NONPARAMETRIC CONFIDENCE BANDS IN WICKSELL'S PROBLEM

Jakub Wojdyła and Zbigniew Szkutnik

AGH University of Science and Technology

Supplementary Material

This supplementary material contains proofs of Lemma 1 and Corollary 1. They refer to formulas, conditions, and theorems by numbers assigned in the main article.

S1 Proof of Lemma 1

Integration by parts, the fact that $K(x)|x|^{1/2}[\log\log|x|]^{1/2} \to 0$, $|x| \to \infty$, and the law of iterated logarithm for the Wiener process yield

$$Y_{n,3}(t) = h^{-3/2} \int K'\left(\frac{t-x}{h}\right) W(x)\, dx.$$

Combining this with (A.6) gives

$$|Y_{n,2}(t) - Y_{n,3}(t)| \leq |I_1| + |I_2|,$$

where

$$I_1 = h^{-1/2} \int K'(x) \frac{g(t-hx)^{1/2} - g(t)^{1/2}}{g(t)^{1/2}} W(t-hx)\, dx,$$

$$I_2 = h^{1/2} \int_{(t-1)/h}^{t/h} K(x) \frac{g'(t-hx)}{2g(t-hx)^{1/2} g(t)^{1/2}} W(t-hx)\, dx.$$

Furthermore, $I_1 = I_{1,1} + I_{1,2} + I_{1,3}$, with

$$I_{1,1} = h^{-1/2} \int_{(t-1)/h}^{t/h} K'(x) \frac{g(t-hx)^{1/2} - g(t)^{1/2}}{g(t)^{1/2}} W(t-hx)\, dx,$$

$$I_{1,2} = -h^{-1/2} \int_{-\infty}^{(t-1)/h} K'(x) W(t-hx)\, dx,$$

$$I_{1,3} = -h^{-1/2} \int_{t/h}^{\infty} K'(x) W(t - hx)\; dx.$$

It follows from assumption (2a) that g is Hölder continuous with any exponent $0 < \beta \leq 1$ and an appropriate constant C. Consequently, for the integral $I_{1,1}$ one obtains

$$|I_{1,1}| \leq C h^{-1/2+\beta} \sup_{x\in[0,1]} |W(x)| \int |K'(x)||x|^{\beta}\; dx.$$

Taking $\beta = 1/2 + \min\{\alpha, 1/2\}$ and using assumption (1b), we deduce that $|I_{1,1}| = O_p(h^{\min\{\alpha,1/2\}})$.

For a constant $\bar{C} \geq 1/(1-b)^{\alpha/2}$,

$$|I_{1,2}| \leq \bar{C} h^{-1/2} \int_{-\infty}^{(t-1)/h} |K'(x)||hx|^{\alpha/2} |W(t - hx)|\; dx$$

and, after another application of the law of iterated logarithm for the Wiener process,

$$|I_{1,2}| \leq O_p(h^{\alpha/2}) \int |K'(x)||x|^{1/2+\alpha/2} [\log\log^{+}|x|]^{1/2}\; dx,$$

uniformly in $t \in [a, b]$, where $\log\log^{+}|x| = 0$, if $|x| < e$ and $\log\log^{+}|x| = \log\log|x|$, otherwise. Therefore, $|I_{1,2}| = O_p(h^{\alpha/2})$. Similarly, one obtains $|I_{1,3}| = O_p(h^{\alpha/2})$.

To complete the proof of Lemma 1, it is sufficient to notice that $|I_2| = O_p(h^{1/2})$, uniformly in $t \in [a, b]$.

S2 Proof of Corollary 1

We begin by introducing the following processes on $[a, b]$:

$$Y_{n,4}(t) = -\frac{n^{1/2} h\pi}{2m\tilde{g}_n(t)^{1/2}} [f_n(t) - E\{f_n(t)\}],$$

$$Y_{n,5}(t) = -\frac{n^{1/2} h\pi}{2m\tilde{g}_n(t)^{1/2}} [f_n(t) - f(t)],$$

$$Y_{n,6}(t) = -\frac{n^{1/2} h\pi}{2\hat{m}\tilde{g}_n(t)^{1/2}} [\hat{f}_n(t) - f(t)].$$

Note that $nh/(\log n)^3 \to \infty$, since $nh^2/\log(1/h) \to \infty$. Thus, the assumptions of Theorem 1 are satisfied. From that theorem, $\|Y_n\| = O_p\{\log(1/h)^{1/2}\}$. Taking the difference of Y_n and $Y_{n,4}$, one has

$$Y_n(t) - Y_{n,4}(t) = Y_n(t) \frac{\tilde{g}_n(t) - g(t)}{\tilde{g}_n(t)^{1/2}[\tilde{g}_n(t)^{1/2} + g(t)^{1/2}]},$$

and, hence, $\|Y_n - Y_{n,4}\| = o_p\{\log(1/h)^{-1/2}\}$ by an application of (2.4). Furthermore, (2.3) and the condition $n^{1/2}h^{k+1}\log(1/h)^{1/2} \to 0$ imply that, uniformly for $t \in [a, b]$,

$$|Y_{n,4}(t) - Y_{n,5}(t)| = \left| \frac{n^{1/2}h\pi}{2m\tilde{g}_n(t)^{1/2}} \right| |f(t) - E\{f_n(t)\}| = o_p\{\log(1/h)^{-1/2}\}.$$

Finally,

$$\begin{aligned} Y_{n,5}(t) - Y_{n,6}(t) &= Y_{n,5}(t)\frac{\hat{m} - m}{m} + \frac{n^{1/2}h\pi}{2\hat{m}\tilde{g}_n(t)^{1/2}}[f_n(t) - f(t)]\frac{\hat{m} - m}{m} \\ &\quad + \frac{n^{1/2}h\pi}{2\hat{m}\tilde{g}_n(t)^{1/2}} f(t)\frac{\hat{m} - m}{m} \\ &= Y_{n,5}(t)\left[\frac{\hat{m} - m}{m}\right]^2 + \frac{n^{1/2}h\pi}{2\hat{m}\tilde{g}_n(t)^{1/2}} f(t)\frac{\hat{m} - m}{m}, \end{aligned}$$

and, hence, $\|Y_{n,5} - Y_{n,6}\| = o_p\{n^{-1+\delta}\log(1/h)^{1/2}\} + o_p(n^\delta h) = o_p\{\log(1/h)^{-1/2}\}$, because of (2.5) and of $n^{-1+\delta}\log(1/h) = O(1)$ and $n^\delta h \log(1/h)^{1/2} = O(1)$.

Therefore, replacing Y_n in Theorem 1 with, consecutively, $Y_{n,4}$, $Y_{n,5}$ and $Y_{n,6}$, one gets

$$P\left([2\log(1/h)]^{1/2} \left[\|Y_{n,6}\|/C_{K,1}^{1/2} - d_n \right] < x \right) \to \exp\{-2\exp(-x)\},$$

and Corollary 1 follows by rearranging the terms.