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STATISTICAL INQUIRY FOR MARKOV CHAINS BY BOOTSTRAP METHOD

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Abstract: In this paper, we propose several different bootstrap algorithms, to get approximate confidence intervals for the parameters of an ergodic Markov chain, each of which can have k possible outcomes. Small-sample comparisons are used to select the best intervals for different parameters. An illustrative example which analyzes a farmers' tenure behavior pattern data set of Taiwan is given.

Key words and phrases: Bootstrap, hitting time, Markov chain, stationary distribution, transition probability.

1. Introduction

Suppose we have a sequence of observations in which each observation has k possible states. Then a common problem is deciding if successive events are independent or if the probabilities of the different outcomes depend on one or more immediately preceding outcomes, in which case a Markov chain model is more appropriate. This type of problem arises in the study of animal behavior, in information science, in sociology and in various other fields. We shall investigate in this paper the simple ergodic (positive recurrent, aperiodic and irreducible) Markov chain for a sequence $X_1, X_2, \ldots, X_n$ of random variables which has a first order dependence with stationary transition probability.

Maximum likelihood methods are commonly used to analyze Markov chain data. The asymptotic properties of maximum likelihood estimators and likelihood ratio tests have been investigated by Anderson and Goodman (1957), Billingsley (1961), and others. The use of bootstrap method to analyze this type of problem are in Kulperger and Prakasa Rao (1990), Athreya and Fuh (1989), Basawa et al. (1990) and Datta and McCormick (1992). A survey paper in this area is in Athreya and Fuh (1992). In these papers, various bootstrap methods, as well as their asymptotic properties, have been verified. Little is known, however, about their small sample properties, nor has much attention been paid to develop appropriate methods for small samples.

The Parametric Bootstrap (PB) and Block Bootstrap (BB) have been given in Athreya and Fuh (1989). In this paper, a new modified bootstrap method, Nested Bootstrap (NB), is proposed. The detailed descriptions of these methods are given in the next section. We shall also examine the small sample properties among all these bootstrap algorithms for estimating the parameters. It is hoped that this paper will stimulate further research along this line.

The remainder of this article is organized as follows. In Section 2, three different bootstrap algorithms are given. Small sample results for approximating the confidence intervals for the parameters (transition probability, stationary probability and hitting time) are in Section 3. An illustrative example which analyzes a farmers' tenure status pattern data set of Taiwan is in Section 4. Concluding remarks are in the last section.

2. Bootstrap Algorithms

Let $\{X_n; n \geq 0\}$ be a homogeneous ergodic Markov chain with state space S and transition probability matrix $P = (p_{ij})$. The problem of estimating the transition probability P, the stationary probability Π, and the distribution of the hitting time T_Δ to a state Δ, arises in several areas of applied probability and statistics. The application of the bootstrap method to a finite state Markov chain case was considered in Fuh (1989), and Kulperger and Prakasa Rao (1990). Athreya and Fuh (1989) discusses the countable state space case.

Here, we consider an ergodic Markov chain with finite state space $S = \{1, 2, \ldots k\}$. The ergodic property implies that the existence of an invariant probability measure $\Pi = (\pi_1, \ldots, \pi_k)$ such that $\pi_j > 0$, $\sum_{j=1}^k \pi_j = 1$, $\pi_j = \sum_i \pi_i p_{ij}$, $j = 1, \ldots, k$, and for all $i \in S$, $p_{ij}^{(n)} \to \pi_j$, as $n \to \infty$, where $p_{ij}^{(n)} \equiv Pr\{X_n = j | X_0 = i\}$ is the probability of $X_n = j$ given $X_0 = i$.

Suppose $\boldsymbol{x} = \{x_0, x_1, \ldots, x_n\}$ is a realization of the process $\{X_j;\, j = 0, \ldots, n\}$ observed up to time n. Let n_{ij} be the number of ij transitions in $\{x_0, \ldots, x_n\}$, and n_i be the number of visits to state i in $\{x_0, \ldots, x_n\}$.

We estimate P by its maximum likelihood estimator $\hat{P}_n \equiv (\hat{p}_n(i,j))$, where

$$\hat{p}_n(i,j) = \begin{cases} n_{ij}/n_i, & \text{if } n_i > 0; \\ \delta_{ij}, & \text{otherwise,} \end{cases}$$

where $\delta_{ij} = 1$ if $i = j$ and $= 0$ if $i \neq j$, and estimate Π by $\hat{\Pi}_n \equiv (\hat{\pi}_n(i))$, where $\hat{\pi}_n(i) = n_i/n$.

