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A NEW LOOK AT OPTIMAL STOPPING PROBLEMS RELATED TO MATHEMATICAL FINANCE

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Abstract: A method is proposed to solve optimal stopping problems. Several examples - classical and new ones - are discussed. Especially, the values of American options (straddle and strangle) with infinite horizon are calculated.

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1. General Ideas

In several papers on sequential Bayes testing and change-point detection (see for instance Beibel (1996), Chapter II of Lerche (1986), or Woodroofe, Lerche, and Keener (1993)) the following argument is used: The Bayes risk $R(T)$ is represented for all stopping times T with $R(T) < \infty$ as

$$R(T) = Eg(L_T), \tag{1}$$

where L_t denotes a certain stochastic process connected to the likelihood process evaluated at time t and where g is a positive function with a unique minimum, let's say at a^*. Then we have $R(T) = Eg(L_T) \geq g(a^*)$. If L_t is a time-continuous process and passes a^* with probability one, the optimal stopping time will be $T^* = \inf\{t > 0 | L_t = a^*\}$. If L_t is discrete in time, one will usually not hit a^* exactly, therefore one has to stop ahead of a^*. This is also the case for the "parking problem" described in Chow, Robbins and Siegmund (1971), pages 45 and 60. There $g(x) = |x|$ and $L_n = X_1 + \cdots + X_n$ where the X_i are geometrically distributed. Therefore, M. Woodroofe has called situations as described above "generalized parking problems" (see Woodroofe, Lerche, and Keener (1993)). Of course for time-continuous processes L_t the solution is trivial, when one has the representation (1). Nevertheless to find a representation of this type is sometimes not obvious (see e.g. Beibel (1996)).

One can combine the above technique with the one recently used by Shepp and Shiryaev (1993). This yields an easy method to handle also some tricky optimal stopping problems.

Since our examples are formulated more naturally as maximization problems, we switch for convenience from minimization to maximization. To explain our technique more thoroughly let $Z = (Z_t; 0 \leq t < \infty)$ denote a continuous stochastic process for which we want to maximize $E(Z_T)$ over all stopping times T with respect to some filtration $\mathcal{F} = (\mathcal{F}_t; 0 \leq t < \infty)$ with $P(T < \infty) = 1$. We will discuss a general approach to transform such a problem to a generalized parking problem. The basic idea is to find another continuous stochastic process Y adapted to $\mathcal{F}$, a function g with a maximum uniquely located at some point y^* and a positive martingale M with $M_0 = 1$ such that $Z_t = g(Y_t)M_t$ for $0 \leq t < \infty$. By the properties of g we have $Z_t \leq g(y^*)M_t$. Since M is a positive martingale we obtain for any stopping time T with $P(T < \infty) = 1$ that $E(Z_T) \leq g(y^*)$. In order to prove the optimality of $T^* = \inf\{t > 0 | Y_t = y^*\}$, one only needs to show that $P(T^* < \infty) = 1$ and $E(M_{T^*}1_{\{T^*<\infty\}}) = M_0 = 1$. This can be seen as follows:

$$E(Z_{T^*}1_{\{T^*<\infty\}}) = E\Big(g(Y_{T^*})M_{T^*}1_{\{T^*<\infty\}}\Big) = g(y^*)E(M_{T^*}1_{\{T^*<\infty\}}) = g(y^*).$$

To reformulate this argument let Q denote the probability measure on $\mathcal{F}_\infty = \sigma(\mathcal{F}_t; t \geq 0)$ with

$$\frac{dQ}{dP}\Big|_{\mathcal{F}_t} = M_t \qquad \text{for } 0 \leq t < \infty.$$

(For our problems such a probability measure always exists since we assume $\mathcal{F}_t = \sigma(X_s, 0 \leq s \leq t)$.) Then we have for all stopping times T with $P(T < \infty) = 1$ that $E(Z_T) = E_Q(g(Y_T)1_{\{T<\infty\}})$. This means that we have transformed the initial stopping problem into a generalized parking problem with respect to a new probability measure Q. To prove $E(M_{T^*}1_{\{T^*<\infty\}}) = 1$ is equivalent to show that $Q(T^* < \infty) = 1$. We note that T^* still maximizes the quantity $E(Z_T 1_{\{T<\infty\}})$ if $P(T^* < \infty) < 1$ but $Q(T^* < \infty) = 1$ holds.

A crucial point in some of our arguments is to establish the martingale property for continuous local martingales. Sufficient conditions for that are given by Protter (1990), p.35, 66. Our technique works especially well for problems with exponentially discounting. Many problems of option pricing have this feature.

In Section 2.1 we discuss a classical result on American put options. In Section 2.2 we take a more general viewpoint which leads also to results on two-sided problems. As a consequence we calculate the values of American straddles and strangles with infinite horizon in Section 2.4. In Section 2.5 we take another look at the classical problem of stopping $E\frac{W_T}{T+1}$. Section 2.6 deals with the problem of maximizing $E(e^{-rT}e^{X_T}1_{\{T<\infty\}})$ where X is a Brownian motion whose drift changes at a random and unknown time τ. This corresponds to an investor who holds one unit of a stock whose price follows e^X and who wants to sell the stock soon after having passed the top. We solve this problem within the framework of Shiryaev (1963).

2. Examples

2.1. Perpetual American put options

Let $\mathbf{R}$ denote the real and $\mathbf{R}^+$ the positive real numbers. Let W denote standard Brownian motion which starts at $W_0 = 0$. Let $\sigma \in \mathbf{R}^+$ and $\mu \in \mathbf{R}$. Let X denote Brownian motion with drift μ given by $X_t = \sigma W_t + \mu t$. We use these notations during the next four sections. The following problem is treated in Jacka (1991) for the case $r = \mu$. (See also Karatzas (1988), McKean (1965) and Samuelson (1965).)

Problem 1. Find a stopping time T of X that maximizes

$$E\Big\{e^{-rT}\left(K - e^{X_T}\right)^+ 1_{\{T<\infty\}}\Big\},$$

where K and r are constants with $K > 0$ and $r > 0$.

