Kernel-Based Adaptive Randomization Toward Balance in Continuous and Discrete Covariates

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Supplementary Material

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S.1 Necessary results for Lemma 1

For a new data point X_{n+1}, (A.1) reduces to

$$U_{nu}(X_{n+1}) = \frac{\sum_{i=1}^{n} K_{h_n}(X_i - X_{n+1}) I_{iu}}{\sum_{i=1}^{n} K_{h_n}(X_i - X_{n+1})},$$

and $\mathbf{U}_n(X_{n+1}) = \{U_{n1}(X_{n+1}), \ldots, U_{n(m-1)}(X_{n+1})\}^{\mathrm{T}}$. We set $\pi_u\{\mathbf{U}_n(X_{n+1})\}$ to be the probability of assigning the new observation to arm u. To derive the asymptotic property of $D_{nu} = \sum_{i=1}^{n}(I_{iu} - \kappa_u)X_i$, we first state useful lemmas that support Lemma 1.

Lemma 1. *Suppose that n subjects have been enrolled in a clinical* *trial. For a new data point X_{n+1} and arms u and v, we have*

$$E\left\{\sum_{i=1}^{n} K_{h_n}(X_i - X_{n+1})(I_{iu} - \kappa_u)\right\}^2 = O(nh_n^{-1})$$

and

$$\Big|\sum_{i=1}^{n} K_{h_n}(X_i - X_{n+1})(I_{iu} - \kappa_u)X_{n+1}\Big| = O_p(n^{1/2}h_n^{-1/2}).$$

Proof: Let $\mathcal{F}_j$ be a sigma field generated by all the event history up to stage j. Suppose that a new participant with covariate $X_{n+1} = x_0$ is to be allocated. We define a function $\eta(x, y) = I(x \neq y)(x - y) + I(x = y)x$, then

$$\begin{aligned}
& E\left(\left[\sum_{i=1}^{j+1} K_{h_n}\{\eta(X_i, x_0)\}(I_{iu} - \kappa_u)\right]^2 \Bigg| \mathcal{F}_j, X_{j+1} = x_0\right) \\
= \; & E\left(\left[\sum_{i=1}^{j} K_{h_n}\{\eta(X_i, x_0)\}(I_{iu} - \kappa_u)\right]^2 \Bigg| \mathcal{F}_j, X_{j+1} = x_0\right) \\
& +2\left[\sum_{i=1}^{j} K_{h_n}\{\eta(X_i, x_0)\}(I_{iu} - \kappa_u)\right] \\
& \times E\left[K_{h_n}\{\eta(x_0, x_0)\}(I_{(j+1)u} - \kappa_u)\Big| \mathcal{F}_j, X_{j+1} = x_0\right] \\
& +E\left[\left\{K_{h_n}(x_0)(I_{(j+1)u} - \kappa_u)\right\}^2 \Big| \mathcal{F}_j, X_{j+1} = x_0\right] \\
\leq \; & E\left(\left[\sum_{i=1}^{j} K_{h_n}\{\eta(X_i, x_0)\}(I_{iu} - \kappa_u)\right]^2 \Bigg| \mathcal{F}_j, X_{j+1} = x_0\right) \\
& +E\left[\left\{K_{h_n}(x_0)(I_{(j+1)u} - \kappa_u)\right\}^2 \Big| \mathcal{F}_j, X_{j+1} = x_0\right] \quad a.s. \qquad \text{(S.1)}
\end{aligned}$$

The last equality holds almost surely, because for any fixed value

x_0, $X_i \neq x_0 (i = 1, \ldots, j)$ $a.s.$, which implies $\eta(X_i, x_0) = X_i - x_0$ $a.s.$ Further, with probability one,

$$\begin{aligned}
& \left[\sum_{i=1}^{j} K_{h_n}\{\eta(X_i, x_0)\}(I_{iu} - \kappa_u)\right] E\left[K_{h_n}\{\eta(x_0, x_0)\}(I_{(j+1)u} - \kappa_u)\middle| \mathcal{F}_j, X_{j+1} = x_0\right] \\
= & \left\{\sum_{i=1}^{j} K_{h_n}(X_i - x_0)(I_{iu} - \kappa_u)\right\} E\left\{K_{h_n}(x_0)(I_{(j+1)u} - \kappa_u)\middle| \mathcal{F}_j, X_{j+1} = x_0\right\} \\
= & \left\{\sum_{i=1}^{j} K_{h_n}(X_i - x_0)\right\} \{U_j(x_0) - \kappa_u\}[\pi_u\{\mathbf{U}_j(x_0)\} - \kappa_u] K_{h_n}(x_0) \\
\leq & \ 0
\end{aligned}$$

by the fact that $U_{nu} - \kappa_u$ and $\pi_u\{\mathbf{U}_n(X_0)\} - \kappa_u$ have opposite signs according to Condition (C1). By taking the expectation on both sides of (S.1), we have

$$\begin{aligned}
& E\left[\sum_{i=1}^{j+1} K_{h_n}\{\eta(X_i, X_{j+1})\}(I_{iu} - \kappa_u)\right]^2 \\
\leq & \ E\left[\sum_{i=1}^{j} K_{h_n}\{\eta(X_i, X_{j+1})\}(I_{iu} - \kappa_u)\right]^2 + E\left\{K_{h_n}(X_{j+1})(I_{(j+1)u} - \kappa_u)\right\}^2.
\end{aligned}$$

