Invariance properties and statistical inference for circular data
Supplementary material

Appendix A : Theoretical applications

In this section, using Theorem 1, we demonstrate that some of the distributions proposed in the literature do not posses the ICID and ICO properties. We show how to verify the invariance properties with respect to the reference system and how to specify the invariant counterpart of any non-invariant circular distribution. To simplify the notation, in this section we avoid the use of the modulus 2π operator, and, if not specified differently, we assume $\Theta^* = \delta(\Theta + \xi) \in \delta(\mathbb{D} + \xi)$.

Wrapped skew normal Let $\Theta \sim WSN(\mu, \sigma^2, \alpha)$, with $\mu \in \mathbb{R}$, $\sigma^2 \in \mathbb{R}^+$ and $\alpha \in \mathbb{R}$,

$$f_\Theta(\theta|\mu, \sigma^2, \alpha) = \sum_{k=-\infty}^{\infty} \frac{1}{\pi\sigma} e^{\frac{(\theta+2\pi k-\mu)^2}{2\sigma^2}} \int_{-\infty}^{\alpha\left(\frac{\theta+2\pi k-\mu}{\sigma}\right)} e^{-\frac{t^2}{2}} dt.$$

Following Theorem 1, the density of Θ^* is

$$f_{\Theta^*}(\theta^*|\mu, \sigma^2, \alpha) = \sum_{k=-\infty}^{\infty} \frac{1}{\pi\sigma} e^{\frac{(\delta\theta^*-\xi+2\pi k-\mu)^2}{2\sigma^2}} \int_{-\infty}^{\alpha\left(\frac{\delta\theta^*-\xi+2\pi k-\mu}{\sigma}\right)} e^{-\frac{t^2}{2}} dt. \tag{1}$$

Since equation (1) can be seen as a density of a WSN with parameters $\delta(\xi + \mu)$, σ^2 and α, the requirement of Theorem 1 is met. Hence the WSN is ICID and ICO.

Discrete circular uniform The density of the discrete circular uniform (Mardia and Jupp, 1999) is: $f_\Theta(\theta) = 1/l$ where l is the number of distinct points equally spaced in $\mathbb{D}$. Clearly Θ^* has the same exact density of Θ and then the discrete circular uniform is ICID and ICO.

Wrapped Poisson A discrete circular variable Θ that can assume l equally spaced points on the circle is said to follow a wrapped Poisson distribution (WP) with parameter $\lambda > 0$ (Mardia and Jupp, 1999), i.e. $\Theta \sim WP_l(\lambda)$, if it has pdf

$$f_\Theta(\theta|\lambda) = \sum_{k=0}^{\infty} \frac{\lambda^{\theta \frac{l}{2\pi} + kl} e^{-\lambda}}{\left(\theta \frac{l}{2\pi} + kl\right)!}.$$

The one of Θ^* is

$$f_{\Theta^*}(\theta^*|\lambda, \delta, \xi) = \sum_{k=0}^{\infty} \frac{\lambda^{(\delta\theta^* - \xi)\frac{l}{2\pi} + kl} e^{-\lambda}}{\left((\delta^*\theta - \xi)\frac{l}{2\pi} + kl\right)!}. \tag{2}$$

The WP is not ICID and ICO. Let $\varphi_\Theta(p)$ and $\varphi_{\Theta^*}(p)$ be the characteristic function of Θ and Θ^*, respectively. From (Girija *et al.*, 2014) we have that

$$\varphi_\Theta(p) = e^{-\lambda\left(1 - \cos\frac{2\pi p}{l}\right)} e^{i\lambda \sin\frac{2\pi p}{l}}$$

and

$$\varphi_{\Theta^*}(p) = e^{i\delta\xi p} e^{-\lambda\left(1 - \cos\frac{\delta 2\pi p}{l}\right)} e^{i\lambda \sin\frac{\delta 2\pi p}{l}} = e^{-\lambda\left(1 - \cos\frac{2\pi p}{l}\right)} e^{i\left(\delta\xi p + \lambda \sin\frac{\delta 2\pi p}{l}\right)}.$$

To be ICID and ICO, the previous characteristics function should have the same functional form, and this is verified only for $\delta = 1$ and $\xi = 0$, i.e. if Θ and Θ^* are the same random variable. It follows that the WP is not ICID and ICO.