Since the state space S is finite, we can consider the non-parametric case as a special case of the parametric case. Therefore, the consistency and asymptotic normality of the maximum likelihood estimators can be deduced by using the analogy with the multinomial distribution. This idea can also be used for the bootstrap estimators of $\hat{P}_n$ given $\boldsymbol{x}$. The central limit theorems for the maximum likelihood estimators $\hat{P}_n$ of P and $\hat{\Pi}_n$ of Π are in Billingsley's book (1961).

Although the validity of the asymptotic normality can be used as an approximation of the sampling distribution of $\hat{P}_n$, the difficulty to compute the asymptotic variance-covariance matrix makes it less suitable for application. Especially, the computation of the distribution of the hitting time T_Δ is extremely difficult. Therefore, we propose several bootstrap algorithms to investigate this type of problem. The Parametric Bootstrap (PB) and the Block Bootstrap (BB) are in Athreya and Fuh (1989). The Nested Bootstrap (NP) will be proposed in this paper. For completeness, we state all three alternative bootstrap algorithms herewith.

I. Parametric Bootstrap (PB)

Let $\boldsymbol{x} = \{x_0, x_1, \ldots, x_n\}$ be a realization of the Markov chain $\{X_n; n \geq 0\}$ with transition probability P. Let $\hat{P}_n \equiv P(n, \boldsymbol{x})$ be an estimator of P based on the observed data $\boldsymbol{x}$. Assume G is a parameter of interest which needs to be estimated and $\hat{G}_n$ its estimator based on the observation $\boldsymbol{x}$. The Parametric Bootstrap to estimate the sampling distribution H_n of $R(\boldsymbol{x}, G) \equiv (\hat{G}_n - G)$ is as follows:

(1) With $\hat{P}_n$ as its transition probability, generate a Markov chain realization of N_n steps $\boldsymbol{x}^* = \{x_0^*, x_1^*, \ldots, x_{N_n}^*\}$. Call this the bootstrap sample, and let $\tilde{G}_n \equiv G(N_n, \boldsymbol{x}^*)$. Note that $\tilde{G}_n$ bears the same relation to $\boldsymbol{x}^*$ as $\hat{G}_n$ to $\boldsymbol{x}$.

(2) Approximate the sampling distribution H_n of $R(\boldsymbol{x}, G)$ by the conditional distribution H_n^* of $R(\boldsymbol{x}^*, \hat{G}_n) \equiv (\tilde{G}_n - \hat{G}_n)$ given $\boldsymbol{x}$, which can be done by Monte-Carlo method.

II. Block Bootstrap (BB)

The existence of a recurrent state Δ which is visited infinitely often (i.o.) for a recurrent Markov chain is well-known. A famous approach to its limit theory is via the embedded renewal process of returns to Δ. This is the so-called regeneration method. For a fixed state Δ, by the strong Markov property, the cycles $\{X_j; j = T_\Delta^{(n)}, \ldots, T_\Delta^{(n+1)} - 1\}$ are i.i.d. for $n = 1, 2, \ldots$, where $T_\Delta^{(n)}$ is the time of the nth return to Δ.

Fix an integer k and observe the chain up to the random time $n = T_\Delta^{(k+1)}$. Let $\boldsymbol{x} = \{x_0, x_1, \ldots, x_n\}$ be a realization of the process. Note that in this situation, $x_n = \Delta$. Fix i, j which are different from Δ. Let $\eta_\alpha \equiv \{x_j; j = T_\Delta^{(\alpha)}, \ldots, T_\Delta^{(\alpha+1)} - 1\}$ be the αth cycle, T_α be the length of η_α, $g(\eta_\alpha)$ be the number of visits to state i during the cycle η_α, and $h(\eta_\alpha)$ be the number of ij transitions during the cycle η_α. Define

$$\hat{\pi}_k(i) \equiv \frac{\sum_{\alpha=1}^k g(\eta_\alpha)}{\sum_{\alpha=1}^k T_\alpha}, \qquad \hat{p}_k(i,j) \equiv \frac{\sum_{\alpha=1}^k h(\eta_\alpha)}{\sum_{\alpha=1}^k g(\eta_\alpha)}$$

be the estimators of π and P respectively.

The Block Bootstrap algorithm is as follows:

(1) Decompose the original sample in the following fashion:

$$\{\eta_0, \eta_1, \eta_2, \ldots, \eta_k\}, \quad \text{where} \quad \eta_0 \equiv \left\{x_0, x_1, \ldots, x_{T_\Delta^{(1)}-1}\right\}.$$

Let $\hat{F}_k$ denote the uniform probability measure on the cycles $\{\eta_\alpha; \alpha = 1, 2, \ldots, k\}$. If $X_0 = \Delta$ w.p.1, then one could take $n = T_\Delta^{(k)}$ and $\hat{F}_k$ to be the uniform probability measure on $\{\eta_\alpha; \alpha = 0, 1, 2, \ldots, k-1\}$.