Summing over j from 1 to n,

$$\begin{aligned}
& \sum_{j=1}^{n} E\left[\sum_{i=1}^{j+1} K_{h_n}\{\eta(X_i, X_{j+1})\}(I_{iu} - \kappa_u) X_{j+1}\right]^2 \\
\leq & \ \sum_{j=1}^{n} E\left[\sum_{i=1}^{j} K_{h_n}\{\eta(X_i, X_{j+1})\}(I_{iu} - \kappa_u) X_{j+1}\right]^2 \\
& + \sum_{j=1}^{n} E\left\{K_{h_n}(X_{j+1})(I_{(j+1)u} - \kappa_u) X_{j+1}\right\}^2,
\end{aligned}$$

and it is readily seen that the $(j-1)$th summand on the left-hand side agrees with the jth summand in the first term on the right-hand side. Further, $X_{n+1} \neq X_i$ *a.s.*, so $K_{h_n}\{\eta(X_i, X_{n+1})\} = K_{h_n}(X_i - X_{n+1}), (i = 1, \ldots, n), a.s.$ As a result, we obtain

$$\begin{aligned} E\left[\sum_{i=1}^{n+1} K_{h_n}\{\eta(X_i, X_{n+1})\}(I_{iu} - \kappa_u)\right]^2 &\leq \sum_{j}^{n+1} E\left[\{K_{h_n}(X_j)(I_{ju} - \kappa_u)\}^2\right] \\ &\leq \sum_{j}^{n+1} E\left[\{K_{h_n}(X_j)\}^2\right] \\ &= \sum_{j}^{n+1} h_n^{-1} \int \{K(t)\}^2 \, dt \sup|f(x)| \\ &= nO_p(h_n^{-1}). \end{aligned}$$

Also by the Minkowski triangle inequality on the L_2 space, we have

$$\begin{aligned} & E\left(\left[\sum_{i=1}^{n} K_{h_n}\{\eta(X_i, X_{n+1})\}(I_{iu} - \kappa_u)\right]^2\right)^{1/2} \\ \leq\ & E\left(\left[\sum_{i=1}^{n+1} K_{h_n}\{\eta(X_i, X_{n+1})\}(I_{iu} - \kappa_u)\right]^2\right)^{1/2} + E\left[\{K_{h_n}(X_{n+1})(I_{iu} - \kappa_u)\}^2\right]^{1/2} \\ =\ & n^{1/2} O_p(h_n^{-1/2}), \end{aligned}$$

which implies

$$E\left[\left\{\sum_{i=1}^{n} K_{h_n}(X_i - X_{n+1})(I_{iu} - \kappa_u)\right\}^2\right] = nO_p(h_n^{-1}),$$

and as a result, by Condition (A6), we have

$$\left\{\sum_{i=1}^{n} K_{h_n}(X_i - X_{n+1})(I_{iu} - \kappa_u)\right\} X_{n+1} = O_p(n^{1/2} h_n^{-1/2}).$$

This proves the result. □

Corollary 1. $\sum_{i=1}^{n}(I_{iu}-\kappa_u)X_i = O_p(nh_n^2 + n^{1/2}h_n^{-1/2})$.

Proof: First we can write

$$\begin{aligned}
&\left|\sum_{i=1}^{n}(I_{iu}-\kappa_u)X_i\right| \\
= &\left|\sum_{j=1}^{n}\sum_{i=1}^{n}\{nf(X_j)\}^{-1}K_{h_n}(X_i-X_j)(I_{iu}-\kappa_u)X_j\right| \\
&+\left|\sum_{i=1}^{n}(I_{iu}-\kappa_u)X_i\right| - \left|\sum_{j=1}^{n}\sum_{i=1}^{n}\{nf(X_j)\}^{-1}K_{h_n}(X_i-X_j)(I_{iu}-\kappa_u)X_j\right|.
\end{aligned}$$

Note that

$$\begin{aligned}
&E\left[\left|\sum_{j=1}^{n}\sum_{i=1}^{n}\{nf(X_j)\}^{-1}K_{h_n}(X_i-X_j)(I_{iu}-\kappa_u)X_j\right|\right] \\
\leq\ & n^{-1}\sum_{j=1}^{n}\left[E\left\{\left|\sum_{i=1}^{n}f(X_j)^{-1}K_{h_n}(X_i-X_j)(I_{iu}-\kappa_u)\right|\right\}^2\right]^{1/2}\{E(X_j^2)\}^{1/2} \\
=\ & O_p(n^{1/2}h_n^{-1/2})
\end{aligned}$$

by Lemma 1.

