \int_0^\theta \frac{\log h(Ax)}{x} dx \tag{32}$$

and $c\theta - \log h_{\zeta,0}(\theta)$ attains its maximal value at $\theta_c$ for $\mu < c$, where $\theta_c$ is the unique positive root of the equation $c\theta - \log h(A\theta) = 0$ on $\theta > 0$.

**Proof.** Let $\log h_\zeta(\theta) = \log h_{\zeta,0}(\theta)$. Since

$$\begin{aligned}
\log h_{\zeta,n}(\theta) &= \sum_{k=0}^{n-1} \log h(A(1-r)^k\theta) = \sum_{k=0}^{n-1}\sum_{m=1}^{\infty} [(1-r)^k A]^m \frac{(\log h(0))^{(m)}}{m!}\theta^m \\
&= \sum_{m=1}^{\infty}\sum_{k=0}^{n-1} [(1-r)^k A]^m \frac{(\log h(0))^{(m)}}{m!}\theta^m
\end{aligned}$$

and

$$\lim_{r\to 0} r \sum_{k=0}^{n-1} ((1-r)^k A)^m = (1 - e^{-m/a}) \frac{A^m}{m},$$

it follows that the second equality of (30) holds for $\log h_{\zeta,n}$. Thus, the first equality of (30) is true as long as we prove that

$$\lim_{r\to 0} r\sum_{k=0}^{n-1} |(C_k(r))^m - ((1-r)^k A)^m| = 0 \tag{33}$$

for $m \geq 1$, since

$$\begin{aligned}
\log h_{Y,n}(\theta) &= \sum_{k=0}^{n-1} \log h(C_k(r)\theta) = \sum_{k=0}^{n-1}\sum_{m=1}^{\infty} (C_k(r))^m \frac{(\log h(0))^{(m)}}{m!}\theta^m \\
&= \sum_{m=1}^{\infty}\sum_{k=0}^{n-1} (C_k(r))^m \frac{(\log h(0))^{(m)}}{m!}\theta^m.
\end{aligned}$$

We first prove that

$$r\sum_{k=0}^{n-1} |C_k(r) - (1-r)^k A| \to 0 \tag{34}$$

as $r \to 0$.

Let $R(p) = (\log r^{-1})^p$ for $p \geq 1$. Taking a small $r$ such that $n > 1/(rR(2p))$, we have

$$\begin{aligned}
\sum_{k=0}^{n-1} C_k(r) &= \sum_{k=0}^{1/(rR(2p))-1} C_k(r) + \sum_{k=1/(rR(2p))}^{n-1} C_k(r) \\
&= \sum_{k=0}^{1/(rR(2p))-1} a_k \sum_{j=0}^{1/(rR(2p))-1-k} (1-r)^j + \sum_{k=1/(rR(2p))}^{n-1} C_k(r).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
& r|\sum_{k=0}^{1/(rR(2p))-1} a_k \sum_{j=0}^{1/(rR(2p))-1-k} (1-r)^j| \\
&\leq \sum_{k=0}^{R(p)} |a_k|[1-(1-r)^{1/(rR(2p))-k}] + \sum_{k=R(p)+1}^{1/(rR(2p))-1} |a_k| \\
&\leq ||A||R(p)[1-e^{-1/R(2p)}] + \frac{1}{R(p)}R(p)\sum_{k=R(p)+1}^{\infty} |a_k| \leq \frac{2||A||}{R(p)} \to 0
\end{aligned}$$

as $r \to 0$. Similarly,

$$r\sum_{k=0}^{1/(rR(2p)-1} (1-r)^k|A| \leq |A|[1-e^{-1/R(2p)}] \leq \frac{|A|}{R(2p)} \to 0$$

Thus,

$$
\begin{aligned}
& r\sum_{k=0}^{n-1}|C_k(r)-(1-r)^kA| \\
\leq\ & \frac{2||A||+|A|}{R(p)}+r\sum_{k=1/(rR(2p))}^{n-1}|C_k(r)-(1-r)^kA| \\
\leq\ & \frac{2||A||+|A|}{R(p)}+r\sum_{k=1/(rR(2p))}^{n-1}\Big((1-r)^k\sum_{j=1}^{R(p)}|a_j|[(1-r)^{-j}-1]+\sum_{j=R(p)+1}^{k}|a_j|[(1-r)^{k-j}-(1-r)^k]\Big) \\
\leq\ & \frac{2||A||+|A|}{R(p)}+r\sum_{k=1/(rR(2p))}^{n-1}(1-r)^k rR(2p)||A||(1-r)^{-R(p)}+\sum_{j=R(p)+1}^{\infty}|a_k| \\
\leq\ & \frac{2||A||+|A|}{R(p)}+2rR(2p)||A||+\frac{1}{R(p)}R(p)\sum_{k=R(p)+1}^{\infty}|a_k| \\
\leq\ & \frac{5||A||+|A|+1}{R(p)}\to 0 \qquad (35)
\end{aligned}
$$