(2) With the original sample fixed, draw a "bootstrap sample" of size k' according to $\hat{F}_k$. Denote this sample by $\eta_1^*, \eta_2^*, \ldots, \eta_{k'}^*$. Then, the bootstrap analogues of $\hat{\pi}_k(i)$, $\hat{p}_k(i,j)$ can be defined as follows:

$$\tilde{\pi}_{k'}(i) \equiv \frac{\sum_{\alpha=1}^{k'} g(\eta_\alpha^*)}{\sum_{\alpha=1}^{k'} T_\alpha^*}, \qquad \tilde{p}_{k'}(i,j) \equiv \frac{\sum_{\alpha=1}^{k'} h(\eta_\alpha^*)}{\sum_{\alpha=1}^{k'} g(\eta_\alpha^*)},$$

where T_α^* is the length of η_α^*.

(3) Approximate the distribution of $\sqrt{k}(\hat{p}_k(i,j) - p_{ij})$ by the conditional distribution of $\sqrt{k'}(\tilde{p}_{k'}(i,j) - \hat{p}_k(i,j))$ given $\boldsymbol{x}$. Similarly for $\sqrt{k}(\hat{\pi}_k(i) - \pi_i)$.

III. Nested Bootstrap (NB)

With the same notation as above, let $\eta_\alpha \equiv \{x_j; j = T_{\Delta_1}^{(\alpha)}, \ldots, T_{\Delta_1}^{(\alpha+1)} - 1\}$ be the αth block of returning to state Δ_1. Fix another state Δ_2, and by the same idea of defining η_α, we define $\eta_{\alpha\beta}$ to be the βth subblock of returning to state Δ_2 within the αth block, where $\beta = 1, 2, \ldots, k_\alpha$, and k_α is the number of subblock during the original αth block. For example, let 132432334231232231 be a realization of a Markov chain with state space {1, 2, 3, 4}. Let $\Delta_1 = 1$, $\Delta_2 = 2$, then,

$$\eta_1 = \{13243233423\}, \qquad \eta_2 = \{123223\}.$$

$$\eta_{11} = \{243\}, \quad \eta_{12} = \{2334\}, \quad \eta_{21} = \{23\}, \quad \eta_{22} = \{2\}.$$

The Nested Bootstrap algorithm is as follows:

(1) Decompose the original sample in the following fashion:

$$\{\eta_0, \eta_1, \eta_2, \ldots, \eta_k\}, \quad \text{where} \quad \eta_0 \equiv \left\{x_0, x_1, \ldots, x_{T_{\Delta_1}^{(1)}-1}\right\}.$$

Let $\hat{F}_k$ denote the uniform probability measure on the cycles $\{\eta_\alpha; \alpha = 1, 2, \ldots, k\}$. If $X_0 = \Delta$ w.p.1, then one could take $n = T_\Delta^{(k)}$ and $\hat{F}_k$ to be the uniform probability measure on $\{\eta_\alpha; \alpha = 0, 1, 2, \ldots, k-1\}$.

(2) With the original sample fixed, draw a first level "bootstrap sample" of size k' according to $\hat{F}_k$. Denote this sample by $\eta_1^*, \eta_2^*, \ldots, \eta_{k'}^*$.

Let $\{\eta_{\alpha\beta}; \alpha = 1, 2, \ldots, k,\ \beta = 1, 2, \ldots, k_\alpha\}$ be the βth subblock within block η_α, and $\tilde{k} = k_1 + \cdots + k_k$. Let $\hat{F}_{\tilde{k}}$ denote the uniform probability measure on the cycles $\{\eta_{\alpha\beta}\}$.

With the first level bootstrap sample $\eta_1^*, \eta_2^*, \ldots, \eta_{k'}^*$ fixed, then according to $\hat{F}_{\tilde{k}}$, we can draw the second level bootstrap sample $\{\eta_{\alpha\beta}^*\}$ of size $\tilde{k}^* \equiv k_1^* + \cdots + k_{k'}^*$, where k_α^* is the number of subblock within the block η_α^*. Then, replace the subblock in the first level bootstrap sample by the second level bootstrap sample. Denote this sample by $\eta_1^{**}, \eta_2^{**}, \ldots, \eta_{k'}^{**}$.