As a result, we have

$$\left|\sum_{j=1}^{n}\sum_{i=1}^{n}\{nf(X_j)\}^{-1}K_{h_n}(X_i-X_j)(I_{iu}-\kappa_u)X_j\right| = O_p(n^{1/2}h_n^{-1/2}). \tag{S.2}$$

Also note that

$$
\begin{aligned}
& \left| \left| \sum_{i=1}^{n} (I_{iu} - \kappa_u) X_i \right| - \left| \sum_{j=1}^{n} \sum_{i=1}^{n} \left\{ nf(X_j) \right\}^{-1} K_{h_n}(X_i - X_j)(I_{iu} - \kappa_u) X_j \right| \right| \\
\leq & \sum_{i=1}^{n} \left| (I_{iu} - \kappa_u) \left[X_i - \sum_{j=1}^{n} \left\{ nf(X_j) \right\}^{-1} K_{h_n}(X_i - X_j) X_j \right] \right| \\
\leq & \sum_{i=1}^{n} \left| X_i - \sum_{j=1}^{n} \left\{ nf(X_j) \right\}^{-1} K_{h_n}(X_i - X_j) X_j \right|.
\end{aligned}
$$

The last equation is of order $O_p(nh_n^2 + n^{1/2}h_n^{-1/2})$, because

$$
\begin{aligned}
& \sum_{j=1}^{n} \left\{ nf(X_j) \right\}^{-1} K_{h_n}(X_i - X_j) X_j \\
= \; & f(X_i) \sum_{j=1}^{n} f(X_i)^{-1} \left\{ nf(X_j) \right\}^{-1} K_{h_n}(X_i - X_j) X_j \\
= \; & f(X_i) E \left[\left\{ f(X_j) \right\}^{-1} X_j \mid X_j = X_i \right] + O_p\{h_n^2 + (nh_n)^{-1/2}\} \\
= \; & X_i + O_p\{h_n^2 + (nh_n)^{-1/2}\}.
\end{aligned}
$$

The second to the last equality holds because $\sum_{j=1}^{n} f(X_i)^{-1} \left\{ nf(X_j) \right\}^{-1} K_{h_n}(X_i - X_j) X_j$ is a fixed design kernel estimator of $X_i f(X_i)^{-1} = E \left[\left\{ f(X_j) \right\}^{-1} X_j \mid X_j = X_i \right]$, while its mean squared error is of order $O\{h_n^4 + (nh_n)^{-1}\}$ (Härdle, 2004).

In conjunction with (S.2), we have

$$
\left| \sum_{i=1}^{n} (I_{iu} - \kappa_u) X_i \right| = O_p(nh_n^2 + n^{1/2}h^{-1/2}).
$$

This proves the result. □

Remark 1. In the above derivations, X_i in $\sum_{i=1}^n (I_{iu} - \kappa_u) X_i$ does not affect the convergence rates. Therefore, the convergence rates are the same when considering $\sum_{i=1}^n (I_{iu} - \kappa_u) Z_i$ for any other integrable random variable Z_i.

Lemma 2. *Let $f(\cdot)$ be the density of X, then we have*

$$\bigg| \int \frac{\sum_{i=1}^n K_{h_n}(X_i - X_{n+1})(I_{iu} - \kappa_u)}{\sum_{i=1}^n K_{h_n}(X_i - X_{n+1})} X_{n+1} f(X_{n+1}) dX_{n+1} - \int \frac{\sum_{i=1}^n K_{h_n}(X_i - X_{n+1})(I_{iu} - \kappa_u) X_{n+1}}{n f(X_{n+1})} f(X_{n+1}) dX_{n+1} \bigg| = O_p\{(nh_n)^{-1}\}.$$