The density (2) arises by applying Proposition 1 and it is the invariant version of the WP, i.e. Θ^* is distributed as an *invariant wrapped Poisson* (IWP) with parameters λ, δ and ξ, i.e. $\Theta^* \sim IWP_l(\lambda, \delta, \xi)$. As for the IWE, the domain of Θ^* depends on its parameters and again, we prefer to write the density as

$$f_{\Theta^*}(\theta^*|\lambda, \delta, \xi) = \sum_{k=0}^{\infty} \frac{\lambda^{[(\delta\theta^* - \xi) \bmod (2\pi)]\frac{l}{2\pi} + kl} e^{-\lambda}}{\left([(\delta\theta^* - \xi) \bmod (2\pi)]\frac{l}{2\pi} + kl\right)!}, \; \theta^* \in \mathbb{D}.$$

The WP trigonometric moments are

$$\alpha_p = e^{-\lambda\left(1 - \cos\frac{2\pi p}{l}\right)} \cos\left(\lambda \sin\frac{2\pi p}{l}\right)$$

and

$$\beta_p = e^{-\lambda\left(1 - \cos\frac{2\pi p}{l}\right)} \sin\left(\lambda \sin\frac{2\pi p}{l}\right),$$

then the ones of the IWP are, respectively,

$$e^{-\lambda\left(1-\cos\frac{2\pi p}{l}\right)}\cos\left(p\xi+\lambda\sin\frac{2\pi p}{l}\right)$$

and

$$\delta e^{-\lambda\left(1-\cos\frac{2\pi p}{l}\right)}\sin\left(p\xi+\lambda\sin\frac{2\pi p}{l}\right).$$

Accordingly, the IWP circular mean is $\mu_1 = \delta\xi + \delta\lambda\sin(2\pi/l)$ and the circular concentration is $c_1 = \exp(-\lambda\,(1-\cos(2\pi/l)))$.

Wrapped Weibull Sarma *et al.* (2011) propose the wrapped Weibull:

$$f_\Theta(\theta|\lambda) = \sum_{k=0}^{\infty}\lambda(\theta+2\pi k)^{\lambda-1}e^{-(\theta+2\pi k)^\lambda},\ \lambda\in\mathbb{R}^+. \tag{3}$$

To show that (3) is not invariant, we use the characteristic functions. The one of the wrapped Weibull is

$$\varphi_\Theta(p) = i\sum_{k\in\mathbb{Z}^+_{odd}}\frac{p^k}{k!}\left(1+\frac{k}{\lambda}\right) - \sum_{k\in\mathbb{Z}^+_{even}}\frac{p^k}{k!}\left(1+\frac{k}{\lambda}\right), \tag{4}$$

where $\mathbb{Z}^+_{odd}$ and $\mathbb{Z}^+_{even}$ are respectively the even (zero included) and odd integer numbers. If Θ and Θ^* have the same distribution, their characteristic function must be of the same functional form. The characteristic function of Θ^* is

$$\begin{aligned}\varphi_{\Theta^*}(p) &= e^{i\delta\xi p}\varphi_\Theta(\delta p) = \cos(\delta\xi p)\varphi_\Theta(\delta p) + i\sin(\delta\xi p)\varphi_\Theta(\delta p)\\ &= \sin(\delta\xi p)\sum_{k\in\mathbb{Z}^+_{odd}}\frac{p^k}{k!}\left(1+\frac{k}{\lambda}\right) - \cos(\delta\xi p)\sum_{k\in\mathbb{Z}^+_{even}}\frac{(\delta p)^k}{k!}\left(1+\frac{k}{\lambda}\right)\\ &+ i\left(\cos(\delta\xi p)\sum_{k\in\mathbb{Z}^+_{odd}}\frac{p^k}{k!}\left(1+\frac{k}{\lambda}\right) - \sin(\delta\xi p)\sum_{k\in\mathbb{Z}^+_{even}}\frac{(\delta p)^k}{k!}\left(1+\frac{k}{\lambda}\right)\right).\end{aligned} \tag{5}$$

The real and imaginary parts of (4) and (5) differ unless $\delta = 1$ and $\xi = 0$, and then the wrapped Weibull is not ICID and ICO