Then, the bootstrap analogues of $\hat{\pi}_k(i)$, $\hat{p}_k(i,j)$ can be defined as follows:

$$\tilde{\pi}_{k'}(i) \equiv \frac{\sum_{\alpha=1}^{k'} g(\eta_\alpha^{**})}{\sum_{\alpha=1}^{k'} T_\alpha^{**}}, \qquad \tilde{p}_{k'}(i,j) \equiv \frac{\sum_{\alpha=1}^{k'} h(\eta_\alpha^{**})}{\sum_{\alpha=1}^{k'} g(\eta_\alpha^{**})},$$

where T_α^{**} is the length of η_α^{**}.

(3) Approximate the sampling distribution of $\sqrt{k}(\hat{p}_k(i,j) - p_{ij})$ by the conditional distribution of $\sqrt{k'}(\tilde{p}_{k'}(i,j) - \hat{p}_k(i,j))$ given $\boldsymbol{x}$. Similarly for $\sqrt{k}(\hat{\pi}_k(i) - \pi_i)$.

3. Monte-Carlo Simulations

A small sample comparison for all three alternative bootstrap methods is given in this section. We compare all the approximate confidence intervals for different parameters, which includes transition probability p_{35}, stationary probability π_5 and hitting time T_5.

For this small sample study, two different sample sizes $k = 20, 50$ ($n = 81, 226$) are included. The bootstrap resample sizes are $k' = 20, 50, 100$ for $k = 20$ and $k' = 50, 100, 150$ for $k = 50$ respectively. Here, we compare 95% confidence interval, empirical coverage probabilities, and their average length for all three alternative bootstrap algorithms.

For each situation (specific k, k' and parameter) 1000 replications Monte-Carlo trials were run. Computations were performed using FORTRAN programs on the VAX-8350 computer of the Institute of Statistical Science, Academia Sinica, Taipei, Taiwan, ROC. The random numbers were generated by using IMSL routines. All the tests were compared on the basis of the same random numbers. Samples of different sizes were nested.

The original sample is a computer simulation from an ergodic Markov chain with transition probability matrix

$$P = \begin{pmatrix} .1 & .2 & .3 & .3 & .1 \\ .5 & .1 & .1 & .1 & .2 \\ .1 & .5 & .1 & .1 & .2 \\ .1 & .2 & .5 & .1 & .1 \\ .2 & .2 & .1 & .4 & .1 \end{pmatrix},$$

and stationary probability

$$\Pi = (.211,\ .241,\ .217,\ .186,\ .146).$$

3.1. Transition and stationary probabilities

The maximum likelihood estimates $\hat{P}_{20}$ ($\hat{\Pi}_{20}$) of P (Π) based on sample size $k = 20$ ($n = 81$) are as follows:

$$\hat{P}_{20} = \begin{pmatrix} .150 & .250 & .200 & .300 & .100 \\ .480 & .120 & .160 & .080 & .160 \\ .154 & .692 & .000 & .000 & .154 \\ .000 & .300 & .500 & .000 & .200 \\ .231 & .385 & .000 & .154 & .231 \end{pmatrix},$$

$$\hat{\Pi}_{20} = (.247,\ .310,\ .160,\ .123,\ .160).$$

The maximum likelihood estimates $\hat{P}_{50}$ ($\hat{\Pi}_{50}$) of P (Π) based on sample size $k = 50$ ($n = 226$) are as follows:

$$\hat{P}_{50} = \begin{pmatrix} .100 & .240 & .180 & .380 & .100 \\ .500 & .107 & .125 & .143 & .125 \\ .087 & .522 & .108 & .087 & .196 \\ .182 & .114 & .500 & .114 & .090 \\ .166 & .300 & .100 & .267 & .167 \end{pmatrix},$$

$$\hat{\Pi}_{50} = (.221,\ .248,\ .204,\ .194,\ .133).$$

The following abbreviated notations will be used in the tables below:

C.I.– confidence interval	n – sample size
NA – normal approximation	k – number of sample block
A.L. – average length	k' – number of resample block

On the basis of Tables 3.1 and 3.2 below, it is suggested that the Parametric Bootstrap be used for a simple parameter like stationary probability.

Table 3.1. Comparison of approximate confidence intervals for $p_{35} = .200$. All coverage probabilities are above 95%.

