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PARAMETER DESIGN WITH MONOTONE LOSS FUNCTIONS

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Abstract: A new technique for parameter design is proposed. It is based on the observation that a loss function can be decomposed into two components each of which is monotone. This allows a separate data modeling and analysis to be applied to the two groups of data (above or below target). Then parameter design optimization can be performed based on the two fitted models. The technique is illustrated with the design of a heat exchanger.

Key words and phrases: Nominal-the-best parameter design, orthogonal array, signal-to-noise ratio.

1. Introduction

Robust parameter design is an important methodology in quality engineering and industrial statistics, which is championed by the eminent quality expert G. Taguchi and eminent statisticians like G. Box, V. Nair and C. F. J. Wu. It employs statistically designed experiments to reduce output variation by making a product or process insensitive to variation in the noise (or hard-to-control) factors. It has been widely practiced in many industries in Japan, US, Europe and other countries. Many industrial case studies provide strong evidence of its successes. A less well known source of case studies is a collection of computer experiments in China in the book *Three-Stage Design of Experiments with Known Transfer Functions* published by the Committee on Three-Stage Design, Chinese Applied Statistics Society. Parameter design has also received a great deal of attention in the industrial statistics research community. A good example is the panel discussion edited by Nair (1992) on parameter design, which covers many aspects of the problem. A central problem in parameter design is how to handle interactions and its connection to the choice and use of loss functions for data analysis. Moorhead and Wu (1998) proposed a strategy for parameter design with a general loss function but did not relate it to the important issue of interactions. In this paper a new approach is proposed which can study very general loss functions by directly analyzing the original data. Because of the direct modeling of data the problem of spurious interactions is minimized.

First we give a brief review of parameter design. Suppose y represents the quality characteristic of a product or process whose target value is T. Then the quality loss is usually represented by

$$L = L(y - T), \tag{1.1}$$

where L is a non-decreasing function over $[0, \infty)$ and a non-increasing function over $(-\infty, 0]$. This monotonicity assumption is very reasonable because the quality loss should not decrease as y moves away from T. By dividing the factors x that influence y into two types, x_C for control factors and x_N for noise factors, we can write $y = f(x_C, x_N)$, where the stochastic variation in y is induced by that in x_N. The expected loss is the average of L over the variation in x_N, i.e.,

$$R = R(x_C) = E_{x_N} L(f(x_C, x_N) - T). \tag{1.2}$$

The purpose of robust parameter design can be formally defined as the task of finding an optimal set of control factor combinations to minimize R in (1.2).

Taguchi (1986) advocates the use of the quadratic loss function

$$L = k(y - T)^2 \tag{1.3}$$

in measuring quality loss. Usually the quadratic loss provides a good approximation to the true quality loss (with k properly determined) and is easy to compute theoretically. As will be demonstrated in Section 3, it is not a good performance measure for parameter design experiments because the $(\cdots)^2$ operation in (1.3) can create spurious interactions among factors that influence y. So instead of using the mean squared error $n^{-1} \sum_{i=1}^{n} (y_i - T)^2$ as a performance measure, Taguchi advocates the use of the signal-to-noise (SN) ratio

$$\eta = \log \Big\{ \frac{1}{n} (S_m - V_e)/V_e \Big\}, \tag{1.4}$$

where

$$S_m = n\overline{y}^2, \quad V_e = \frac{1}{n-1} \sum_{i=1}^{n} (y_i - \overline{y})^2, \qquad \overline{y} = \frac{1}{n} \sum_{i=1}^{n} y_i,$$

and $y_1, \ldots, y_n$ represent a sample of y with x_C fixed and x_N varying according to its distribution (or an experimental plan in parameter design experiment). Taguchi's two-step procedure consists of maximizing the SN ratio in (1.4) and then adjusting $\overline{y}$ toward the target T. Detailed discussions on this and other two step procedures can be found in León, Shoemaker and Kacker (1987) and León and Wu (1992). Taguchi's use of the SN ratio η has stirred some controversies. As shown by León et al. (1987), it can only be justified in very limited

situations. An extensive simulation study and analytic work by Bérubé (1997) further supported this claim. Even when η can be justified, it requires a further assumption that an adjustment factor (among the x_C's) exists, which influences the $\overline{y}$ value but not the η value. When an adjustment factor does not exist, the two-step approach would not work. Moorhead in his 1995 University of Waterloo Ph.D. Thesis demonstrated this on a brake rotor experiment and proposed an alternative strategy.