as $r\to 0$ for $n>1/(rR(2p)$. This implies (34). Furthermore, (33) follows from

$$
\begin{aligned}
r\sum_{k=0}^{n-1}|(C_k(r))^m-((1-r)^kA)^m| &= r\sum_{k=0}^{n-1}|(C_k(r)-(1-r)^kA)(\sum_{j=0}^{m-1}(C_k(r))^{m-1-j}[(1-r)^kA]^j)| \\
&\leq m||A||^{m-1}r\sum_{k=0}^{n-1}|(C_k(r)-(1-r)^kA|\to 0
\end{aligned}
$$

as $r\to 0$ for each $m>1$.

Similarly, it can be checked that

$$
\lim_{r\to 0} r\Big(\log h_{Y,n}(\theta)\Big)'=\lim_{r\to 0} r\Big(\log h_{\zeta,n}(\theta)\Big)'=\Big(\log h_{\zeta,a}(\theta)\Big)'. \qquad (36)
$$

Moreover, by (30), (31) and (36) we have

$$
h'_\zeta(\theta)/h_\zeta(\theta)=\frac{1}{\theta}\log h(A\theta).
$$

This means (32). Note that $c-h'_\zeta(0)/h_\zeta(0)=c-\mu>0$ and $h'_\zeta(\theta)/h_\zeta(\theta)$ is strictly increasing since $h'(\theta)/h(\theta)$ is strictly increasing (see Durrett (2005, P.70-73)). Then, there is a unique positive number, $\theta_c$, such that $c-h'_\zeta(\theta_c)/h_\zeta(\theta_c)=0$, or equality, $c\theta_c-\log h(A\theta_c)=0$, and therefore, $c\theta-\log h_\zeta(\theta)$ attains its maximal value at $\theta_c$. This completes the proofs.

**Proof of Theorem 2.** (i). Let $D_{n,k}(r) = \sum_{j=0}^{n-1} a_{n+k-j}(1-r)^j$. The statistics $E_m(r)$ of the EWMA can be rewritten as

$$E_m(r) = rX_m + (1-r)E_{m-1}(r) = r\sum_{k=0}^{m-1}(1-r)^k X_{m-k} = r[Y_{m,n} + Z_{m,n} + R_{m,n}]$$

where

$$Y_{m,n} = \sum_{k=0}^{n-1} C_k(r)\xi_{m-k}, \qquad Z_{m,n} = \sum_{k=0}^{\infty} D_{n,k}(r)\xi_{m-n-k}$$

and

$$R_{m,n} = (1-r)^n \sum_{k=0}^{m-n-1}(1-r)^k X_{m-n-k}, \qquad R_{m,m} = 0$$

for $m \geq n$. Let $n = 3r^{-1}\log r^{-1}$ for small $r$. Note that $Y_{kn,n}, k \geq 1$, are i.i.d. random variables and $Z_{kn,n}, k \geq 1$, are identically distributed. For large $m$ and any small $\epsilon > 0$, we have

$$\begin{aligned}
& \mathbf{P}_\mu(T_E > m) \leq \mathbf{P}_\mu\Big(Y_{kn,n} + Z_{kn,n} + R_{kn,n} < \frac{c}{r}, \ \ 1 \leq k \leq m/n,\Big) \\
\leq \ & \mathbf{P}_\mu\Big(Y_{kn,n} < \frac{c}{r} + \epsilon, \ \ 1 \leq k \leq m/n,\Big) + \mathbf{P}_\mu\Big(\max_{1\leq k\leq m/n}|Z_{kn,n} + R_{kn,n}| \geq \epsilon\Big) \\
\leq \ & [\mathbf{P}_\mu\Big(Y_{n,n} < \frac{c}{r} + \epsilon\Big)]^{m/n} + m/n\mathbf{P}_\mu\Big(|Z_{n,n}| \geq \epsilon/2\Big) + \sum_{k=1}^{m/n}\mathbf{P}_\mu\Big(|R_{kn,n}| \geq \epsilon/2\Big). \qquad (37)
\end{aligned}$$

Next, we prove that

$$\mathbf{P}_\mu\Big(Y_{n,n} < \frac{c}{r} + \epsilon\Big) \leq 1 - \exp\{-\frac{1}{r}(c\theta_c - \log h_\zeta(\theta_c)) + o(1)\} \qquad (38)$$

for small $r > 0$.