	$k=20,\ k'=20$ 95% C.I.	A.L.	$k=20,\ k'=50$ 95% C.I.	A.L.	$k=20,\ k'=100$ 95% C.I.	A.L.
True	(.000, .353)	.353	(.046, .253)	.207	(.077, .227)	.149
NA	(.000, .351)	.351				
PB	(.063, .447)	.384	(.127, .376)	.248	(.103, .270)	.167
BB	(.001, .287)	.285	(.130, .326)	.196	(.020, .156)	.136
NB	(.040, .403)	.363	(.010, .242)	.232	(.055, .217)	.161
	$k=50,\ k'=50$		$k=50,\ k'=100$		$k=50,\ k'=150$	
True	(.075, .299)	.223	(.120, .274)	.153	(.129, .254)	.125
NA	(.081, .311)	.230				
PB	(.033, .254)	.220	(.060, .223)	.163	(.076, .200)	.123
BB	(.015, .210)	.195	(.104, .250)	.146	(.105, .210)	.105
NB	(.042, .272)	.229	(.111, .313)	.201	(.099, .284)	.184

Table 3.2. Comparison of approximate confidence intervals for $\pi_5 = .146$. All coverage probabilities are above 95%.

	$k=20,\ k'=20$ 95% C.I.	A.L.	$k=20,\ k'=50$ 95% C.I.	A.L.	$k=20,\ k'=100$ 95% C.I.	A.L.
True	(.076, .212)	.135	(.101, .185)	.083	(.116, .174)	.057
NA	(.080, .240)	.160				
PB	(.077, .242)	.164	(.105, .215)	.109	(.118, .197)	.079
BB	(.071, .250)	.178	(.102, .219)	.116	(.121, .200)	.079
NB	(.068, .234)	.166	(.105, .219)	.113	(.118, .200)	.082
	$k=50,\ k'=50$		$k=50,\ k'=100$		$k=50,\ k'=150$	
True	(.101, .186)	.085	(.113, .172)	.058	(.119, .171)	.051
NA	(.088, .177)	.089				
PB	(.087, .179)	.092	(.099, .163)	.064	(.106, .157)	.051
BB	(.087, .179)	.091	(.101, .164)	.063	(.106, .158)	.052
NB	(.078, .172)	.093	(.092, .158)	.065	(.098, .156)	.057

3.2. Hitting time

With the notation above, let T_5 be the first hitting time up to state 5. Let $\Pr(t; P) \equiv \Pr(T_5 \le t | X_0 = 1, P)$ denote the probability that $T_5 \le t$ for $t = 1, 2, 3, \ldots$, where P is the transition probability matrix of the above given Markov chain with initial state 1.

For any stochastic matrix P, let $A \equiv A(P)$ be the stochastic matrix which is the same as P except that the last row is replaced by (0, 0, 0, 0, 1). Note that $\Pr(t; P) = (A^t)_{1,5}$. The bootstrap estimate of the distribution $\Pr(t; P)$ of the hitting time T_5 is $\Pr(t; \hat{P}_n)$. All three bootstrap methods for estimating the distribution of the hitting time T_5 are illustrated in the following figures. Here the sample block size and the resample block size are both 50. Note that the Block Bootstrap has the smallest average length.

The following abbreviated notations will be used.

– – true distribution	□ – true confidence band
△ – PB confidence band	◇ – BB confidence band
$*$ – NB confidence band	A.L. – average length

Here, A.L. is computed for each $t = 1, 2, \ldots, 30$, and then take the average.