By recognizing that the loss function L can be decomposed into two monotone functions L^+ and L^- on each side of the origin (see Section 2), we propose that the observed data y be decomposed into two groups: $y_i \geq T$ and $y_i \leq T$. Then a separate modeling and analysis is done for each group, resulting in the two fitted models $\hat{B}_R$ and $\hat{B}_L$ given in Section 3. Because of the monotonicity of L^+ and L^-, the results of analysis can be translated into the loss function L. In Section 3 we also give a simple argument to show why the quadratic loss in (1.3) and the absolute error loss (denoted by L_1 in Section 3) can create spurious interactions when the underlying model is strictly additive. (Wu (1992) made a similar comment that the SN ratio (1.4) can create spurious interactions but gave no details.) By contrast our proposed approach does not suffer from this shortcoming since it models the data directly. We use a heat exchanger example to demonstrate the modeling technique in Section 3 and show how it can be used for parameter design optimization in Section 4. Concluding remarks and extensions are given in Section 5.

2. Decomposition of Loss Function Into Two Monotone Functions

The main idea here is to decompose the L function into two functions, each of which is monotone on $y \geq T$ and on $y \leq T$. Formally we define

$$\begin{aligned} B_R &= y - T = f(x_C, x_N) - T, \\ B_L &= T - y = T - f(x_C, x_N). \end{aligned} \tag{2.1}$$

Let D be the domain of f and

$$D^+ = \{(x_C, x_N)\epsilon D | f(x_C, x_N) \geq T\},$$

$$D^- = \{(x_C, x_N)\epsilon D | f(x_C, x_N) \leq T\}.$$

Then $D = D^+ \cup D^-$ and L can be rewritten as

$$L = L(y - T) = \begin{cases} L^+(B_R), (x_C, x_N)\epsilon D^+, \\ L^-(B_L), (x_C, x_N)\epsilon D^-, \end{cases} \tag{2.2}$$

where $L^+(z) = L(z)$ for $z \geq 0$ and $L^-(z) = L(-z)$ for $z \geq 0$. Because of the monotonicity assumption on L in (1.1), both L^+ and L^- in (2.2) are non-decreasing functions in B_R and B_L respectively. If L is a symmetric loss function, then $L^+ = L^-$. The decomposition in (2.2) plays a key role in the proposed approach. It allows parameter design optimization to be performed on two monotone loss functions, instead of on a generally non-monotone loss function. As will be demonstrated in Section 3, non-monotonicity of loss function like the quadratic loss can induce spurious interactions.

For asymmetric loss L, one can use

$$K = \int_0^A L^+(x)dx / \int_0^A L^-(x)dx, \tag{2.3}$$

where $A = \min\{\max B_R, \max B_L\}$, to compute the ratio of losses for deviations above target and below target. If both L^+ and L^- are linear loss functions, i.e.,

$$\begin{aligned} L^+(B_R) &= a_1 B_R + b_1 \\ L^-(B_L) &= a_2 B_L + b_2 \end{aligned} \tag{2.4}$$

the previous interpretation of K has a clearer and stronger justification. Without loss of generality, we can assume $L(0) = 0$, which implies $L^+(0) = b_1 = L^-(0) = b_2 = 0$. It is then easy to show that $K = a_1/a_2$. That is, K is the ratio of the rates of losses above and below T.

In general, because L^+ and L^- are monotone functions in B_R and B_L, we can reduce the study of L over D to the much easier problems of studying KB_R over D^+ and B_L over D^-. Since both B_R and B_L are linear functions in y, it is easier to model B_R and B_L as a function of x_C and the parameter design optimization results are more reliable. The problem of having spurious interactions as mentioned before can also be avoided.