Put

$$\gamma = \frac{\mu}{\sigma^2} + \sqrt{\frac{\mu^2}{\sigma^4} + \frac{2r}{\sigma^2}} \quad \text{and} \quad C^* = \max_{-\infty < x \le \log K} \left\{(K - e^x)^+ e^{\gamma x}\right\}.$$

Let x^* denote the unique point in $(-\infty, \log K)$ where the function $(K - e^x)^+ e^{\gamma x}$ attains its maximum C^*. A straightforward computation yields

$$x^* = \log \frac{\gamma K}{\gamma + 1} \quad \text{and} \quad C^* = \frac{1}{\gamma + 1}\left(\frac{\gamma}{\gamma + 1}\right)^\gamma K^{1+\gamma}.$$

Note that $\gamma > 0$. We will solve the above problem under the additional assumption $x^* < 0$.

Theorem 1. *Let $r > 0$ and $K < 1 + 1/\gamma$, then*

$$\sup_T E\left\{e^{-rT}\left(K - e^{X_T}\right)^+ 1_{\{T<\infty\}}\right\} = E\left\{e^{-rT^*}\left(K - e^{X_{T^*}}\right)^+ 1_{\{T^*<\infty\}}\right\} = C^*,$$

for $T^* = \inf\{t > 0 | X_t = x^*\}$. Under the additional assumption $\mu \le 0$ it follows that $P(T^* < \infty) = 1$. In this case T^* also maximizes $E\{e^{-rT}(K - e^{X_T})^+\}$ among all stopping times T.

Proof. Let M_t denote the process $e^{-rt}e^{-\gamma X_t}$. Since $\frac{(\gamma\sigma)^2}{2} + \gamma\mu = r$, it follows that $M_t = \exp\{-\gamma\sigma W_t - \frac{(\gamma\sigma)^2}{2} t\}$ and so M is a positive martingale with $M_0 = 1$. By the choice of C^* we have for all $0 \le t < \infty$ that

$$e^{-rt}(K - e^{X_t})^+ = (K - e^{X_t})^+ e^{\gamma X_t} M_t \le C^* M_t.$$

This yields for all stopping times T that $E(e^{-rT}(K - \exp(X_T))^+ 1_{\{T<\infty\}}) \le C^*$. For the stopping time T^* we have

$$E\left(e^{-rT^*}(K - e^{X_{T^*}})^+ 1_{\{T^*<\infty\}}\right) = C^* E\left(M_{T^*} 1_{\{T^*<\infty\}}\right).$$

Let Q denote the probability measure on $\sigma(W_s; 0 \le s < \infty)$ defined by

$$\frac{dQ}{dP}\Big|_{\mathcal{F}_t^W} = M_t$$

for $0 \le t < \infty$, where $\mathcal{F}_t^W = \sigma(W_s; 0 \le s \le t)$. Under the probability measure Q the process W is a Brownian motion with drift $-\gamma\sigma$. Therefore X is a Brownian motion with drift $-\gamma\sigma^2 + \mu = -\sigma^2\sqrt{\frac{\mu^2}{\sigma^4} + \frac{2r}{\sigma^2}} < 0$. This yields $Q(T^* < \infty) = 1$ since $x^* < 0$. Hence we have $E(M_{T^*} 1_{\{T^* < \infty\}}) = 1$.

2.2. Exponentially discounted functions of Brownian motion with drift – one-sided boundaries

The arguments in Section 2.1 can be put in a more general context. Let h denote a measurable real-valued function. Let r be a strictly positive constant. The following problem is treated for $\mu = 0$ in van Moerbeke (1974a) and van Moerbeke (1974b).

Problem 2. Find a stopping time T with respect to $\mathcal{F}^X$ that maximizes

$$E\left\{e^{-rT} h(X_T) 1_{\{T<\infty\}}\right\}.$$

Let

$$\alpha_1 = -\frac{\mu}{\sigma^2} + \sqrt{\frac{\mu^2}{\sigma^4} + \frac{2r}{\sigma^2}} \quad \text{and} \quad \alpha_2 = -\frac{\mu}{\sigma^2} - \sqrt{\frac{\mu^2}{\sigma^4} + \frac{2r}{\sigma^2}}$$

denote the two solutions of the quadratic equation $(\alpha\sigma)^2/2 + \alpha\mu = r$. Of course $\alpha_2 < 0 < \alpha_1$. Therefore the processes $M_t^{(1)}$ and $M_t^{(2)}$ given by $M_t^{(1)} = e^{-rt} e^{\alpha_1 X_t}$ and $M_t^{(2)} = e^{-rt} e^{\alpha_2 X_t}$ are positive martingales.

Theorem 2. *If* $0 < C_1 = \sup_{x \in \mathbf{R}} (e^{-\alpha_1 x} h(x)) < \infty$ *and* $C_1 = e^{-\alpha_1 x_1} h(x_1)$ *for some* $x_1 > 0$, *then*

$$\sup_T E\left\{e^{-rT} h(X_T) 1_{\{T<\infty\}}\right\} = C_1$$

and the supremum is attained for $T^* = \inf\{t > 0 | \ X_t = x_1\}$.

Proof. Let $Q^{(1)}$ denote the probability measure on $\sigma(W_s; 0 \le s < \infty)$ with

$$\frac{dQ^{(1)}}{dP}\Big|_{\mathcal{F}_t^W} = M_t^{(1)}$$

for all $0 \le t < \infty$. We have for all stopping times T

$$\begin{aligned} E\left\{e^{-rT} h(X_T) 1_{\{T<\infty\}}\right\} &= E\left\{M_T^{(1)} e^{-\alpha_1 X_T} h(X_T) 1_{\{T<\infty\}}\right\} \\ &= E_{Q^{(1)}}\left\{e^{-\alpha_1 X_T} h(X_T) 1_{\{T<\infty\}}\right\}. \end{aligned}$$

By the definition of C_1 we obtain for all stopping times T

$$E_{Q^{(1)}}\left\{e^{-rT}h(X_T)1_{\{T<\infty\}}\right\} \leq C_1.$$

On the set $\{T^* < \infty\}$ we have $\exp\{-\alpha_1 X_{T^*}\}h(X_{T^*}) = C_1$ and so

$$E_{Q^{(1)}}\left\{e^{-rT^*}h(X_{T^*})1_{\{T^*<\infty\}}\right\} = C_1 Q^{(1)}(T^* < \infty).$$

To complete the proof it is therefore sufficient to show $Q^{(1)}(T^* < \infty) = 1$. Under $Q^{(1)}$ the process W is a Brownian motion with drift $\alpha_1\sigma > 0$. This yields the desired result.