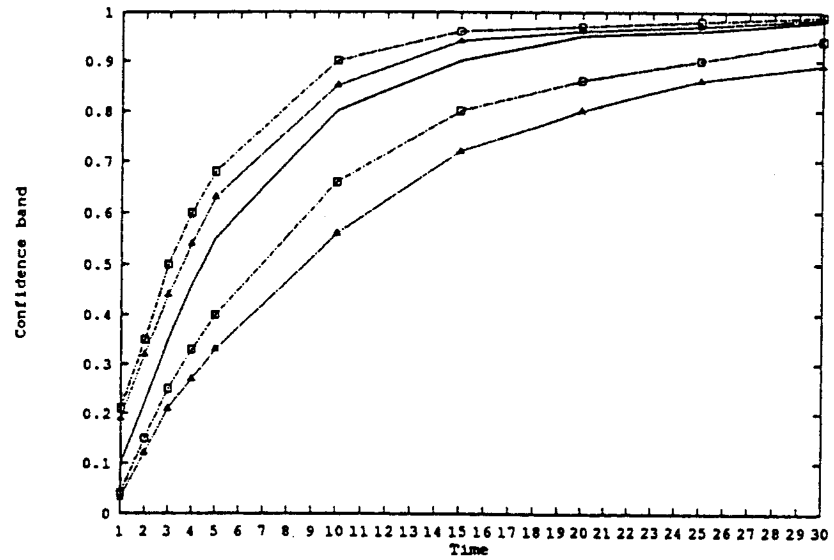

Figure 3.1. PB approximate confidence band for T_5, A.L.= .197

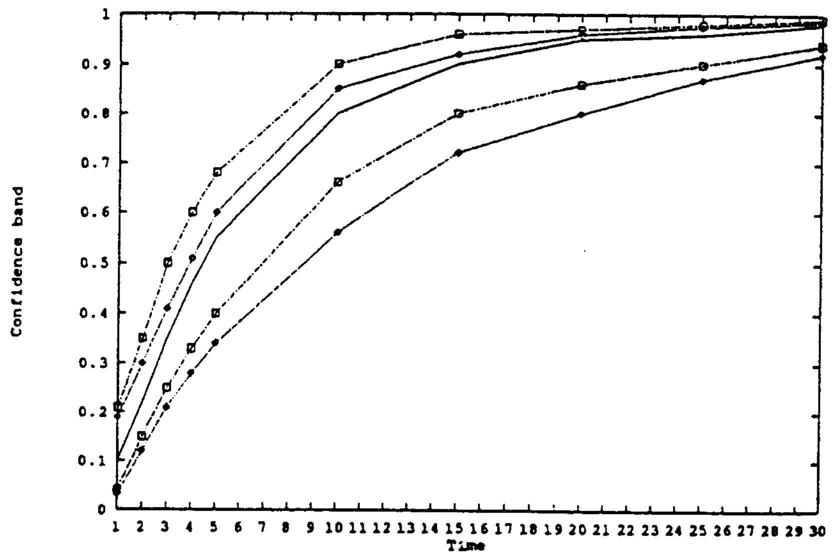

Figure 3.2. BB approximate confidence band for T_5, A.L.= .182

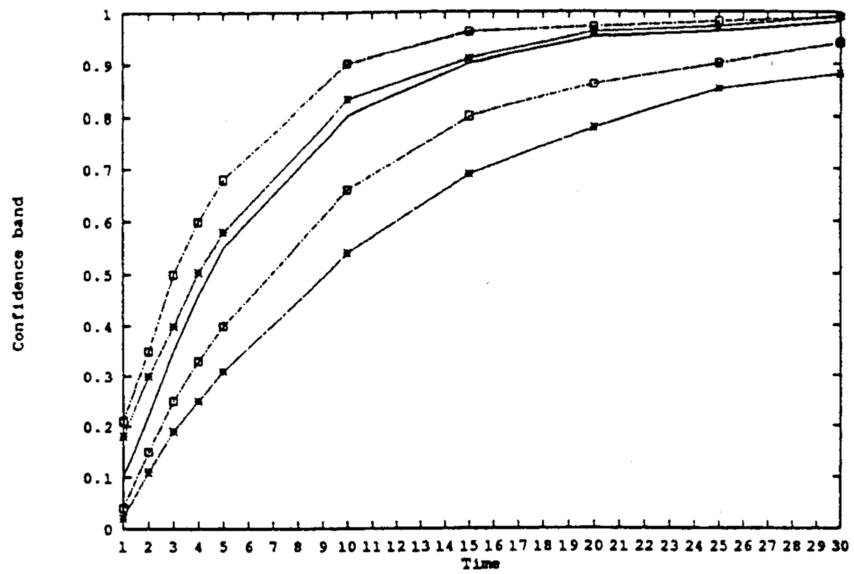

Figure 3.3. NB approximate confidence band for T_5, A.L.= .200

4. An Illustrative Example

In this section, we illustrate the bootstrap algorithms of Section 2 by applying them to a real example, which is the problem of the tenure behavior pattern of farmers in Taiwan. In this example, a Markov chain is used for modeling, and bootstrap techniques are used to make statistical inference for the accuracy of the estimators.

4.1. Land reform problem in Taiwan

This example is concerned with the problem of the tenure behavior pattern before and after the land reform program in Taiwan in the early 1950's. We study, within the Markovian framework as well as by the bootstrap techniques stated in the present paper, the behavior systems reflecting the time ordered changes in the tenure status of farmers before and after the new land policy. The data concerning tenure status in Taiwan from 1941 to 1966 are given in Table 4.1.

Although time ordered tenure status data are not available, annual proportional data for the island's tenant, part owner and owner-operator are available from 1941 to 1966 as shown in the table. Note that the land reform program which started in 1949 actually only got effectively underway in 1953. Thus, we assume these aggregate data were from a first order Markov chain, and the transition probabilities are stationary from 1941 to 1952. After 1952, due to the land reform, we assume the transition probabilities changed from the previous period but remain stationary from 1953 to 1966.