To complete the definitions of B_R and B_L, we further define $B_R(x) = 0$ for $x \epsilon D^-$ and $B_L(x) = 0$ for $x \epsilon D^+$, where $x = (x_C, x_N)$. Then we can write

$$|y - T| = B_R + B_L. \tag{2.5}$$

3. Modeling of Monotone Loss Functions

In this section we shall study the statistical modeling of

$$B_R = f(x_C, x_N) - T, \quad (x_C, x_N) \epsilon D^+,$$

and

$$B_L = T - f(x_C, x_N), \quad (x_C, x_N) \epsilon D^-.$$

Because the loss functions L^+ and L^- are monotone in B_R and B_L respectively, the modeling of B_R and B_L allows the investigators to separately optimize L^+ and L^-, which make up the loss function L. Suppose the observed data are $(y_1, x_1), \ldots, (y_n, x_n)$, where x_i represent the ith setting of x_C and x_N in a parameter design experiment. Based on this data some regression models can be built as follows:

$$\hat{y} = \hat{f}(x_C, x_N), \quad (x_C, x_N)\epsilon D \tag{3.1}$$

$$\hat{B}_R = \hat{f}_R(x_C, x_N) - T, \quad (x_C, x_N)\epsilon D^+ \tag{3.2}$$

$$\hat{B}_L = T - \hat{f}_L(x_C, x_N), \quad (x_C, x_N)\epsilon D^- \tag{3.3}$$

where $\hat{f}, \hat{f}_R$ and $\hat{f}_L$ are parametric or semi-parametric models depending on the nature of data and problem. The fitted model $\hat{f}$ is based on all the data, $\hat{f}_R$ is based on those with $y_i \geq T$ and $\hat{f}_L$ on those with $y_i \leq T$.

Based on (3.1), for any values of x_C and x_N, even those not in the experiment, we can predict

$$\hat{y} = \hat{f}(x_C, x_N).$$

With this prediction model, we can now give a classification of D into D^+ and D^- based on the observed data: D^+ consists of those (x_C, x_N) with $\hat{f}(x_C, x_N) \geq T$ and D^- consists of those (x_C, x_N) with $\hat{f}(x_C, x_N) < T$. Therefore, we can use the more reliable model (3.2) to do model fitting and prediction for any values of (x_C, x_N) in D^+. Similarly, we can use model (3.3) to do model fitting and prediction for D^-. Therefore we can use $\hat{B}_R$ and $\hat{B}_L$ to identify factor settings that minimize the loss L^+ and L^- respectively.

Note that the accuracy in $\hat{B}_R$ and $\hat{B}_L$ depends on the accuracy in the fitted model for $\hat{y}$. Errors in $\hat{y}$ may lead to using $\hat{B}_R$ to predict B_L or vice versa. This point will be addressed in the numerical study in Section 4.

Because $B_R = y - T$ for $x\epsilon D^+$ and $B_L = T - y$ for $x\epsilon D^-$, apart from translation and restriction to subset (e.g., D^+ or D^-), B_R and B_L keep the same relationships as y. The same is not true for the quadratic loss

$$L_2 = (y - T)^2,$$

or the absolute error loss

$$L_1 = |y - T|.$$

First we take the simplest case of a linear relation between y and $x : y = 2 + x$, and $T = 1$. The functional relationships between L_2 and x, and between L_1 and x are depicted in Figures 1 and 2. Obviously L_2 is a quadratic function in x. So a spurious curvature effect is created. Even by taking the square root of L_2, L_1 still

has a quadratic curvature effect in x, which can be verified by fitting a quadratic curve (dashed curve in Figure 2) to the L_1 function.

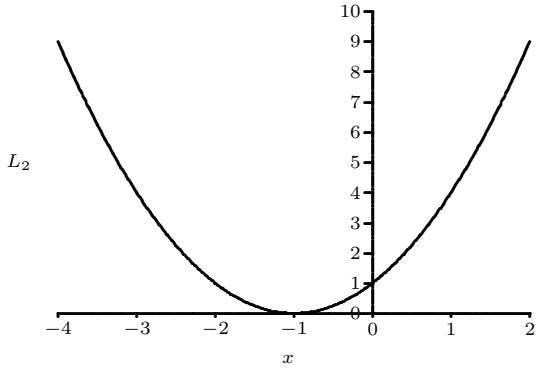

Figure 1. L_2 as a quadractic function in x

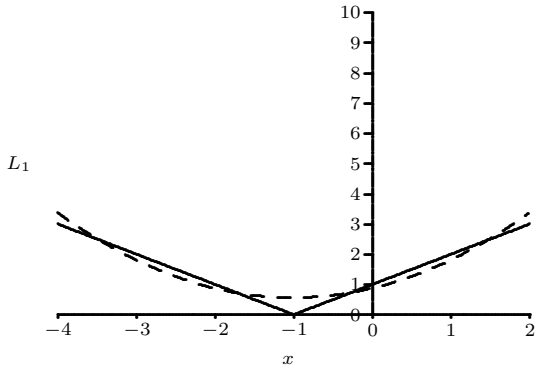

Figure 2. L_1 as a function of x (solid line). The best quadractic approximation, $(\frac{5}{16})(x+1)^2 + (\frac{9}{16})$ (dotted curve) over $[-4, 2]$.