Remark. If $P(T^* < \infty) = 1$ holds, then T^* also maximizes $Ee^{-rT}h(X_T)$ among all stopping times T.

Example. (See van Moerbeke (1974b), p. 553-554) For $\mu = 0$ and $\sigma = 1$ we have $\alpha_1 = \sqrt{2r}$. For a sufficiently smooth function h the point x_1 satisfies the equation $(h(x)e^{-\sqrt{2r}x})' = 0$. Since $(h(x)e^{-\sqrt{2r}x})' = (h'(x) - \sqrt{2r}h(x))e^{-\sqrt{2r}x}$, the optimal threshold x_1 also solves

$$\frac{d}{dx}\log h(x) = \sqrt{2r}\ .$$

For $h(x) = x$ it is easy to check the above conditions. We obtain

$$\sup_T E\left(e^{-rT}W_T\right) = E\left(e^{-rT^*}W_{T^*}\right) = \frac{1}{\sqrt{2r}}e^{-1}$$

for $T^* = \inf\{t > 0 \Big| W_t = \frac{1}{\sqrt{2r}}\}$.

With similar arguments as above one can also prove the following theorem.

Theorem 3. *If* $0 < C_2 = \sup_{x\in\mathbf{R}}(e^{-\alpha_2 x}h(x)) < \infty$ *and* $C_2 = e^{-\alpha_2 x_2}h(x_2)$ *for some* $x_2 < 0$*, then*

$$\sup_T E\Big\{e^{-rT}h(X_T)1_{\{T<\infty\}}\Big\} = C_2$$

and the supremum is attained for $T^* = \inf\{t > 0 | X_t = x_2\}$.

Note that the conditions of Theorem 2 and Theorem 3 are mutually exclusive. Suppose for example that the conditions of Theorem 2 are satisfied, that is

$$0 < C_1 = \sup_{x\in\mathbf{R}}(e^{-\alpha_1 x}h(x)) < \infty$$

and there exists a point $x_1 > 0$ with $C_1 = \exp\{-\alpha_1 x_1\}h(x_1)$. Then for all $x < 0$ with $h(x) > 0$ we have

$$e^{-\alpha_1 x_1}h(x_1) \geq e^{-\alpha_1 x}h(x) > e^{-\alpha_2 x}h(x).$$

Since $x_1 > 0$ we have $e^{-\alpha_2 x_1} h(x_1) > e^{-\alpha_1 x_1} h(x_1)$ and so we obtain $\exp\{-\alpha_2 x_1\}$ $h(x_1) > e^{-\alpha_2 x} h(x)$ for all $x < 0$ with $h(x) > 0$. This inequality clearly also holds for $x < 0$ with $h(x) \leq 0$. Therefore $\sup_{x\in\mathbf{R}} (e^{-\alpha_2 x} h(x))$ cannot be attained at some point $x_2 < 0$.

2.3. Exponentially discounted functions of Brownian motion with drift – two-sided boundaries

The method of Section 2.2 can be extended to treat problems with two-sided boundaries. We will now consider the problem to maximize $E\{e^{-rT} h(X_T) 1_{\{T<\infty\}}\}$ for a function h with the properties:

$$\sup_{x\leq 0} \big(e^{-\alpha_1 x} h(x)\big) > \sup_{x\geq 0} \big(e^{-\alpha_1 x} h(x)\big) > 0 \tag{2}$$

and

$$\sup_{x\geq 0} \big(e^{-\alpha_2 x} h(x)\big) > \sup_{x\leq 0} \big(e^{-\alpha_2 x} h(x)\big) > 0. \tag{3}$$

In this case we can neither apply Theorem 2 nor Theorem 3. Examples for such functions are given by $h(x) = x^2$ or $h(x) = \max\{(L-e^x)^+, (e^x-K)^+\}$ (if $\alpha_1 > 1$). The basic idea is to replace the martingales $M^{(1)}$ and $M^{(2)}$ by the martingale $M = pM_t^{(1)} + (1-p)M_t^{(2)}$ where $p \in (0,1)$ is suitable chosen. We have for all stopping times T

$$E\{e^{-rT} h(X_T)\} = E\left\{M_T \frac{h(X_T)}{pe^{\alpha_1 X_T} + (1-p)e^{\alpha_2 X_T}}\right\}.$$

To find a proper value for p we now study the function

$$G_p(x) = h(x)/[pe^{\alpha_1 x} + (1-p)e^{\alpha_2 x}] \tag{4}$$

more closely.

Lemma 1. *If* (2) *and* (3) *hold, then there exists a number* $p^* \in (0,1)$ *with*

$$\sup_{x\geq 0} G_{p^*}(x) = \sup_{x\leq 0} G_{p^*}(x).$$

Proof. Since the sets $\{x \geq 0 | h(x) > 0\}$ and $\{x \leq 0 | h(x) > 0\}$ are both nonempty, we have

$$\sup_{x\geq 0} G_p(x) = \sup_{x\geq 0; h(x)>0} G_p(x) = \Big(\inf_{x\geq 0; h(x)>0} \frac{pe^{\alpha_1 x} + (1-p)e^{\alpha_2 x}}{h(x)}\Big)^{-1}$$

and

$$\sup_{x\leq 0} G_p(x) = \sup_{x\leq 0; h(x)>0} G_p(x) = \Big(\inf_{x\leq 0; h(x)>0} \frac{pe^{\alpha_1 x} + (1-p)e^{\alpha_2 x}}{h(x)}\Big)^{-1}.$$