Proof: We have that

$$\begin{aligned}
& \bigg| \int \frac{\sum_{i=1}^n K_{h_n}(X_i - X_{n+1})(I_{iu} - \kappa_u)}{\sum_{i=1}^n K_{h_n}(X_i - X_{n+1})} f(X_{n+1}) dX_{n+1} \\
& - \int \frac{\sum_{i=1}^n K_{h_n}(X_i - X_{n+1})(I_{iu} - \kappa_u)}{n f(X_{n+1})} f(X_{n+1}) dX_{n+1} \bigg| \\
\leq & \left\{ \int \left| \sum_{i=1}^n K_{h_n}(X_i - X_{n+1})(I_{iu} - \kappa_u) \right|^2 f(X_{n+1}) dX_{n+1} \right\}^{1/2} \\
& \times \left\{ \int \left| \frac{f(X_{n+1}) - n^{-1} \sum_{i=1}^n K_{h_n}(X_i - X_{n+1})}{\sum_{i=1}^n K_{h_n}(X_i - X_{n+1}) f(X_{n+1})} \right|^2 f(X_{n+1}) dX_{n+1} \right\}^{1/2} \\
= & O_p(n^{1/2} h_n^{-1/2}) \left\{ \int \left| \frac{f(X_{n+1}) - n^{-1} \sum_{i=1}^n K_{h_n}(X_i - X_{n+1})}{\sum_{i=1}^n K_{h_n}(X_i - X_{n+1}) f(X_{n+1})} \right|^2 f(X_{n+1}) dX_{n+1} \right\}^{1/2} \\
= & O_p(n^{1/2} h_n^{-1/2}) O_p(h_n^2 + n^{-1/2} h_n^{-1/2}) O_p(n^{-1}) \\
= & O_p(n^{-1/2} h_n + n^{-1} h_n^{-1}) \\
= & O_p(n^{-1} h_n^{-1}).
\end{aligned}$$

The first equality is a result from Lemma 1. The second equality holds because for each $X_i, i = 1, \ldots, n$,

$$\left| \sum_{i=1}^{n} K_{h_n}(X_i - X_{n+1}) \right| = O_p(n)$$

and

$$\left| f(X_{n+1}) - n^{-1} \sum_{i=1}^{n} K_{h_n}(X_i - X_{n+1}) \right| = O_p\{h_n^2 + (nh_n)^{-1/2}\},$$

which follows the uniform convergence of the kernel density estimator (Silverman, 1978). With Condition (A6), we obtain the desired result. □

Lemma 3. *For a constant ρ_0 and $n > n_0 > \rho_0$, we define $A_n = \prod_{l=n_0}^{n}(1 - \rho_0/l)^{-1}$, then we have $\lim_{n\to\infty} n^{-\rho_0} A_n = A_0$, where $A_0 = n_0^{-\rho_0}$.*

Proof: The limiting result shown below follows the convergence of the product integral. We define $t_l = l/n, l = n_0 - 1, \ldots, n$, $n(t) = l$, for $t_l \le t < t_{l+1}$. For $t \ge t_{n_0}$, let $P(t) = \sum_{t_{n_0} \le t_l \le t} 1/n(t_l) = \int_{s\in[t_{n_0},t]} n(s)^{-1} dn(s)$. For $t < t_{n_0}$, define $P(t) = 0$. Note that $n(s)$ changes its values only when $s = t_l$, so $dn(s)$ is nonzero only at

$s = t_l, l = n_0, \ldots, n$. Therefore, we can write

$$\begin{aligned}
\lim_{n\to\infty} A_n^{-1} &= \lim_{\sup_{n_0 \le l \le n} |t_l - t_{l-1}| \to 0} A_{n(t_n)}^{-1} \\
&= \lim_{\sup_{n_0 \le l \le n} |t_l - t_{l-1}| \to 0} \prod_{t_l = t_{n_0}}^{t_n} \{1 - \rho_0 P'(t_l) dt_l\} \\
&= \lim_{\sup_{n_0 \le l \le n} |t_l - t_{l-1}| \to 0} \prod_{t_l = t_{n_0}}^{t_n} \{1 - \rho_0 \int_{t_{l-1}}^{t_l} P'(t) dt\}.
\end{aligned}$$

As $n \to \infty$, or $\sup_{n_0 \le l \le n} |t_l - t_{l-1}| \to 0$, the above form is a product limit in Definition 1 in Gill and Johansen (1990). Similar to Example 2.5.6 in Slavík and Karlova (2007), this product limit can be written as

$$\begin{aligned}
& \exp\left(-\rho_0 \int_{t\in[t_{n_0}, t_n]} dP(t)\right) \\
= & \exp\left(-\rho_0 \int_{s\in[t_{n_0}, t_n]} n(t)^{-1} dn(t)\right) \\
= & \exp\left(-\rho_0 [\log\{n(t_n)\} - \log\{n(t_{n_0})\}]\right) \\
= & \exp[-\rho_0\{\log(n) - \log(n_0)\}] \\
= & n_0^{\rho_0} n^{-\rho_0}.
\end{aligned}$$

Therefore, $\lim_{n\to\infty} n^{-\rho_0} A_n = A_0$. This proves the result. $\square$