To extend this conclusion to the multi-variable case, consider next the two-factor model:

$$y = 2 + x_1 + x_2 \text{ and } T = 1.$$

Obviously $L_2 = (y-T)^2 = (1+x_1+x_2)^2 = 1 + 2x_1 + 2x_2 + 2x_1x_2 + x_1^2 + x_2^2$ has both the curvature effects (i.e., x_1^2 and x_2^2) and the interaction effect (i.e. x_1x_2). Even by taking its square root,

$$L_1 = \begin{cases} 1 + x_1 + x_2, & \text{for } x_1 + x_2 \geq -1, \\ -1 - x_1 - x_2, & \text{for } x_1 + x_2 \leq -1, \end{cases}$$

still has both the curvature and interaction effects. To see this, first rewrite $w = x_1 + x_2$, then $L_1 = 1 + w$ for $w \geq -1, L_1 = -1 - w$ for $w \leq -1$, which has exactly the same relationship as in the one factor case described above. As shown

there, L_1 has a curvature effect in w and can be approximated by a quadratic model $c_0+c_1(w+1)^2$. Since $w = x_1+x_2, (w+1)^2 = 1+2x_1+2x_2+2x_1x_2+x_1^2+x_2^2$, therefore, a curvature effect in w amounts to both the curvature effects (i.e., x_1^2 and x_2^2) and the interaction effect x_1x_2. These two simple examples demonstrate how *the non-monotonicity in the loss function L_1 and L_2 can create spurious curvature and interaction effects in the models that fit L_1 or L_2 directly.*

Because spurious curvature and interaction effects will make it more difficult to perform successful parameter design experiments, it is advised that direct statistical modeling of non-monotone loss functions like L_1, L_2 or η should be avoided. Another important observation is that the modeling of f_R and f_L in (3.2) and (3.3) is usually easier than the modeling of f because a simple parsimonious model can better fit the data in a smaller region like D^+ or D^- than over the larger region D. Both points will be confirmed in the following empirical study.

Example. Design of Heat Exchanger

The inlet temperature T_1 of a heat exchanger fluctuates in the range 670° C ± 30° C, and the flow rate V fluctuates in the range $(42000 \pm 2000) \times 1/21 \times [1 + (T_1/273)]$. The three control factors are d (outside diameter of pipe), D (diameter of heat exchanger), and L/D, where L is the length of pipe. The objective of the design is to select the control factor levels so that the outlet temperature T_2 does not deviate from the target temperature 360° C by more than 15° C for values of T_1 and V in their respective ranges.

According to the derivations in Cui et al. (1983), T_2 is related to the other factors by the following equations:

$$T_2 = (T_1 - T_g)e^{-A} + T_g, \qquad T_g = 222.7^\circ C$$

and

$$A = \frac{57.1(L/D)D^3\lambda_t}{Vd^2\rho_t C_{pm}}[1.53 \times 10^{-3}\frac{d_i\rho_t}{\mu_t}\frac{V}{D^2}(\frac{d}{d_i})^2]^{0.8}[\frac{C_{pm}\mu_t}{\lambda_t}]^{0.4}, \qquad (3.5)$$

where d_i = inside diameter of pipe $(m), \rho_t$ = gaseous density $(Kg/m^3), \mu_t$ = gaseous viscosity, λ_t = thermal conductivity $(KCal/s \cdot m \cdot^\circ D) = 3.335 \times 10^{-5}, C_{pm}$ = specific heat $(KCal/Kg \cdot^\circ C)$.