Note that for all $p \in (0,1)$ we have

$$0 < \sup_{x\geq 0; h(x)>0} G_p(x) \leq \frac{1}{p} \sup_{x\geq 0} \left(e^{-\alpha_1 x} h(x)\right) < \infty$$

and

$$0 < \sup_{x\leq 0; h(x)>0} G_p(x) \leq \frac{1}{1-p} \sup_{x\leq 0} \left(e^{-\alpha_2 x} h(x)\right) < \infty.$$

For fixed x with $h(x) > 0$ the function $p : [pe^{\alpha_1 x} + (1-p)e^{\alpha_2 x}]/h(x)$ is linear. Therefore, the functions $m_1(p)$ and $m_2(p)$ given by

$$m_1(p) = \inf_{x\geq 0; h(x)>0} \frac{pe^{\alpha_1 x} + (1-p)e^{\alpha_2 x}}{h(x)} \text{ and } m_2(p) = \inf_{x\leq 0; h(x)>0} \frac{pe^{\alpha_1 x} + (1-p)e^{\alpha_2 x}}{h(x)}$$

are concave functions on $(0,1)$ with values in $(0,\infty)$. The function m_1 is nondecreasing and the function m_2 is nonincreasing. Condition (2) and (3) yield

$$\lim_{p\to 1} m_1(p) = \frac{1}{\sup_{x\geq 0}(e^{-\alpha_1 x}h(x))} < \infty \text{ and } \lim_{p\to 0} m_2(p) = \frac{1}{\sup_{x\leq 0}(e^{-\alpha_2 x}h(x))} < \infty.$$

We have further

$$\lim_{p\to 0} m_1(p) = \frac{1}{\sup_{x\geq 0}(e^{-\alpha_2 x}h(x))} \quad \text{and} \quad \lim_{p\to 1} m_2(p) = \frac{1}{\sup_{x\leq 0}(e^{-\alpha_1 x}h(x))},$$

with the convention that $\frac{1}{+\infty} = 0$. Since $\sup_{x\geq 0}(e^{-\alpha_2 x}h(x)) > \sup_{x\leq 0}(e^{-\alpha_2 x}h(x))$ we obtain $\lim_{p\to 0}(m_1(p) - m_2(p)) < 0$. In a similar way we can show that $\lim_{p\to 1}(m_1(p) - m_2(p)) > 0$. Therefore $m_1(p) - m_2(p)$ has at least one zero in $(0,1)$.

Remark. In general p^* is not unique. We will now show that p^* is unique, if there exists a point $\tilde{x} > 0$ with $e^{-\alpha_1 \tilde{x}} h(\tilde{x}) = \sup_{x\geq 0}(e^{-\alpha_1 x}h(x))$. Suppose there exist p^* and p^{**} with $0 < p^* < p^{**} < 1$ such that

$$m_1(p^*) - m_2(p^*) = 0 = m_1(p^{**}) - m_2(p^{**}).$$

This implies $0 \geq m_1(p^*) - m_1(p^{**}) = m_2(p^*) - m_2(p^{**}) \geq 0$ and so

$$m_1(p^*) - m_1(p^{**}) = m_2(p^*) - m_2(p^{**}) = 0.$$

Since m_1 is concave and nondecreasing this yields $m_1(p^{**}) = m_1(p)$ for all $p \in (p^{**}, 1)$. Therefore we have $m_1(p^{**}) = \lim_{p\to 1} m_1(p) = 1/\sup_{x\geq 0}(e^{-\alpha_1 x}h(x))$. This is a contradiction to

$$m_1(p^{**}) \quad \leq \quad \frac{p^{**}e^{\alpha_1 \tilde{x}} + (1-p^{**})e^{\alpha_2 \tilde{x}}}{h(\tilde{x})} < \frac{e^{\alpha_1 \tilde{x}}}{h(\tilde{x})} = \frac{1}{\sup_{x\geq 0}(e^{-\alpha_1 x}h(x))}.$$

Theorem 4. *Let p^* be chosen according to Lemma 1 and let $C^* = \sup_{x\in\mathbf{R}} G_{p^*}(x)$. If there exist points $x_1 > 0$ and $x_2 < 0$ such that $G_{p^*}(x_1) = C^* = G_{p^*}(x_2)$, then*

$$\sup_T \left\{ Ee^{-rT} h(X_T) 1_{\{T<\infty\}} \right\} = C^*$$

and the supremum is attained for $T^* = \inf\{t > 0 | X_t = x_1 \text{ or } X_t = x_2\}$.

Proof. For all stopping times T we have

$$E\left\{e^{-rT} h(X_T) 1_{\{T<\infty\}}\right\} = E\left\{M_T G_{p^*}(X_T) 1_{\{T<\infty\}}\right\} \leq C^* E\left(M_T 1_{\{T<\infty\}}\right),$$

where $M_t = p^* M_t^{(1)} + (1-p^*) M_t^{(2)}$. Therefore it only remains to show that $E(M_{T^*} 1_{\{T^*<\infty\}}) = 1$. Let Q denote the probability measure on $\sigma(W_s; 0 \leq s < \infty)$ with

$$\frac{dQ}{dP}\Big|_{\mathcal{F}_t^W} = M_t$$

for all $0 \leq t < \infty$, where $\mathcal{F}_t^W = \sigma(W_s; 0 \leq s \leq t)$. Let B denote a standard Brownian motion and Θ a random variable that is independent of B with $P(\Theta = \alpha_1 \sigma) = p^* = 1 - P(\Theta = \alpha_2 \sigma)$. Under the probability measure Q the process W has the same distribution as $(B_t + \Theta t; 0 \leq t < \infty)$ and so $Q(T^* < \infty) = 1$.