Wrapped Lévy Following Fisher (1996), if a circular random variable Θ has pdf

$$f_\Theta(\theta|\mu,\sigma^2)=\sum_{k:\theta\frac{l}{2\pi}+2kl-\mu>0}^{\infty}\sqrt{\frac{\sigma^2}{2\pi}}\frac{e^{-\frac{\sigma^2}{2\left(\theta\frac{l}{2\pi}+2kl-\mu\right)}}}{\left(\theta\frac{l}{2\pi}+2kl-\mu\right)^{\frac{3}{2}}},\mu\in\mathbb{R},\sigma^2\in\mathbb{R}^+, \quad (6)$$

it is said to be distributed as a wrapped Lévy.
Here again to prove that (6) does not hold the ICID and ICO properties we exploit the characteristic function of Θ and Θ^* that are respectively $\varphi_\Theta(p)=\exp\left(i\mu p-\sqrt{-2i\sigma^2 p}\right)$ and $\varphi_{\Theta^*}(p)=\exp\left(ip\delta(\mu+\xi)-\sqrt{-2i\sigma^2\delta p}\right)$. The two characteristic functions are of the same kind if $\mu^*=\delta(\mu+\xi)$ and $\sigma^{2*}=\sigma^2\delta$ but if $\delta=-1$ then σ^{2*} is negative while, by definition of the Wrapped Lévy distribution, it must be positive. The distribution is not ICID and ICO.

Wrapped geometric Another example is the wrapped geometric proposed by Jayakumar and Jacob (2012). The pdf of Θ is

$$f_\Theta(\theta|\lambda)=\frac{\lambda(1-\lambda)^{\frac{\theta l}{2\pi}}}{1-(1-\lambda)^l}. \quad (7)$$

We can see that (7) has not the two properties because it is a decreasing function in Θ, its maximum value is reached always at $\Theta=0$ and it cannot satisfy the two properties given in Definitions 1 and 2.

Wrapped skew Laplace on integers The wrapped skew Laplace on integers of Jayakumar and Jacob (2012) has pdf

$$f_\Theta(\theta|d,q)=\sum_{k=0}^{\infty}\frac{(1-d)(1-q)}{1-dq}\left(d^{\theta\frac{l}{2\pi}+2kl}+q^{\left|-\theta\frac{l}{2\pi}-2kl\right|}\right),\ d,q\in(0,1).$$

In each interval $[l(k-1),l(k-1)+(l-1))$ the maximum value of the term inside the sum is reached at $l(k-1)$ that, over the circle, corresponds to the point $\Theta=0$. Here again, as with the wrapped geometric example, since the maximum value of the density is fixed on the point $\Theta=0$, the density cannot be ICID and ICO.

Wrapped binomial The wrapped binomial of Girija *et al.* (2014) has pdf

$$f_\Theta(\theta|q,n) = \sum_{k=0}^{k:n-(\theta\frac{l}{2\pi}+kl)\geq 0} \left(\begin{array}{c} n \\ \theta\frac{l}{2\pi}+kl \end{array} \right) q^{\theta\frac{l}{2\pi}+kl}(1-q)^{n-(\theta\frac{l}{2\pi}+kl)},$$

and characteristic function

$$\varphi_\Theta(p) = (1-q+qe^{ip})^n. \tag{8}$$

while the characteristic function of Θ^* is

$$\varphi_{\Theta^*}(p) = e^{i\delta\xi p}(1-q+qe^{i\delta p})^n. \tag{9}$$

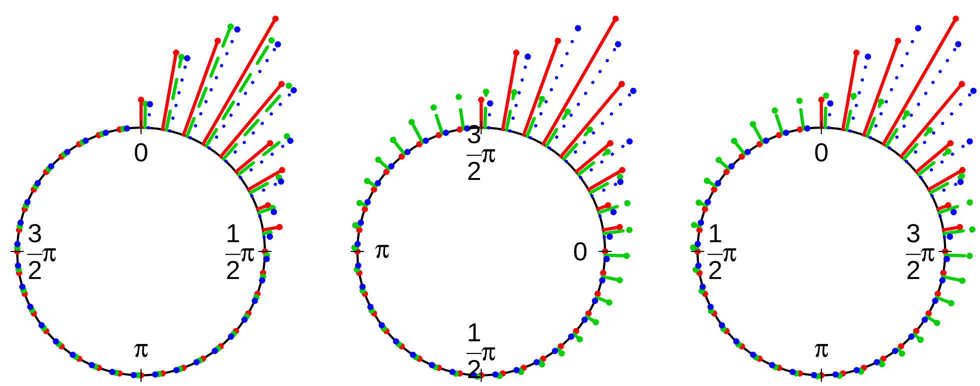
Equations (8) and (9) are not of the same functional form and then the wrapped binomial is not ICID and ICO.