Since T_1 and V cannot be fixed at the design stage and can vary during operation of the exchanger, they are treated as noise factors. The layouts of control factors and noise factors are given in Tables 1 and 2. The parameter d_i in (3.5) is a deterministic function of d (Table 3); ρ_t, C_{pm}, and μ_t in (3.5) are deterministic functions of T_1 (Table 4). The same example was used by Wu, Ma and Mao (1990) to illustrate a different technique they proposed.

Table 1. Levels of control factors

Level	d	D	L/D
1	.025	.8	3
2	.032	1.0	4
3	.038	1.2	5

Table 2. Levels of noise factors

Level	T_1	V
1	640	$\frac{40000}{21} \times (1 + \frac{T_1}{273})$
2	670	$\frac{42000}{21} \times (1 + \frac{T_1}{273})$
3	700	$\frac{44000}{21} \times (1 + \frac{T_1}{273})$

Table 3. d_i as function of d

d	d_i
.025	.019
.032	.025
.038	.031

Table 4. ρ_t, C_{pm}, and μ_t as functions of T_1

T_1	ρ_t	C_{pm}	μ_t
640	5.286	1.024	2.83×10^{-5}
670	5.185	1.029	2.89×10^{-5}
700	5.089	1.031	2.93×10^{-5}

We use the following quadratic regression model to approximate the relationship between T_2 and the five factors

$$y = a_0 + \sum_{1}^{5} a_i x_i + \sum_{1}^{5} b_i x_i^2 + \sum_{i>j}^{5} c_{ij} x_i x_j + \epsilon, \tag{3.6}$$

where $(x_1, x_2, x_3) = (d, D, \frac{L}{D})$ are the control factors, $(x_4, x_5) = (T_1, V)$ are the noise factors, and ϵ represents the modeling error. There is no experimental error because it is a mathematical model. From the model (3.5), we generated a set of data $(x_{i1}, \ldots, x_{i5}, y_i), i = 1, 2, \ldots, 243$, for all the $3^5 = 243$ combinations of factor levels, where $x = (x_1, \ldots, x_5)$ are the values of the five input variables in (3.5) and y is the output T_2. By fitting the model (3.6) to these data, we obtain the fitted model for $\hat{T}_2$

$$\begin{aligned}\hat{T}_2 = {} & 313.6 + 31.14x_1 - 31.98x_2 - 28.72x_3 + 6.77x_4 + 1.05x_5 \\ & +0.34x_1^2 - 4.05x_1x_2 - 3.62x_1x_3 + 1.87x_1x_4 + 4.61x_2^2 + 3.46x_2x_3 \\ & +1.91x_2x_4 + 4.54x_3^2 - 1.72x_3x_4, \quad \text{for } x \epsilon D. \end{aligned} \tag{M.1}$$

Similarly by applying (3.6) to the data in D^+ (i.e. $y \geq 360$) and D^- (i.e., $y \leq 360$), we have the fitted models for B_R and B_L,

$$\begin{aligned}\hat{B}_R = {} & 34.15 + 25.70x_1 - 30.74x_2 - 32.23x_3 + 10.19x_4 + 1.31x_5 - 1.28x_1^2 \\ & +0.46x_1x_2 + 0.56x_1x_3 + 1.30x_1x_4 - 0.02x_1x_5 + 1.43x_2^2 - 0.54x_2x_3 - 1.45x_2x_4 \\ & +2.13x_3^2 - 1.50x_3x_4 - 0.16x_4^2 + 0.06x_4x_5 - 0.02x_5^2, \quad \text{for } x \epsilon D^+. \end{aligned} \tag{M.2}$$

$$\begin{aligned}\hat{B}_L = {} & 69.42 - 28.59x_1 + 26.46x_2 + 24.39x_3 - 5.41x_4 - 0.97x_5 - 1.29x_1^2 \\ & +5.25x_1x_2 + 4.88x_1x_3 - 1.78x_1x_4 - 0.19x_1x_5 - 4.77x_2^2 - 4.53x_2x_3 \\ & +1.69x_2x_4 + 0.17x_2x_5 - 4.79x_3^2 + 1.51x_3x_4 + 0.16x_3x_5 + 0.15x_4^2 \\ & -0.07x_4x_5, \quad \text{for } x \epsilon D^-. \end{aligned} \tag{M.3}$$