Remark. In general it is not possible to determine p^*, x_1 and x_2 explicitly. One particular situation in which it is straightforward to determine p^* is, when h is symmetric around zero (i.e. $h(x) = h(-x)$ for all x) and the Brownian motion X has drift zero. Then we have $\alpha_2 = -\alpha_1 = -\sqrt{2r/\sigma^2}$, $p^* = \frac{1}{2}$ and $x_2 = -x_1$. Even in this particular case there seems to be no explicit expression for x_1. If in addition h is sufficiently smooth, the point x_1 is a solution of the differential equation

$$h'(x)\left\{e^{\alpha x} + e^{-\alpha x}\right\} = \alpha h(x)\left\{e^{\alpha x} - e^{-\alpha x}\right\}$$

with $\alpha = \sqrt{2r/\sigma^2}$.

2.4. Perpetual straddle and strangle options

Theorem 4 above can be used to determine the value and optimal exercise strategy of a perpetual straddle or strangle option. A strangle (straddle) is a combination of a put with exercise prize L (K) and a call with exercise prize K on the same security, where $L \leq K$. If we model the price of the underlying asset by a geometric Brownian motion

$$\exp\left\{\sigma W_t + (\tilde{\mu} - \frac{\sigma^2}{2})t\right\},$$

with W a standard Brownian motion, then we have to solve the following problem. Let $X_t = \sigma W_t + \mu t$, with $\mu = \tilde{\mu} - \frac{\sigma^2}{2}$ and

$$h(x) = \begin{cases} L - e^x, & x \le \log L, \\ 0, & \log L \le x \le \log K, \\ e^x - K, & x \ge \log K. \end{cases}$$

The task is to find a stopping time T that maximizes $E\{e^{-rT}h(X_T)\}$ for some $r > 0$. Let

$$\alpha_1 = -\Big(\frac{\tilde{\mu}}{\sigma^2} - \frac{1}{2}\Big) + \sqrt{\frac{2r}{\sigma^2} + \Big(\frac{\tilde{\mu}}{\sigma^2} - \frac{1}{2}\Big)^2}$$

and

$$\alpha_2 = -\Big(\frac{\tilde{\mu}}{\sigma^2} - \frac{1}{2}\Big) - \sqrt{\frac{2r}{\sigma^2} + \Big(\frac{\tilde{\mu}}{\sigma^2} - \frac{1}{2}\Big)^2}.$$

Then $\alpha_2 < 0 < \alpha_1$. Since the value of a straddle or strangle is larger than the value of the corresponding call option, we assume that the inflation factor r satisfies $r > \mu$. This implies $\alpha_1 > 1$ and so the constants α_1 and α_2 and the function h fulfill the conditions of Theorem 4. Moreover we assume that $\log L \le 0 = X_0 \le \log K$. Under these conditions we obtain from Theorem 4 the following result.

Corollary 1. *Let x_1, x_2 and p^* be the unique solutions with $x_1 > \log K$, $x_2 < \log L$ and $0 < p < 1$ of the following system of equations:*

$$\begin{aligned} \frac{e^{x_1} - K}{p^* e^{\alpha_1 x_1} + (1-p^*)e^{\alpha_2 x_1}} &= \frac{L - e^{x_2}}{p^* e^{\alpha_1 x_2} + (1-p^*)e^{\alpha_2 x_2}} \\ \frac{e^{x_1}}{e^{x_1} - K} &= \frac{p^*\alpha_1 e^{\alpha_1 x_1} + (1-p^*)\alpha_2 e^{\alpha_2 x_1}}{p^* e^{\alpha_1 x_1} + (1-p^*)e^{\alpha_2 x_1}} \\ \frac{-e^{x_2}}{L - e^{x_2}} &= \frac{p^*\alpha_1 e^{\alpha_1 x_2} + (1-p^*)\alpha_2 e^{\alpha_2 x_2}}{p^* e^{\alpha_1 x_2} + (1-p^*)e^{\alpha_2 x_2}}. \end{aligned}$$

Let C^* denote the common value of

$$\frac{e^{x_1} - K}{p^* e^{\alpha_1 x_1} + (1-p^*)e^{\alpha_2 x_1}} \quad \textit{and} \quad \frac{L - e^{x_2}}{p^* e^{\alpha_1 x_1} + (1-p^*)e^{\alpha_2 x_1}}.$$

Then

$$\sup_T E\Big(e^{-rT} \max\Big\{L - e^{\sigma W_T + (\tilde{\mu} - \frac{\sigma^2}{2})T}, 0, e^{\sigma W_T + (\tilde{\mu} - \frac{\sigma^2}{2})T} - K\Big\}\Big) = C^*$$

and the supremum is attained for

$$T^* = \inf\Big\{t > 0 | \sigma W_t + (\tilde{\mu} - \frac{\sigma^2}{2})t = x_1 \quad \textit{or} \quad \sigma W_t + (\tilde{\mu} - \frac{\sigma^2}{2})t = x_2\Big\}.$$

Proof. Let $G_p(\cdot)$ be as in (4). Since $\sup_{x\geq 0} e^{-\alpha_1 x}h(x) = e^{-\alpha_1\tilde{x}}h(\tilde{x})$ for $\tilde{x} = \log\frac{\alpha_1}{\alpha_1-1}+\log K$, there exists a unique p^* with $\sup_{x\geq 0}G_{p^*}(x) = C^* = \sup_{x\leq 0}G_{p^*}(x)$. For any $p\in(0,1)$ we have $\lim_{x\to\infty} G_p(x) = 0$. As $h(x)\geq 0$ for all x and $h(\log K) = 0$, the function $G_p(\cdot)$ assumes for any fixed $p\in(0,1)$ its maximum over $(\log K,\infty)$ at some point x in $(\log K,\infty)$. Each such point is a solution of $G_p(x)' = 0$. On $(\log K,\infty)$ this equation is equivalent to

$$\frac{e^x}{e^x-K} = \frac{p\alpha_1 e^{\alpha_1 x}+(1-p)\alpha_2 e^{\alpha_2 x}}{pe^{\alpha_1 x}+(1-p)e^{\alpha_2 x}}.$$

The function $\frac{e^x}{e^x-K}$ is strictly decreasing on $(\log K,\infty)$ and the function

$$\frac{p\alpha_1 e^{\alpha_1 x}+(1-p)\alpha_2 e^{\alpha_2 x}}{pe^{\alpha_1 x}+(1-p)e^{\alpha_2 x}}$$

is strictly increasing. Therefore there is at most one solution of $G_p(x)' = 0$ in $(\log K,\infty)$. With similar arguments one can show that for any fixed $p\in(0,1)$ the function $G_p(x)$ assumes its maximum over the interval $(-\infty,\log L)$ at the point, which is the unique solution of

$$\frac{-e^x}{L-e^x} = \frac{p\alpha_1 e^{\alpha_1 x}+(1-p)\alpha_2 e^{\alpha_2 x}}{pe^{\alpha_1 x}+(1-p)e^{\alpha_2 x}}$$

in $(-\infty,\log L)$.