Appendix B: Data examples

In this section we report two more examples, one simulated and one based on real data. The aim is to show some inferential problems that arise when non-invariant distribution are chosen

	WP			IWP				
	$\hat{\lambda}_i$	$\hat{\mu}_{1,i}$	$\hat{c}_{1,i}$	$\hat{\lambda}_i^*$	$\hat{\delta}_i^*$	$\hat{\xi}_i^*$	$\hat{\mu}_{1,i}^*$	$\hat{c}_{1,i}^*$
RS1	3.158	0.5484	0.9532	3.158	1	0	0.5484	0.9532
RS2	30.1403	5.2338	0.6326	3.158	1	4.7124	5.2608	0.9532
RS3	32.8657	5.7071	0.607	3.158	-1	0	5.7348	0.9532

Table 1: Simulated example - MLE of the WE and IWE parameters, circular mean and circular concentration



(a) RS1 (b) RS2 (c) RS3

Figure 1: Simulated example - Density used to simulate the data (solid line), wrapped Poisson (dashed line) and invariant wrapped Poisson (dotted line) density computed using the MLE of the parameters in the three reference systems

Artificial data - Wrapped Poisson We simulate 500 observations in the RS1 from a $WP_{36}(3)$. We show in Table 1 the MLEs and in Figure 1 the density used to simulate the data and the WP and IWP densities evaluated with the MLEs in the three reference systems are reported.

Data - Drosophila change of direction As illustration, we investigate the movements of a larva of the wild fruit fly Drosophila melanogaster whose position was recorded once per second over three minutes (Suster, 2000). Larval movement is characterized by linear locomotion, pausing and turning episodes. Here, we focus on the pauses as they often imply a change in the direction. Indeed, during a pause, the larva makes one or more sharp turns to identify a new direction to resume its locomotion. Most of the directions are concentrated and, accordingly, the distribution is unimodal, but there are also several observations spread over the circle, suggesting a non negligible variability. To describe the larva movements, the WE distribution is considered.

	WE			IWE				
	$\hat{\lambda}_i$	$\hat{\mu}_{1,i}$	$\hat{c}_{1,i}$	$\hat{\lambda}_i^*$	$\hat{\delta}_i^*$	$\hat{\xi}_i^*$	$\hat{\mu}_{1,i}^*$	$\hat{c}_{1,i}^*$
RS1	0.1392	1.4325	0.1379	0.7739	1	5.4127	0.0417	0.1101
RS2	·	·	·	0.7739	1	3.8419	4.7541	0.1101
RS3	·	·	·	0.7739	-1	5.4127	6.2415	0.1101

Table 2: Drosophila example - MLE of the WE and IWE parameters, circular mean and circular concentration

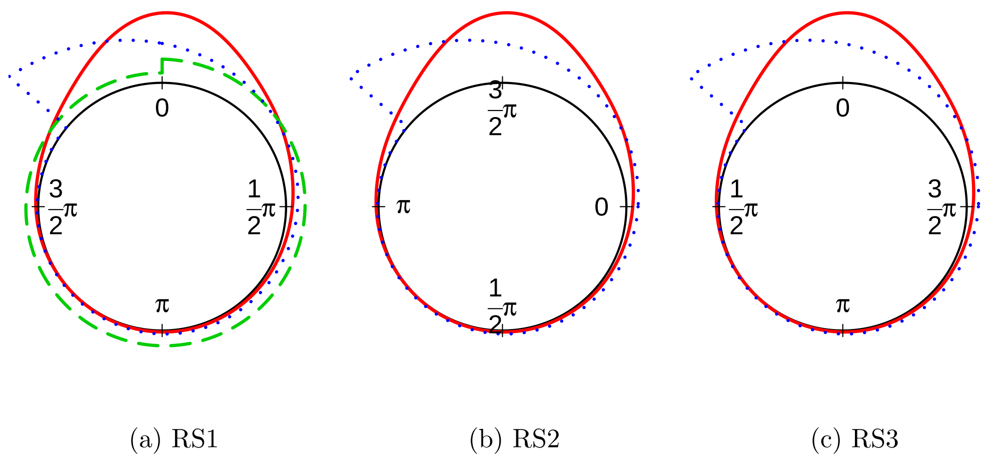


Figure 2: Drosophila example - Density estimate of observed data (solid line), wrapped exponential (dashed line) and invariant wrapped exponential (dotted line) density computed using the MLE of the parameters in the three reference systems

Table 2 and Figure 2 show, respectively, the MLEs of the parameters and the plot of the observed data with the WE and IWE densities evaluated using the MLEs, in the three reference systems.

References

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