S.2 Proof of Lemma 1

To assess the properties of D_{nu}, we first note that for $n > n_0$ and $u < m$,

$$
\begin{aligned}
& E(I_{(n+1)u}|\mathcal{F}_n, X_{n+1}) - \kappa_u \\
= \; & \pi_u\{\mathbf{U}_n(X_{n+1})\} - \kappa_u \\
= \; & \pi_u(\boldsymbol{\kappa}) + \pi_u^{'}(\boldsymbol{\kappa})\{\mathbf{U}_n(X_{n+1}) - \boldsymbol{\kappa}\} + 1/2\{\mathbf{U}_n(X_{n+1}) - \boldsymbol{\kappa}\}^{\mathrm{T}}\pi_u^{''}(\boldsymbol{\kappa}) \\
& \times\{\mathbf{U}_n(X_{n+1}) - \boldsymbol{\kappa}\}(\{1 + o_p(1)\} - \kappa_u \\
= \; & \pi_{uu}'(\boldsymbol{\kappa})\{U_{nu}(X_{n+1}) - \kappa_u\} + \sum_{r\neq u, r=1}^{m-1} \pi_{ur}'(\boldsymbol{\kappa})\{U_{nr}(X_{n+1}) - \kappa_r\} \\
& + 1/2\{\mathbf{U}_n(X_{n+1}) - \boldsymbol{\kappa}\}^{\mathrm{T}}\pi_u^{''}(\boldsymbol{\kappa})\{\mathbf{U}_n(X_{n+1}) - \boldsymbol{\kappa}\}\{1 + o_p(1)\} \\
= \; & -\rho\{U_{nu}(X_{n+1}) - \kappa_u\} + 1/2\{\mathbf{U}_n(X_{n+1}) - \boldsymbol{\kappa}\}^{\mathrm{T}}\pi_u^{''}(\boldsymbol{\kappa})\{\mathbf{U}_n(X_{n+1}) - \boldsymbol{\kappa}\}\{1 + o_p(1)\}.
\end{aligned}
$$

The third equality holds by Remark 1 that $\pi_u(\boldsymbol{\kappa}) = \kappa_u$. The last equality holds because by Remark 2, $\pi_{ur}'(\boldsymbol{\kappa}) = 0$ when $r = 1, \ldots, m-1, r \neq u$, and by Remark 3, $\pi_{uu}' = -\rho$.

Multiplying the above equation by X_{n+1} and taking expectation with respect to X_{n+1}, we have

$$E\{(I_{(n+1)u} - \kappa_u)X_{n+1}|\mathcal{F}_n\} = -\rho E[\{U_{nu}(X_{n+1}) - \kappa_u\}X_{n+1}|\mathcal{F}_n] + \gamma_{1nu},$$

where

$$\gamma_{1nu} \equiv E\left[1/2\{\mathbf{U}_n(X_{n+1}) - \boldsymbol{\kappa}\}^{\mathrm{T}}\pi_u^{''}(\boldsymbol{\kappa})\{\mathbf{U}_n(X_{n+1}) - \boldsymbol{\kappa}\}X_{n+1}\middle|\mathcal{F}_n\right]\{1 + o_p(1)\}.$$

Also, for $u = 1, \ldots, m-1$, we have

$$\begin{aligned}
& E[\{U_{nu}(X_{n+1}) - \kappa_u\}X_{n+1}|\mathcal{F}_n] \\
= & \int \{U_{nu}(X_{n+1}) - \kappa_u\}X_{n+1} f(X_{n+1}) dX_{n+1} \\
= & n^{-1}\sum_{i=1}^{n} \int K_{h_n}(X_i - X_{n+1})(I_{iu} - \kappa_u)X_{n+1} dX_{n+1} + \gamma_{2nu} \\
= & n^{-1}\sum_{i=1}^{n} (I_{iu} - \kappa_u)X_i + \gamma_{2nu},
\end{aligned}$$

where

$$\begin{aligned}
\gamma_{2nu} \equiv & \int \frac{\sum_{i=1}^{n} K_{h_n}(X_i - X_{n+1})(I_{iu} - \kappa_u)}{\sum_{i=1}^{n} K_{h_n}(X_i - X_{n+1})} X_{n+1} f(X_{n+1}) dX_{n+1} \\
& - \int \frac{\sum_{i=1}^{n} K_{h_n}(X_i - X_{n+1})(I_{iu} - \kappa_u)X_{n+1}}{n f(X_{n+1})} f(X_{n+1}) dX_{n+1}.
\end{aligned}$$

This gives

$$E\{(I_{(n+1)u} - \kappa_u)X_{n+1}|\mathcal{F}_n\} = -\rho n^{-1}\sum_{i=1}^{n}(I_{iu} - \kappa_u)X_i + \gamma_{1nu} - \rho\gamma_{2nu},$$

for $u = 1, \ldots, m-1$.