2.5. Parabolic boundaries

Let h be a measurable function such that $\sup_{x\in\mathbf{R}}\{h(x)/H(x)\}<\infty$, where

$$H(x) = \int_0^\infty e^{ux-\frac{u^2}{2}}u^{2\beta-1}du,$$

with $\beta\in\mathbf{R}^+$. We further assume that the supremum of $h(x)/H(x)$ over $\mathbf{R}$ is attained at a unique point x^* and that this supremum is strictly positive. Let $C^* = \{h(x^*)/H(x^*)\}$. Let $x_0 < x^*$ and let $X_t = W_t + x_0$ for $0\leq t<\infty$, where W is a standard Brownian motion with $W_0 = 0$.

Problem 3. Find a stopping time T of X that maximizes

$$E\left\{(T+1)^{-\beta}h\Big(\frac{X_T}{\sqrt{T+1}}\Big)\right\}.$$

This problem is treated in van Moerbeke (1974a) (under different assumptions on h).

Theorem 5. *Under the above assumptions we have*

$$\sup_T E\left\{(T+1)^{-\beta}h\Big(\frac{X_T}{\sqrt{T+1}}\Big)\right\} = E\left\{(T^*+1)^{-\beta}h\Big(\frac{X_{T^*}}{\sqrt{T^*+1}}\Big)\right\} = H(x_0)C^*,$$

where

$$T^* = \inf\left\{t>0\Big|\frac{X_t}{\sqrt{t+1}} = x^*\right\}.$$

Proof. Let M_t denote the process $(t+1)^{-\beta}H(X_t/\sqrt{t+1})/H(x_0)$. We have

$$(t+1)^{\beta}\int_0^\infty e^{uX_t-\frac{u^2}{2}t}e^{-\frac{u^2}{2}}u^{2\beta-1}du = H\left(\frac{X_t}{\sqrt{t+1}}\right).$$

Moreover $\exp\{uX_t-\frac{u^2}{2}t\} = \exp\{ux_0\}\exp\{uW_t-\frac{u^2}{2}t\}$ and $E(\exp\{uX_t-\frac{u^2}{2}t\}) = \exp\{ux_0\}$. Therefore $(M_t; 0\le t<\infty)$ is a positive martingale with $EM_0 = 1$ and by the definition of C^* we have

$$(t+1)^{-\beta}h\Big(\frac{X_t}{\sqrt{t+1}}\Big) = H(x_0)\frac{h\Big(\frac{X_t}{\sqrt{t+1}}\Big)}{H\Big(\frac{X_t}{\sqrt{t+1}}\Big)}M_t \le H(x_0)C^*M_t.$$

This implies

$$E\Big\{(T+1)^{-\beta}h\Big(\frac{X_T}{\sqrt{T+1}}\Big)\Big\} \le H(x_0)C^*E(M_T) \le H(x_0)C^*$$

for all $\mathcal{F}^X$–stopping times T. On the set $\{T^*<\infty\}$ one has $h(\frac{X_{T^*}}{\sqrt{T^*+1}})/H(\frac{X_{T^*}}{\sqrt{T^*+1}})$ $= C^*$ and so

$$(T^*+1)^{-\beta}h\left(\frac{X_{T^*}}{\sqrt{T^*+1}}\right) = C^*H(x_0)M_{T^*}.$$

In order to complete the proof it is therefore sufficient to show $P(T^*<\infty) = 1$ and $E(M_{T^*}) = 1$. The law of the iterated logarithm immediately yields $P(T^* < \infty) = 1$. Let ρ denote the probability measure on $\mathbf{R}^+$ with Lebesgue–density $\exp\{ux_0-\frac{u^2}{2}\}u^{2\beta-1}/H(x_0)$. Let Q denote the probability measure on $\sigma(W_s; 0\le s<\infty)$ with

$$\frac{dQ}{dP}\Big|_{\mathcal{F}_t^W} = \int_0^\infty e^{uX_t-\frac{u^2}{2}t}\rho(du) = \frac{1}{H(x_0)}\int_0^\infty e^{uX_t-\frac{u^2}{2}t}e^{-\frac{u^2}{2}}u^{2\beta-1}du = M_t,$$

for $0\le t<\infty$, where $\mathcal{F}_t^W = \sigma(W_s; 0\le s\le t)$. Let B denote a standard Brownian motion and Θ a random variable with distribution ρ that is independent of B. Under Q the process $(X_t; 0\le t<\infty)$ has the same distribution as $(x_0+B_t+\Theta t; 0\le t<\infty)$. Therefore $Q(T^*<\infty) = 1$ and so the assertion follows.