Let $F_j(x)$ denote the distributions of $C_j(r)\xi_{j+1}, 0 \leq j \leq n-1$. Let $Y_j^\lambda, 0 \leq j \leq n-1$, be independent variables with the distributions $F_j^\lambda(y)$ for some $\lambda > \theta_c + r\epsilon$ and the moment-generating functions $h_j^\lambda(\theta)$ defined in (10). Denote by $F^n$ and $F_\lambda^n$ the distributions of $S_n = \sum_{j=0}^{n-1} C_j(r)\xi_{j+1}$ and $S_n^\lambda = \sum_{j=0}^{n-1} Y_j^\lambda$, respectively.

Taking $v > c + r\epsilon$ and $\tilde{n} = 1/r$, it follows from Lemma 1 that

$$\begin{aligned}
& \mathbf{P}_\mu\Big(Y_{n,n} \geq \frac{c}{r} + \epsilon\Big) \geq \exp\{-\tilde{n}\lambda v + \sum_{j=0}^{n-1}\log h(C_j(r)\lambda) + \log(F_\lambda^n(\tilde{n}v) - F_\lambda^n(\tilde{n}(c + r\epsilon))\} \\
= \ & \exp\{-\frac{1}{r}\Big(\lambda v + r\sum_{j=0}^{n-1}\log h(C_j(r)\lambda) + r\log(F_\lambda^n(\tilde{n}v) - F_\lambda^n(\tilde{n}(c + r\epsilon))\} \qquad (39)
\end{aligned}$$

By (21), we have

$$\log h_j^\lambda(\theta) = C_j(r)\frac{h'(C_j(r)\lambda)}{h(C_j(r)\lambda)}\theta + o(\theta)$$

and

$$r\sum_{j=0}^{n-1} C_j(r)\frac{h'(C_j(r)\lambda)}{h(C_j(r)\lambda)} \to \frac{h'_\zeta(\lambda)}{h_\zeta(\lambda)}$$

as $r \to 0$. Hence, as in (20), we can show that

$$\mathbf{P}\Big(\{S_n^\lambda > \tilde{n}v\} \cup \{S_n^\lambda < \tilde{n}(c+r\epsilon)\}\Big) \to 0;$$

that is,

$$F_\lambda^n(\tilde{n}v) - F_\lambda^n(\tilde{n}(c+r\epsilon)) \to 1$$

as $r \to 0$ for $\theta_c < h'_\zeta(\lambda)/h_\zeta(\lambda) < v$.

It follows from (39) and Lemma 3 that

$$\mathbf{P}_\mu\Big(Y_{n,n} \geq \frac{c}{r} + \epsilon\Big) \geq \exp\{-\frac{1}{r}\Big(\lambda v - \log h_\zeta(\lambda) + o(1)\Big)\}. \tag{40}$$

Moreover, $\lambda, v$ $(\lambda > \theta_c, v > h'_\zeta(\lambda)/h_\zeta(\lambda))$ are arbitrary and $h'_\zeta(\theta_c)/h_\zeta(\theta_c) = c$. Let $\lambda \searrow \theta_c$ and $v \searrow h'_\zeta(\lambda)/h_\zeta(\lambda)$ in (40), we obtain (38).

Let $m = 3tr^{-1}\log(1/r)\exp\{\frac{1}{r}(c\theta_c - \log h_\zeta(\theta_c))\}$ for $t > 0$. By (38), we have

$$[\mathbf{P}_\mu\Big(Y_{n,n} < \frac{c}{r} + \epsilon\Big)]^{m/n} \leq \Big(1 - \exp\{-\frac{1}{r}(c\theta_c - \log h_\zeta(\theta_c)) + o(1)\}\Big)^{m/n} \to e^{-t} \tag{41}$$

as $r \to 0$.

Note that

$$\begin{aligned}
\frac{1}{r}\sum_{k=0}^{\infty}|D_{n,k}(r)| &\leq \frac{1}{r}\sum_{k=0}^{\infty}\sum_{j=0}^{n-1}|a_{n+k-j}|(1-r)^j \\
&= \frac{1}{r}\sum_{j=0}^{1/r}(||A||-||A_j||)(1-r)^{n-1-j} + \frac{1}{r}\sum_{j=1/r+1}^{n-1}(||A||-||A_j||)(1-r)^{n-1-j} \\
&\leq ||A||\frac{1}{r^2}(1-r)^{3r^{-1}\log r^{-1}} + \frac{1}{r}\sum_{j=1/r+1}^{\infty}|a_j| \to 0
\end{aligned} \tag{42}$$