Define $\alpha_{nu} = 1 - \rho/n$ for $n \geq n_0$, and $\alpha_{nu} = 1$ otherwise, and let $\beta_{nu} = \gamma_{1nu} - \rho\gamma_{2nu}$ for $n \geq n_0$, and $\beta_{nu} = 0$ otherwise, for $u = 1, \ldots, m-1$. We have

$$E(D_{(n+1)u}|\mathcal{F}_n) = \alpha_{nu} D_{nu} + \beta_{nu}.$$

Combining the results of $D_{(n+1)u}, u = 1, \ldots, m-1$, we have

$$E(\mathbf{D}_{n+1}|\mathcal{F}_n) = \boldsymbol{\alpha}_n \mathbf{D}_n + \boldsymbol{\beta}_n,$$

where $\boldsymbol{\alpha}_n = \text{diag}\{\alpha_{n1}, \ldots, \alpha_{n(m-1)}\}$ and $\boldsymbol{\beta}_n = (\beta_{n1}, \ldots, \beta_{n(m-1)})^{\mathrm{T}}$. Let $A_{nu} = \prod_{l=1}^{n-1} \alpha_{lu}^{-1} = \prod_{l=n_0}^{n-1} \alpha_{lu}^{-1}, B_{nu} = \sum_{l=1}^{n-1} A_{(l+1)u}\beta_{lu} = \sum_{l=n_0}^{n-1} A_{(l+1)u}\beta_{lu}$, and define $M_{nu} = A_{nu}D_{nu} - B_{nu}$. It is easy to verify that $M_{iu} = D_{iu}$ for $i \leq n_0$. For $n > n_0$, we have

$$\begin{aligned} E(M_{(n+1)u}|\mathcal{F}_n) &= A_{(n+1)u}(\alpha_{nu}D_{nu} + \beta_{nu}) - \sum_{l=n_0}^{n} A_{(l+1)u}\beta_{lu} \\ &= A_{nu}D_{nu} - \sum_{l=n_0}^{n-1} A_{(l+1)u}\beta_{lu} \\ &= M_{nu}. \end{aligned}$$

Further, X_i and $I_i, i = 1, \ldots, n$, and their continuous functions, D_{nu} and B_{nu}, have finite second moments by Condition (A6). Therefore, $E(|M_{nu}|) < \infty$, which implies M_{nu} is a martingale. We further define $\Delta M_{nu} = M_{nu} - M_{(n-1)u}$ to be a martingale difference. Combining the results for arm u, the vector $\mathbf{M}_n = (M_{n1}, \ldots, M_{n(m-1)})^{\mathrm{T}}$ is a martingale vector, and $\Delta\mathbf{M}_n = (\Delta M_{n1}, \ldots, \Delta M_{n(m-1)})^{\mathrm{T}}$ is a vector of martingale differences. We further define $\mathbf{A}_n = \text{diag}(A_{n1}, \ldots, A_{n(m-1)})$, and $\mathbf{B}_n = (B_{n1}, \ldots, B_{n(m-1)})^{\mathrm{T}}$.

Now we assess the asymptotic properties of $\mathbf{D}_n$ through $\mathbf{M}_n$ by utilizing martingale techniques. We first derive the asymptotic properties of $\mathbf{z}^{\mathrm{T}}\mathbf{M}_n$, where $\mathbf{z}$ is an arbitrary $m-1$ dimensional vector, and then we show that the term $\mathbf{B}_n$ is ignorable because it converges faster to 0 than $\mathbf{M}_n$. Note that $\mathbf{z}^{\mathrm{T}}\mathbf{M}_n$ is a martingale while $\mathbf{z}^{\mathrm{T}}\Delta\mathbf{M}_n$ is a martingale difference, because the linear function does not alter the expectation and boundedness properties.

Let $\mathbf{s}_n = E(\mathbf{M}_n\mathbf{M}_n^{\mathrm{T}})$, according to the martingale invariance principle introduced on page 99 in Hall and Heyde (1980), if we have

$$(\mathbf{z}^{\mathrm{T}}\mathbf{s}_n\mathbf{z})^{-1}\sum_{i=1}^{n}\mathbf{z}^{\mathrm{T}}\Delta\mathbf{M}_i(\Delta\mathbf{M}_i)^{\mathrm{T}}\mathbf{z} \xrightarrow{p} 1 \tag{S.3}$$

$$(\mathbf{z}^{\mathrm{T}}\mathbf{s}_n\mathbf{z})^{-1}\sum_{i=1}^{n}E[\mathbf{z}^{\mathrm{T}}\Delta\mathbf{M}_i(\Delta\mathbf{M}_i)^{\mathrm{T}}\mathbf{z}\ I\{|\mathbf{z}^{\mathrm{T}}\Delta\mathbf{M}_i| > \epsilon(\mathbf{z}^{\mathrm{T}}\mathbf{s}_n\mathbf{z})^{1/2}\}]$$
$$\to 0, \forall\epsilon > 0 \tag{S.4}$$

as $n \to \infty$, then $(\mathbf{z}^{\mathrm{T}}\mathbf{s}_n\mathbf{z})^{-1/2}\mathbf{z}^{\mathrm{T}}\mathbf{M}_n$ converges weakly to a standard normal random variable, and in turn $\mathbf{s}_n^{-1/2}\mathbf{M}_n$ converges to a multivariate standard normal vector. Thus, (S.3) readily holds by Chebyshev's inequality for the uncorrelated random variables and