For the purpose of comparison, we also used the same data and model to fit L_1 and L_2 over D, resulting in the fitted models,

$$\begin{aligned}\hat{L}_1 &= 43.11 - 14.34x_1 + 13.91x_2 + 12.26x_3 - 1.23x_4 \\ &\quad +5.58x_1^2 - 10.91x_1x_2 - 9.67x_1x_3 + 2.15x_1x_4 + 4.06x_2^2 + 11.32x_2x_3 \\ &\quad -2.76x_2x_4 + 2.64x_3^2 - 2.04x_3x_4, \quad \text{for } x\epsilon D. \qquad \text{(M.4)}\\ \hat{L}_2 &= 2347.79 - 1834.58x_1 + 1757.18x_2 + 1540.18x_3 - 159.98x_4 \\ &\quad +849.15x_1^2 - 1572.79x_1x_2 - 1404.85x_1x_3 + 314.45x_1x_4 + 640.46x_2^2 \\ &\quad +1515.53x_2x_3 - 349.29x_2x_4 + 464.34x_3^2 - 317.53x_3x_4, \quad \text{for } x\epsilon D. \qquad \text{(M.5)}\end{aligned}$$

It is interesting to note the similarity between the models in (M.4) and (M.5). Both the terms in the models and their signs are identically the same. The models in (M.1) - (M.5) were selected by using stepwise regression with 0.01 as the critical level for entering and deleting variables.

To compare the prediction errors of the five models (M.1) - (M.5), we use the mean absolute error (MAE) criterion

$$e = \frac{1}{n}\sum_{i=1}^{n}|y_i - \hat{y}_i|$$

and the maximum error (MXE) criterion

$$q = \max_{1\le i\le n}|y_i - \hat{y}_i|.$$

Because L_2 does not have the same dimension as y, we need to modify the previous criteria for L_2 as follows,

$$e_{L_2} = \Big(\frac{1}{n}\sum_{i=1}^{n}|L_{2i} - \hat{L}_{2i}|\Big)^{\frac{1}{2}}$$

and

$$q_{L_2} = \max_{1\le i\le n}|L_{2i} - \hat{L}_{2i}|^{\frac{1}{2}},$$

so that the e and q values for the five approaches are on the same scale. Here $n = 243$.

Table 5. Comparison of prediction errors of the five approaches

	MAE (e value)	MXE (q value)	F value
B_R	0.065	0.207	243305.5
B_L	0.623	2.084	16604.6
T_2	1.641	6.933	10190.5
L_1	13.727	47.348	58.2
L_2	29.977*	69.026*	209.9

From Table 5 it is clear that B_R and B_L have much smaller prediction errors than those based on either the loss function L_2 or L_1. This confirms our previous theoretical argument that use of L_1 and L_2 can create higher order effects like curvatures and interactions which are not captured by the quadratic model (3.6). Unless the fitted model has higher order terms, the prediction error cannot be further reduced. On the other hand use of B_R and B_L does not create these higher order effects. Therefore a lower order model like (3.6) can be effectively used for prediction.

Another impressive finding from Table 5 is that $\hat{B}_R$ and $\hat{B}_L$ outperform $\hat{T}_2$. Again this supports our previous argument (before the Example) in favor of $\hat{B}_R$ and $\hat{B}_L$ that, by working over a smaller region of x, B_R and B_L can be better approximated by a lower order model. To further explain this finding, we examine the signs of terms in the models (M.1) - (M.3). Because $B_R = T_2 - T$ for $x \epsilon D^+$, a term with positive (and respectively negative) coefficient in $\hat{B}_R$ is said to have a positive (and respectively negative) correlation with $\hat{T}_2$. On the other hand, because $B_L = T - T_2$ for $x \epsilon D^-$, a term with positive (and respectively negative) coefficient in $\hat{B}_L$ is said to have a negative (and respectively positive) correlation with $\hat{T}_2$. These terms are tabulated in Table 6. From the table it becomes clear that each of the terms $x_1^2, x_1x_2, x_1x_3, x_1x_5$ and x_2x_3 has the opposite effects on $\hat{T}_2$ depending on $x \epsilon D^+$ or $x \epsilon D^-$. (These are the terms that appear in the diagonal blocks or off-diagonal blocks of Table 6.) On the other hand this opposite effect on T_2 cannot be seen from a single model like $\hat{T}_2$. For example, the term x_1^2 has the coefficient -1.28 in $\hat{B}_R$ and -1.29 in $\hat{B}_L$. Although these two coefficients are for data over the two disjoint regions D^+ and D^-, they almost get "canceled" in the model $\hat{T}_2$ which has 0.34 as the coefficient of x_1^2. By using two separate models over D^+ and D^- we can detect a nonlinear effect of x_1^2 on T_2, which is lost in the model $\hat{T}_2$.