Example (A classical stopping problem). We now consider the special case $h(x) = x$, $x_0 = 0$, and $\beta = \frac{1}{2}$. That means we want to maximize $E\{W_T/(T+1)\}$. This problem is treated in Shepp (1969) and Taylor (1968) and was initiated by Chow and Robbins (1965) and Dvoretzky (1965). An easy calculation shows that

$$\int_0^\infty e^{ux-\frac{u^2}{2}}du = e^{\frac{x^2}{2}} \int_{-\infty}^x e^{-\frac{z^2}{2}}dz.$$

Differentation yields the following transcendental equation for the threshold x^* (see Shepp (1969))

$$0 = (1-x^2)\int_0^\infty e^{ux-\frac{u^2}{2}}du - x.$$

Remark. Let T_a denote the stopping time $T_a = \inf\{t > 0 | x_0 + W_t \geq a\sqrt{t+1}\}$. Since M_t is a martingale the optional stopping theorem yields

$$E\,(T_a+1)^{-\beta} = \frac{H(x_0)}{H(a)}$$

for $a > x_0$ and $\beta > 0$. Note that $\sup_{-\infty<z\leq a} H(z) = H(a) < \infty$. This is a special case of the results of Novikov (1971) and Shepp (1967).

2.6. When to sell a stock?

Let W denote standard Brownian motion and τ a positive random variable independent of W with $P(\tau > t) = e^{-\lambda t}$ for some $\lambda > 0$ and all $t > 0$. Let

$$X_t = \begin{cases} \theta_0 t + \sigma W_t, & \text{for} \quad t \leq \tau, \\ \theta_0\tau + \theta_1(t-\tau) + \sigma W_t, & \text{for} \quad t > \tau, \end{cases}$$

where $\sigma > 0$ and $\theta_0 \neq \theta_1 \in \mathbf{R}$. This model has been considered by Shiryaev (1963) who studied the problem of detecting the change of drift as soon and as reliable as possible if one observes X sequentially.

Now let $r \geq 0$ and θ_0 and θ_1 be such that $\theta_0+\sigma^2/2 > r > \theta_1+\sigma^2/2$. Obviously e^X is a geometric Brownian motion whose mean changes at the random time point τ. We will discuss the problem of maximizing $E(e^{-rT}e^{X_T}1_{\{T<\infty\}})$ over all stopping times T of X. This corresponds to an investor who owns one unit of a stock whose price follows the process e^X. We assume a constant rate of inflation which is given by r. Initially $e^{-rt}e^{X_t}$ has increasing mean and holding this particular stock is favourable. At the random and unknown time τ the mean of $e^{-rt}e^{X_t}$ starts decreasing and holding the stock is no longer favourable. The investor therefore wants to sell the stock soon after having passed the top. To make things tractable we assume that λ is known as well as $\theta_0, \theta_1, \sigma$ and r.

If $\theta_0 + \sigma^2/2 - r > \lambda$, we have for any $t > 0$ that

$$E\left(e^{-rt}e^{X_t}\right) \geq P(\tau \geq t)e^{(\theta_0 - r)t}E\left(e^{\sigma W_t}\right) = e^{(\theta_0 - r + \sigma^2/2 - \lambda)t}$$

and $\limsup_{t\to\infty} E(e^{-rt}e^{X_t}) = +\infty$. This implies $\sup_T E(e^{-rT}e^{X_T}1_{\{T<\infty\}}) = +\infty$ and therefore our problem is trivial for $\theta_0 + \sigma^2/2 - r > \lambda$ since the change occurs "too late".

If $\theta_0 + \sigma^2/2 - r = \lambda$ and $\theta_0 - \theta_1 \leq \lambda$, then

$$E\left(e^{-rt}e^{X_t}\right) = e^{\left(\frac{\sigma^2}{2} - r + \theta_0\right)t}E\left(e^{(\theta_1 - \theta_0)(t-\tau)^+}\right) \geq e^{\left(\frac{\sigma^2}{2} - r + \theta_0\right)t}e^{-\lambda t}\int_0^t \lambda du = t.$$

Therefore we have again $\lim_{t\to\infty} E(e^{-rt}e^{X_t}) = +\infty$. The case "$\theta_0 + \sigma^2/2 - r = \lambda$ and $\theta_0 - \theta_1 > \lambda$" is less obvious. Unfortunately the method which we use below cannot be applied in that case. Therefore we shall assume from now on that $\theta_0 + \sigma^2/2 - r < \lambda$.

Problem 4. Let $\theta_0 + \sigma^2/2 - r < \lambda$. Find a stopping time T of X that maximizes $E(e^{-rt}e^{-X_T}1_{\{T<\infty\}})$.

To solve this problem we first note that $e^{-rt}e^{X_t}$ is a submartingale with respect to the filtration $\sigma(W_s; 0 \leq s \leq t; \tau)$ for $0 \leq t \leq \tau$ and a supermartingale for $\tau \leq t < \infty$. If we knew τ, we would stop at τ. Since $\theta_0 + \sigma^2/2 - r < \lambda$, we have $E(e^{-r\tau}e^{X_\tau}) < \infty$ and $\sup_T E(e^{-rT}e^{X_T}1_{\{T<\infty\}}) < \infty$. Moreover the process $e^{-rt}e^{X_t}$ is uniformly integrable.

Let $\tilde{X}_t = X_t - \theta_0 t$. Clearly we have for all $t \geq 0$ that $\sigma(\tilde{X}_s; 0 \leq s \leq t) = \sigma(X_s; 0 \leq s \leq t)$. Let $\mathcal{F}_t = \sigma(X_s; 0 \leq s \leq t)$ and $\pi_t = P(\tau \leq t|\mathcal{F}_t)$. It is easy to see that $(\pi_t, \mathcal{F}_t; 0 \leq t < \infty)$ is a strong Markov process. Moreover (see Shiryaev (1963), p. 33)

$$d\pi_t = \lambda(1 - \pi_t)dt + \frac{\theta_1 - \theta_0}{\sigma}\pi_t(1 - \pi_t)d\overline{W}_t,$$

where $\overline{W}_t = (1/\sigma)(\tilde{X}_t - (\theta_1 - \theta_0)\int_0^t \pi_s ds)$. The process $(\overline{W}_t, \mathcal{F}_t; 0 \leq t < \infty)$ is a standard Brownian motion. We can rewrite $e^{-rt}e^{X_t}$ as $e^{\sigma\overline{W}_t - \frac{\sigma^2}{2}t}e^{-A(t)}$ with

$$A(t) = \int_0^t \left\{r - \theta_0 - \frac{\sigma^2}{2} - (\theta_1 - \theta_0)\pi_s\right\}ds. \tag{5}$$

Let Q denote the probability measure on $\sigma(X_s, 0 \leq s < \infty)$ given by

$$\frac{dQ}{dP}\Big|_{\mathcal{F}_t} = \exp\left\{\sigma\overline{W_t} - \frac{\sigma^2}{2}t\right\}$$

for all $t \geq 0$. Then we have for any stopping time T of X that

$$E\left(e^{-rT}\exp\{X_T\}1_{\{T<\infty\}}\right) = E_Q\left(\exp\{-A(T)\}1_{\{T<\infty\}}\right).$$

This leads to the problem to find a stopping time T of X that maximizes $E_Q(e^{-A_T}1_{\{T<\infty\}})$. Solving this problem yields the following theorem.