$$\begin{aligned}
&\mathbf{z}^{\mathrm{T}}\mathbf{s}_n\mathbf{z}\\
= \ & E(\mathbf{z}^{\mathrm{T}}\mathbf{M}_n\mathbf{M}_n^{\mathrm{T}}\mathbf{z})\\
= \ & E\left[\mathbf{z}^{\mathrm{T}}\left\{\begin{array}{ccc}\sum_{i=1}^{n}(\Delta M_{i1})^2 & \ldots & \sum_{i=1}^{n}(\Delta M_{i1}\Delta M_{i(m-1)})\\ \vdots & \ddots & \vdots\\ \sum_{i=1}^{n}(\Delta M_{i1}\Delta M_{i(m-1)}) & \ldots & \sum_{i=1}^{n}(\Delta M_{i(m-1)})^2\end{array}\right\}\mathbf{z}\right]\\
= \ & \sum_{i=1}^{n}E\{\mathbf{z}^{\mathrm{T}}\Delta\mathbf{M}_i(\Delta\mathbf{M}_i)^{\mathrm{T}}\mathbf{z}\}.
\end{aligned}$$

The second equality holds because for $i < j$ and arms u and v, $E(\Delta M_{iu}\Delta M_{jv}) = E\left\{\Delta M_{iu}E(\Delta M_{jv}|\mathcal{F}_{j-1})\right\} = 0$.

If (S.4) holds, then the martingale invariance principle allows us to show the asymptotic properties of $\mathbf{z}^{\mathrm{T}}\mathbf{M}_n$ through accessing the convergence of $\mathbf{s}_n$. Therefore, in the following, we proceed to find the exact form of $\mathbf{s}_n$ and verify (S.4).

Let $s_{nuu} = \sum_{i=1}^{n} E\{(\Delta M_{iu})^2\}$ and $s_{nuv} = \sum_{i=1}^{n} E(\Delta M_{iu}\Delta M_{ju})$ we examine the convergence of each term s_{nuv} in the matrix $\Delta\mathbf{M}_i(\Delta\mathbf{M}_i)^{\mathrm{T}}$. Note that for $n > n_0$,

$$A_{nu}^{-1}\Delta M_{nu} = (I_{nu} - \kappa_u)X_n + \rho D_{(n-1)u}/(n-1) - \beta_{(n-1)u}. \quad \text{(S.5)}$$

By Corollary 1 and Condition (A4) that $nh_n^2 \to \infty$, we have

$$\rho D_{(n-1)u}/(n-1) = O_p(h_n^2 + n^{-1/2}h_n^{-1}) = o_p(1).$$

Next, $\gamma_{2nu} = O_p\{(nh_n)^{-1}\} = o_p(1)$ by Lemma 2. In addition, from

$$\gamma_{1nu} = E\left[1/2\{\mathbf{U}_n(X_{n+1}) - \boldsymbol{\kappa}\}^{\mathrm{T}}\pi_u^{''}(\boldsymbol{\kappa})\{\mathbf{U}_n(X_{n+1}) - \boldsymbol{\kappa}\}X_{n+1}\middle|\mathcal{F}_n\right]\{1 + o_p(1)\},$$

by the boundedness of $\pi_u^{''}$, γ_{1nu} has the same order as

$$\begin{aligned} & E\left[\left\{\sum_{i=1}^{n} K_{h_n}(X_i - X_{n+1})\right\}^{-2}\left\{\sum_{i=1}^{n} K_{h_n}(X_i - X_{n+1})(I_{iu} - \kappa_u)\right\}^2 X_{n+1}\middle|\mathcal{F}_n\right] \\ = \; & O_p(n^{-2})O_p(nh_n^{-1}) \\ = \; & O_p\{(nh_n)^{-1}\} \end{aligned}$$

by Lemma 1, and the fact that $\sum_{i=1}^n K_{h_n}(X_i - X_{n+1}) = O_p(n)$.

These together with Lemma 2 imply $|\beta_{nu}| = o_p(1)$. Therefore,

$$A_{nu}^{-1}\Delta M_{nu} = (I_{nu} - \kappa_u)X_n + o_p(1). \tag{S.6}$$

Further note that

$$\begin{aligned}
& n^{-1-2\rho}\sum_{i=1}^n(\Delta M_{iu})^2 \\
= \;& n^{-1-2\rho}\sum_{i=1}^n A_{iu}^2(A_{iu}^{-1}\Delta M_i)^2 \\
= \;& n^{-1-2\rho}\sum_{i=1}^n A_{iu}^2\{(I_{iu} - \kappa_u)X_i + o_p(1)\}^2 \\
= \;& \Bigg[n^{-1-2\rho}\sum_{i=1}^n A_{iu}^2\left\{(1-\kappa_u)\kappa_u X_i^2\right\} \\
& +n^{-1}\sum_{i=1}^n (A_{iu}/n^\rho)^2\{(I_{iu}-\kappa_u)(1-2\kappa_u)X_i^2\}\Bigg]\{1+o_p(1)\} \\
= \;& \left\{n^{-1-2\rho}(1-\kappa_u)\kappa_u\sum_{i=1}^n A_{iu}^2 X_i^2 + o_p(1)\right\}\{1+o_p(1)\}. \qquad \text{(S.7)}
\end{aligned}$$