The maximum likelihood estimates, Lee et al. (1977), for the first period (1941–1952) are as follows:

$$\hat{P}_1 = \begin{pmatrix} .862 & .058 & .080 \\ .181 & .819 & .000 \\ .000 & .070 & .930 \end{pmatrix},$$

$$\hat{\Pi}_1 = (.347,\ .263,\ .390).$$

The maximum likelihood estimates for the second period (1953–1966) are as follows:

$$\hat{P}_2 = \begin{pmatrix} .663 & .337 & .000 \\ .206 & .543 & .251 \\ .000 & .075 & .925 \end{pmatrix},$$

$$\hat{\Pi}_2 = (.123,\ .202,\ .675).$$

Here, state 1, 2, 3 refers to Tenant, Part-owner and Owner, respectively.

The estimators of the expected hitting time ET_{13} from tenant to owner status based on the two different Markov chain models are, in units of year,

$$\hat{ET}^1_{13} = 13.8 \quad \text{and} \quad \hat{ET}^2_{13} = 8.5.$$

To gain information about the accuracy of these estimators, we would like to find approximate confidence interval for $\hat{\pi}_3$, the stationary probability for owner, and the sampling distribution of the hitting time $\hat{T}_{13}$. Here, we use bootstrap methods to answer these questions partially. Based on the maximum likelihood estimators $\hat{P}_1$ and $\hat{P}_2$, we can generate bootstrap Markov chain data by the methods (PB, BB and NB) described above for this problem. Although time ordered tenure status data are not available, annual proportional data for the island's tenant, part owner and owner-operator are available in Table 4.1. The Block Bootstrap and Nested Bootstrap can be modified herewith. That is, we use the estimator $\hat{P}_1$ (or $\hat{P}_2$) of P_1 (or P_2) to generate first level Markov chain data, called $\boldsymbol{x} \equiv \{x_1, \ldots, x_n\}$. Then, based on this presumed Markov chain data, we apply bootstrap methods (BB and NB) to approximate the confidence intervals of $\hat{\pi}_3$ and the sampling distribution of $\hat{T}_{13}$. The approximate confidence intervals of $\hat{\pi}_3$ are given below in Tables 4.2 and 4.3, and the confidence bands for $\hat{T}_{13}$ are shown in Figures 4.1 and 4.2 for the two Markov chain models.

By comparing the stationary probabilities, expected hitting times from tenant to owner status as well as their corresponding confidence intervals for these two Markov chain models, the result appear to be consistent with expected economic outcome and would suggest that the land reform program has provided tenants with a good opportunity of changing their tenant status. In particular, the estimated stationary probability for owner has increased from .390 to .675 and the expected hitting time from tenant to owner status decreased from 13.8 to 8.5 years.

Table 4.1. Number of farm families and percentages of classified farmers, Taiwan, 1941-1966.

Year	Total number of farm families	Percentages		
		Owner	Part-owner	Tenant
1941	440,105	31	31	38
42	452,462	31	31	38
43	470,374	31	30	39
44	482,776	31	30	39
45	500,533	30	29	41
46	527,016	33	28	39
47	553,308	32	27	41
48	597,333	35	26	39
49	620,875	36	25	39
50	638,062	36	26	38
51	661,125	38	25	37
52	676,750	39	26	35
1953	702,325	55	24	21
54	716,582	57	24	19
55	732,555	59	24	17
56	746,318	60	23	17
57	759,234	60	23	17
58	769,925	61	23	16
59	780,402	62	23	15
60	785,592	64	21	15
61	800,835	65	21	14
62	809,917	65	21	14
63	824,560	66	21	13
64	834,827	66	21	13
65	847,242	67	20	13
1966	854,203	67	21	12

Source: Taiwan Agricultural Yearbook (1941-1966).

Table 4.2. Approximate confidence intervals for $\hat{\pi}_3^1 = .390$ with $k = 20$

	$k' = 20$		$k' = 50$		$k' = 100$	
	95% C.I.	A.L.	95% C.I.	A.L.	95% C.I.	A.L.
PB	(.038, .690)	.651	(.064, .629)	.565	(.148, .562)	.414
BB	(.019, .821)	.801	(.223, .771)	.547	(.357, .714)	.357
NB	(.355, .821)	.465	(.386, .773)	.386	(.393, .718)	.324

Table 4.3. Approximate confidence intervals for $\hat{\pi}_3^2 = .675$ with $k = 20$

	$k' = 20$		$k' = 50$		$k' = 100$	
	95% C.I.	A.L.	95% C.I.	A.L.	95% C.I.	A.L.
PB	(.209, .802)	.593	(.467, .770)	.303	(.541, .750)	.209
BB	(.256, .825)	.568	(.223, .771)	.547	(.357, .714)	.357
NB	(.257, .823)	.565	(.584, .792)	.208	(.631, .780)	.148

The following notation will be used in Figures 4.1 and 4.2.