and

$$\frac{1}{r}(1-r)^n\sum_{j=0}^{kn-n-1}(1-r)^j \to 0 \tag{43}$$

as $r \to 0$. As in (24) and (25), it can be shown that

$$m/n\mathbf{P}_\mu\Big(|Z_{n,n}| \geq \epsilon/2\Big) \to 0, \qquad m/n\mathbf{P}_\mu\Big(|R_{m,n}| \geq \epsilon/2\Big) \to 0 \tag{44}$$

as $r \to 0$. Thus, by (37), (41) and (44) we have

$$\mathbf{P}_\mu(T_E > m) \le e^{-t} \tag{45}$$

as $r \to 0$ for $m = 3tr^{-1}\log(1/r)\exp\{\frac{1}{r}(c\theta_c - \log h_\zeta(\theta_c))\}$. This implies the upward inequality of (5).

Let $n = r^{-1}\log r^{-1}$ and $m = t\exp\{\frac{1}{r}(c\theta_c - \log h_\zeta(\theta_c))\}$ for $t > 0$. Using Theorem 5.1 in Esary, Proschan and Walkup (1967), we have

$$\begin{aligned}\mathbf{P}_\mu(T_E > m) &\ge \prod_{k=1}^{m}\mathbf{P}_\mu(E_k(r) < c)\\ &= \prod_{k=1}^{n-1}\mathbf{P}_\mu\Big(Y_{k,k} + Z_{k,k} < c/r\Big)\prod_{k=n}^{m}\mathbf{P}_\mu\Big(Y_{k,n} + Z_{k,n} + R_{k,n} < c/r\Big).\end{aligned}$$

Furthermore, by Chebyshev's inequality and as in (38) and (44), it follows that

$$\mathbf{P}_\mu\Big(Y_{k,n} + Z_{k,n} + R_{k,n} \ge c/r\Big) \le \exp\{-\frac{1}{r}\Big(c\theta_c - \log h_\zeta(\theta_c) + o(1)\Big)\}$$

for $k \ge n$ and small $r$. Hence

$$\prod_{k=n}^{m}\mathbf{P}_\mu\Big(Y_{k,n} + Z_{k,n} + R_{k,n} < c/r\Big) \ge \Big(1 - \exp\{-\frac{1}{r}\Big(c\theta_c - \log h_\zeta(\theta_c) + o(1)\Big)\}\Big)^{m-n} \to e^{-t}.$$

as $r \to 0$.

On the other hand, by Lemma 3, we know that $c\theta - \log h_\zeta(\theta)$ attains its maximal value at $\theta_c$ since $h'_\zeta(\theta)/h_\zeta(\theta)$ is strictly increasing and $c - h'_\zeta(0)/h_\zeta(0) = c - \mu > 0$. As in (38) and (44), we can similarly obtain

$$\mathbf{P}_\mu\Big(Y_{k,k} + Z_{k,k} < c/r\Big) \ge \Big(1 - \exp\{-\frac{1}{r}\Big(c\theta_c - \log h_\zeta(\theta_c) + o(1)\Big)\}\Big), \tag{46}$$

and therefore

$$\prod_{k=1}^{n-1}\mathbf{P}_\mu\Big(Y_{k,k} + Z_{k,k} < c/r\Big) \ge \Big(1 - \exp\{-\frac{1}{r}\Big(c\theta_c - \log h_\zeta(\theta_c) + o(1)\Big)\}\Big)^{n-1} \to 1$$

as $r \to 0$. Thus, $\mathbf{P}_\mu(T_E > m) \ge e^{-t}$ for $m = t\exp\{\frac{1}{r}(c\theta_c - \log h_\zeta(\theta_c))\}$ as $r \to 0$. This proves the downward inequality of (5).

(ii). Let $Y_{m,m}(r) = \sum_{j=0}^{m-1}[C_j(r)\xi_{m-k} - \mu(1-r)^j]$. Take $N = r^{-1}\log(1 - c/\mu)^{-1}$ and $m = Nt$ for $t > 1$. It follows that

$$\mu\sum_{j=0}^{m-1}(1-r)^j = \frac{\mu}{r}[1 - (1-r)^{Nt}] \ge \frac{\mu}{r}[1 - (1 - \frac{c}{\mu})^t]$$