The last column in Table 5 gives the value of the F statistics for the fitted model for each of the five methods. Not surprisingly both B_R and B_L give a much better fit than the T_2 model. The F values for L_1 and L_2 are much smaller, indicating a very poor model fit of the L_1 and L_2 loss functions.

Table 6. Terms in $\hat{B}_R$ and $\hat{B}_L$ that have positive or negative correlations with $\hat{T}_2$.

	Positive	Negative
	Correlation Terms	
$\hat{B}_R$	$x_1, x_4, x_5, x_1x_2, x_1x_3$	$x_2, x_3, x_1^2, x_1x_5, x_2x_3$
	$x_1x_4, x_2^2, x_3^2, x_4x_5$	$x_2x_4, x_3x_4, x_4^2, x_3^2$
$\hat{B}_L$	$x_1, x_4, x_5, x_1^2, x_1x_4$	$x_2, x_3, x_1x_2, x_1x_3, x_2x_4$
	$x_1x_5, x_2^2, x_2x_3, x_3^2, x_4x_5$	$x_2x_5, x_3x_4, x_3x_5, x_4^2$

Finally we note that, for this example, when use of $\hat{T}_2$ results in the misclassification of D^+ as D^- (or D^- as D^+), the maximum prediction error over the misclassified x

$$\max\{|\hat{B}_R(x) - B_L(x)|, |\hat{B}_L(x) - B_R(x)|\} = 0.640,$$

is smaller than the maximum 3.68 of $|\hat{T}_2 - T_2|$ over the same set of misclassified x. Similarly the mean absolute error over the misclassified x,

$$\frac{1}{\#}\sum |\hat{B}_R(x) - B_L(x)| + \frac{1}{\#}\sum |\hat{B}_L(x) - B_R(x)| = 0.415,$$

is still smaller than the corresponding value 3.3 for $|\hat{T}_2 - T_2|$. This provides further evidence that $\hat{B}_R$ and $\hat{B}_L$ outperform $\hat{T}_2$.

4. Parameter Design Based on $\hat{B}_R$ and $\hat{B}_L$

In this section we use the heat exchanger example to illustrate an efficient parameter design procedure based on the fitted models $\hat{B}_R$ and $\hat{B}_L$.

We arranged the three control factors in a control (or inner) array as part of the orthogonal array $L_9(3^4)$ (see Table 7) and the two noise factors in a noise (i.e. outer) array according to $L_9(3^2)$.

Table 7.

i	$x_1(D)$	$x_2(d)$	$x_3(L/D)$
1	1	1	3
2	1	2	1
3	1	3	2
4	2	1	2
5	2	2	3
6	2	3	1
7	3	1	1
8	3	2	2
9	3	3	3

For each of the nine settings in the control array, say setting i, we can compute $\hat{y}_{i1}, \cdots, \hat{y}_{i9}$ for the nine settings in the noise array according to the fitted model (M.1), where $\hat{y} = \hat{T}_2$. According to whether $\hat{y}_{ij} \geq T$ or $\leq T$, we classify the factor settings as in D^+ or D^- and then compute the corresponding $\hat{B}_R$ and $\hat{B}_L$ values (using (M.2) and (M.3) respectively). Then we use

$$B^*_{Ri} = \max_{x_{ij} \in D^+} \{\hat{B}_R(x_{ij})\}, \tag{4.1}$$

and

$$B^*_{Li} = \max_{x_{ij} \in D^-} \{\hat{B}_L(x_{ij})\}, \tag{4.2}$$

as performance measures for the ith control factor setting, where x_{ij} denotes the x value for the ith control setting and jth noise setting, $i = 1, \ldots, 9; j = 1, \ldots, 9$.