□ - Markov chain model 1

△ - Markov chain model 2

The middle solid (or dashed) line is the estimator of T_{13} under Markov chain model 1 (or 2).

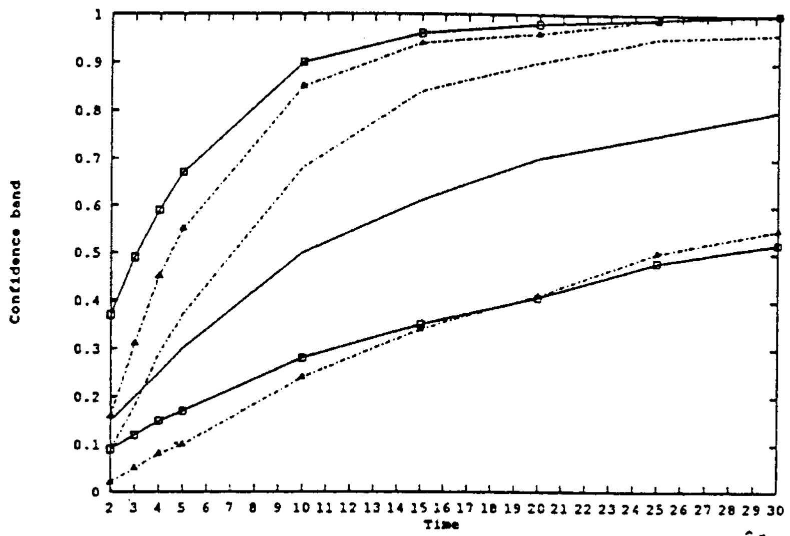

Figure 4.1. Comparison of the approximate confidence band for $\hat{T}_{13}^1$ and $\hat{T}_{13}^2$ by PB

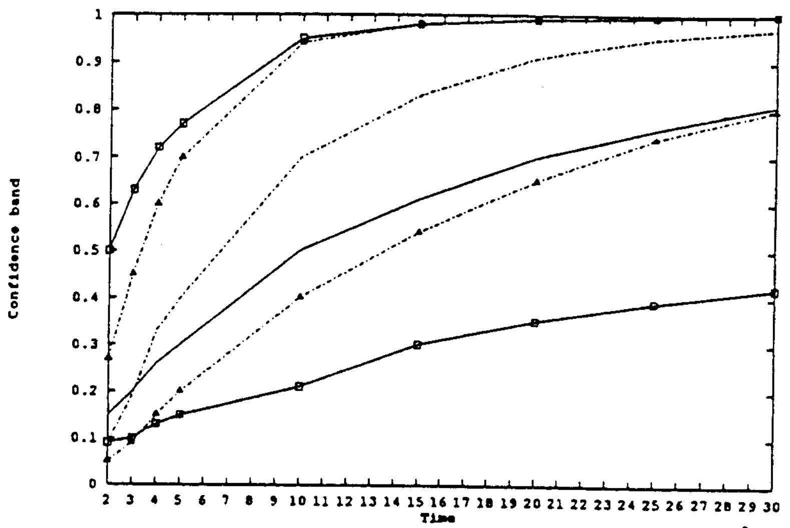

Figure 4.2. Comparison of the approximate confidence band for $\hat{T}_{13}^1$ and $\hat{T}_{13}^2$ by BB

5. Concluding Remarks

In this paper, we state three alternative bootstrap algorithms, and use an empirical approach to construct approximate confidence intervals or confidence bands for the parameters. A real example was presented to illustrate the application of these methods. In the example pertaining to the Taiwan land reform problem, we found the effect of the new land policy by Markov chain modeling and bootstrap methods. Finally, we have the following remarks:

1. By computer simulation, the Parametric Bootstrap is recommended in general, especially for a simple parameter like stationary probability. Another reason is that this method uses all the data information.

2. The bootstrap theory for Markov chains developed by Athreya and Fuh (1989), among others, are only for micro data. Here, we apply them to the macro data case with modification. For consistency of the bootstrap methods, rigorous theoretical investigation, involving the central limit theorem for both the maximum likelihood estimator and its bootstrap versions, is required.

3. Theoretical study of the accuracy of Parametric Bootstrap, Block Bootstrap and Nested Bootstrap, which involves the Edgeworth expansion for Markov chains and corresponding block, is need.

4. Investigation of the method of Pao-Zhuan Yin-Yu proposed by Fu and Li (1992) to Markov chain should be an interesting problem.

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