for small $r$. Then,

$$\begin{aligned}
\mathbf{P}_\mu(T_E > m) &\leq \mathbf{P}_\mu\Big(Y_{m,m}(r) + Z_{m,m} < \frac{c}{r} - \mu \sum_{j=0}^{m-1} (1-r)^j\Big) \\
&\leq \mathbf{P}_\mu\Big(Y_{m,m}(r) + Z_{m,m} < \frac{c}{r} - \frac{\mu}{r}[1 - (1 - \frac{c}{\mu})^t]\Big) \\
&= \mathbf{P}_\mu\Big( - Y_{m,m}(r) - Z_{m,m} > \frac{\mu}{r}[1 - \frac{c}{\mu} - (1 - \frac{c}{\mu})^t]\Big) \\
&\leq \exp\{-\theta\frac{\mu}{r}[1 - \frac{c}{\mu} - (1 - \frac{c}{\mu})^t] + \log f_{Y,m}(-\theta) + \log f_{Z,m}(-\theta)\},
\end{aligned}$$

where

$$\begin{aligned}
\log f_{Y,m}(-\theta) &= \sum_{j=0}^{m-1} [\log h(-C_j(r)\theta) + \theta\mu(1-r)^j], \\
\log f_{Z,m}(-\theta) &= \sum_{j=0}^{\infty} \log h(-D_{m,j}(r)\theta).
\end{aligned}$$

Let $d = t(\mu[1 - c/\mu - (1 - c/\mu)^t])^{-1}$. Taking $\theta = rd$, it follows from (33) and (42) that

$$\begin{aligned}
&\sum_{j=0}^{m-1} [\log h(-C_j(r)rd) + rd\mu(1-r)^j] = (1 + o(1))rd \sum_{j=0}^{m-1} [A(1-r)^j - C_j(r)] \to 0 \\
&\sum_{j=0}^{\infty} \log h(-D_{m,j}(r)rd) = -(1 + o(1))rd \sum_{j=0}^{\infty} D_{m,j}(r) \to 0
\end{aligned}$$

as $r \to 0$. Thus,

$$\mathbf{P}_\mu(T_E > m) \quad \leq \quad e^{-t(1+o(1))}$$

as $r \to 0$. This implies the upward equality of (7).

Let $M = r^{-1} \log(1 - c/\mu)^{-1}(1 - [\log r^{-1}]^{-p})$, where $p > 0$. Then,

$$\begin{aligned}
\mathbf{E}_\mu(T_E) &\geq \sum_{m=1}^{M} \mathbf{P}_\mu(T_E > m) \\
&\geq \sum_{m=1}^{M} \prod_{n=1}^{m} \mathbf{P}_\mu\Big(Y_{n,n}(r) + Z_{n,n} < \frac{c}{r}\Big) \\
&= \sum_{m=1}^{M} \prod_{n=1}^{m} \mathbf{P}_\mu\Big(Y_{n,n}(r) + Z_{n,n} < \frac{c}{r} - \frac{\mu}{r}[1 - (1-r)^n]\Big) \\
&\geq \sum_{m=1}^{M} \Big[\mathbf{P}_\mu\Big(Y_{M,M}(r) + Z_{M,M} < \frac{c}{r} - \frac{\mu}{r}[1 - (1-r)^M]\Big)\Big]^{mM}.
\end{aligned}$$

Since

$$\frac{c}{r} - \frac{\mu}{r}[1-(1-r)^M] = (1+o(1))\frac{(\mu-c)\log\frac{\mu}{\mu-c}}{r(\log r^{-1})^p}$$

for small $r$, by taking $\theta = 3r(\log r^{-1})^{p+1}[(\mu-c)\log(1-c/\mu)^{-1}]^{-1}$ and using (35), we have

$$\log f_{Y,m}(\theta) \to 0, \qquad \log f_{Z,m}(\theta) \to 0$$

as $r \to 0$, and therefore,

$$\begin{aligned} & \mathbf{P}_\mu\Big(Y_{M,M}(r) + Z_{M,M} > \frac{c}{r} - \frac{\mu}{r}[1-(1-r)^M]\Big) \\ \geq\ & \exp\{-\theta(1+o(1))\frac{(\mu-c)\log(1-c/\mu)^{-1}}{r(\log r^{-1})^p} + \log f_{Y,m}(\theta) + \log f_{Z,m}(\theta)\} \\ =\ & \exp\{-3\log r^{-1}(1+o(1)) + o(1)\} = (1+o(1))r^{3(1+o(1))} \end{aligned}$$

for small $r$. Thus,

$$\mathbf{E}_\mu(T_C) \ \ \geq\ \ \sum_{m=1}^{M}\Big[1-(1+o(1))r^{3(1+o(1))}\Big]^{mM} \to M.$$

as $r \to 0$. This is the downward inequality of (7). This completes the proof of Theorem 2.

# References


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