The objective of design in the example is to identify appropriate control factor setting to minimize B^*_{Ri} and B^*_{Li} and ensure that they are smaller than 15°C. The detailed steps are as follows:

1. Compute the B^*_{Ri} and B^*_{Li} values for each of the control factor settings. The results are given in Table 8.

Table 8. Values of B^*_R and B^*_L for each control factor setting i

i	1	2	3	4	5	6	7	8	9
B^*_{Ri}	0.000	0.000	0.000	16.798	0.000	0.000	107.733	4.319	0.000
B^*_{Li}	81.469	58.561	113.975	8.213	81.856	50.289	0.000	18.554	83.766

2. For each level of a control factor, compute the mean of the B^*_{Ri} and B^*_{Li} values over the other factors, which are denoted by $\overline{B}^*_R$ and $\overline{B}^*_L$ in Table 9. Note that this computation is the same as the one for computing factor main effects. Then we compute the sum of squares of these mean values for each control factor and its percent contribution (denoted by ρ^+ for $\overline{B}^*_R$ and ρ^- for $\overline{B}^*_L$) toward the total sum of squares. Again these computations are analogous to those in ANOVA.

Table 9. Values of $\overline{B}^*_R$ and $\overline{B}^*_L$ for each control factor level and the percent contribution ρ^+ and ρ^- for each control factor

	$\overline{B}^*_R$			$\rho^+(\%)$	$\overline{B}^*_L$			$\rho^-(\%)$
	1	2	3		1	2	3	
x_1	0.000	16.798	112.052	24.194	254.005	140.358	102.320	33.747
x_2	124.531	4.319	0.000	33.102	89.682	158.971	248.030	34.141
x_3	107.733	21.117	0.000	21.592	108.850	140.743	247.090	28.389

3. From Table 8, identify the setting(s) with small values of both B^*_R and B^*_L. For the example, $i = 4, B^*_{R4} = 16.798, B^*_{L4} = 8.213$ and the levels are $x_{1,2}$ $x_{2,1}$ and $x_{3,2}$. Obviously we need to further reduce the B^*_R value (to be smaller than 15) by selecting a different factor combination. From Table 9, factor x_2 (i.e. D) has the largest ρ^+ value, that is, the best potential to affect or reduce the B^*_R value. From the $\overline{B}^*_R$ and $\overline{B}^*_L$ values in Table 9 for x_2, it is clear that level 2 is the compromise choice, because level 1 has the largest $\overline{B}^*_R$ value while level 3 has the largest $\overline{B}^*_L$ value. In order to avoid the problem

of increasing the B_L^* value when B_R^* is being reduced, we can choose another factor x_1 or x_3 to control or reduce the B_L^* value. From Table 9 level 1 of x_3 or level 3 of x_1 appears to be the best choice. So this adjustment of levels in the original setting for $i = 4$ leads to

$$x_{1,2}\ x_{2,2}\ x_{3,1}, B_R^* = 12.43, B_L^* = 12.33,$$

$$x_{13}, x_{22}, x_{32}, B_R^* = 4.335, B_L^* = 19.318.$$

Obviously, x_{12}, x_{22}, x_{31} satisfy the design requirement that $B_R^* \leq 15$ and $B_L^* \leq 15$.

5. Concluding Remarks

Using B_R and B_L as twin performance measures for the control factor settings has a strong statistical justification. Their linearity (or more generally monotonicity) renders a simpler and more parsimonious model and better prediction results. They are particularly suited for paramter designs with mathematical models. If empirical models need to be established for the relationship between y and x, the proposed approach needs to be further developed. The procedure for choosing factor levels as in Step 3 of Section 4 needs a more careful study. One other advantage of the proposed approach is that it does not require the existence of an adjustment factor. It should therefore be applicable to parameter designs for general loss functions (see Moorhead and Wu (1996)) even when an adjustment factor cannot be identified empirically or on physical ground.

In the heat exchanger example, because of the engineering design requirement, we used the maximum deviations B_R^* and B_L^* as performance measures. In other situations, use of average deviations may be more appropriate, i.e.,

$$EB_{Ri} = \frac{1}{m_i} \sum_{x_{ij} \epsilon D^+} B_R(x_{ij}),$$

and

$$EB_{Li} = \frac{1}{n_i} \sum_{x_{ij} \epsilon D^-} B_L(x_{ij}),$$

$m_i + n_i = n, j = 1, \ldots, n$. These alternative measures should be kept in mind when dealing with actual design problems.

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