Theorem 6. *Let $\theta_0+\sigma^2/2-r<\lambda$. There exists a number $p^*\in(0,1)$ such that for $T^*=\inf\{t>0|\pi_t=p^*\}$ we have*

$$\sup_T E\left(e^{-rT}e^{X_T}1_{\{T<\infty\}}\right)=E\left(e^{-rT^*}e^{X_{T^*}}1_{\{T^*<\infty\}}\right).$$

Proof. The process $\overline{W}_t-\sigma t$ is a standard Brownian motion under Q. Moreover

$$d\pi_t=\Big(\lambda(1-\pi_t)+(\theta_1-\theta_0)\pi_t(1-\pi_t)\Big)dt+\frac{\theta_1-\theta_0}{\sigma}\pi_t(1-\pi_t)d\left(\overline{W}_t-\sigma t\right).$$

This yields that $(\pi_t,\mathcal{F}_t;0\le t<\infty)$ is a diffusion under Q. $A(t)$ as defined in (5) is an additive functional of $(\pi_t;0\le t<\infty)$ which takes on both, negative and positive values. Let $Q^{(x)}$ denote the distribution of $(\pi_t;0\le t<\infty)$ if $\pi_0=x$. Let $E_Q^{(x)}$ denote the expectation with respect to $Q^{(x)}$. Note that $Q^{(0)}=Q$ and $E_Q^{(0)}=E_Q$. For $0<x<y<1$ let

$$\phi(x,y)=E_Q^{(x)}\left(e^{-A(T_y)}1_{\{T_y<\infty\}}\right),$$

where $T_x=\inf\{t>0|\pi_t=x\}$. Obviously $\phi(0,y)=E(e^{-rT_y}e^{X_{T_y}}1_{\{T_y<\infty\}})<\infty$. Now fix $b>0$. The strong Markov property of $(\pi_t;0\le t<\infty)$ yields for all stopping times S with $0\le S\le T_b$

$$\begin{aligned}E_Q^{(0)}\left(e^{-A(T_b)}1_{\{T_b<\infty\}}\Big|\mathcal{F}_S\right)&=e^{-A(S)}E_Q^{(0)}\left(e^{-[A(T_b)-A(S)]}1_{\{T_b<\infty\}}\Big|\mathcal{F}_S\right)\\&=e^{-A(S)}E_Q^{(\pi_S)}\left(e^{-A(T_b)}1_{\{T_b<\infty\}}\right)=e^{-A(S)}\phi(\pi_S,b).\end{aligned}$$

This implies for any pair of stopping times S and S' with $0\le S\le S'\le T_b$ that

$$E_Q\Big(e^{-A(S')}\phi(\pi_{S'},b)\Big|\mathcal{F}_S\Big)=E_Q\Big[E_Q\Big(e^{-A(T_b)}1_{\{T_b<\infty\}}\Big|\mathcal{F}_{S'}\Big)\Big|\mathcal{F}_S\Big]=e^{-A(S)}\phi(\pi_S,b).$$

In particular we may put $S=s\wedge T_b$ and $S'=s'\wedge T_b$ where $0\le s\le s'<\infty$. Hence $\exp\{-A(t\wedge T_b)\}\phi(\pi_{t\wedge T_b},b)$ is a uniformly integrable martingale under $Q^{(0)}$ for any $b>0$. Let $\psi(x)=1/\phi(0,x)$. We have $\phi(0,b)=\phi(0,x)\phi(x,b)$ for $0<x<b$. This yields $\phi(x,b)=\phi(0,b)\psi(x)$ and so $M_t=e^{-A_t}\psi(\pi_t)$ is a positive local martingale with respect to $Q^{(0)}=Q$.

Since $\pi_0=0$ we have $A_t<0$ for $0\le t\le T_{\tilde{x}}$, where $\tilde{x}=(-r+\theta_0+\sigma^2/2)/(\theta_0-\theta_1)$. We have $\tilde{x}\in(0,1)$ since $-r+\theta_0+\sigma^2/2>0>-r+\theta_1+\sigma^2/2$. Hence ϕ is strictly increasing on $(0,\tilde{x})$. The uniform integrability of $e^{-rt}e^{X_t}$ yields $\lim_{x\to1}E(e^{-rT_x}e^{X_{T_x}}1_{\{T_x<\infty\}})=0$ and so $\lim_{x\to1}\phi(0,x)=0$. A similar argument shows that $\phi(0,.)$ is continuous. This implies the existence of a point $x^* \in [\tilde{x}, 1)$ such that $\phi(0, x^*) = \sup_{x\in(0,1)} \phi(0, x)$. We obtain for any stopping time T that

$$E_Q\left(e^{-A_T}1_{\{T<\infty\}}\right) \leq \frac{1}{\psi(x^*)} E_Q\left(M_T 1_{\{T<\infty\}}\right) \leq \frac{1}{\psi(x^*)}.$$

For any $x \in (0,1)$ holds $E_Q(M_{T_x} 1_{\{T_x<\infty\}}) = \psi(0) = 1$ and therefore

$$E_Q\left(e^{-A_{T^*}}1_{\{T^*<\infty\}}\right) = \frac{1}{\psi(x^*)}$$

with $T^* = \inf\{t > 0 | \pi_t = x^*\}$.

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