The second equality holds by directly plugging in (S.6). Strictly speaking, (S.6) can be used only when i is large. However, since the value on the first finitely many terms do not affect the final asymptotic results, we do not make distinction here. This practice also applies similarly in the remaining text. The last equality follows Remark 1 and the fact that A_{iu}/n^ρ is bounded due to Lemma 3. For a given $\xi > 0$

$$\begin{aligned} &\lim_{n\to\infty} \Pr\left[n^{-1-2\rho}\left| \sum_{i=1}^{n} A_{iu}^2 X_i^2 - E(X_i^2)\sum_{i=1}^{n} A_{iu}^2 \right| > \xi \right] \\ \leq\ &\lim_{n\to\infty} n^{-2-4\rho}\xi^{-2}\sum_{i=1}^{n} A_{iu}^4 \mathrm{var}(X_i^2). \end{aligned} \tag{S.8}$$

To inspect the right-hand side of (S.8), using Lemma 3 and following the same argument to that in the proof of Theorem 1 in Smith (1984), we have

$$\frac{n^{-2-4\rho}\sum_{i=1}^{n} A_{iu}^4}{n^{-2-4\rho}\sum_{i=1}^{n} A_{0u}^4 i^{4\rho}} \to 1 \tag{S.9}$$

in probability, as $n \to \infty$. Further,

$$n^{-2}\sum_{i=1}^{n}(i/n)^{4\rho} \to n^{-1}\int_0^1 x^{4\rho}dx = n^{-1}(1+4\rho)^{-1}.$$

Thus, the right-hand side of (S.8) goes to 0. This shows that

$$n^{-1-2\rho}\left| \sum_{i=1}^{n} A_{iu}^2 X_i^2 - E(X_i^2)\sum_{i=1}^{n} A_{iu}^2 \right|$$

converges to 0 in probability.

Now, we assess the limit of $n^{-1-2\rho}E(X_i^2)\sum_{i=1}^{n} A_{iu}^2$. Similar to the previous argument, as $n \to \infty$,

$$\frac{n^{-1-2\rho}\sum_{i=1}^{n} A_{iu}^2}{n^{-1-2\rho}A_{0u}^2\sum_{i=1}^{n} i^{2\rho}} \xrightarrow{p} 1,$$

and

$$n^{-1}\sum_{i=1}^{n}(i/n)^{2\rho} \to \int_0^1 x^{2\rho}dx = (1+2\rho)^{-1}.$$

Therefore,

$$n^{-1-2\rho}E(X_i^2)\sum_{i=1}^{n}A_{iu}^2 \to (1+2\rho)^{-1}A_{0u}^2E(X_i^2),$$

and hence

$$n^{-1-2\rho}\sum_{i=1}^{n}A_{iu}^2X_i^2 \xrightarrow{p} (1+2\rho)^{-1}A_{0u}^2Q.$$

Plugging the result into (S.7), we have

$$\begin{aligned} & n^{-1-2\rho}\sum_{i=1}^{n}(\Delta M_{iu})^2 \to (1+2\rho)^{-1}A_0^2(1-\kappa_u)\kappa_u E(X_i^2) \\ = \quad & (1+2\rho)^{-1}(1-\kappa_u)\kappa_u A_{0u}^2 Q \qquad \text{(S.10)} \end{aligned}$$

in probability as $n \to \infty$.

Similarly, for $\sum_{i=1}^{n} \Delta M_{iu} \Delta M_{iv}$, $u \neq v$, we have

$$\begin{aligned}
& n^{-1-2\rho} \sum_{i=1}^{n} \Delta M_{iu} \Delta M_{iv} \\
= \;& n^{-1-2\rho} \left(\sum_{i=1}^{n} A_{iu} A_{iv} A_{iu}^{-1} \Delta M_{iu} A_{iv}^{-1} \Delta M_{iv} \right) \\
= \;& n^{-1-2\rho} \left\{ \sum_{i=1}^{n} A_{iu} A_{iv} (-I_{iu}\kappa_v - I_{iv}\kappa_u + \kappa_u \kappa_v) X_i^2 \right\} \{1 + o_p(1)\} \\
= \;& \Bigg[- n^{-1-2\rho} \left(\sum_{i=1}^{n} A_{iu} A_{iv} \kappa_u \kappa_v X_i^2 \right) - n^{-1-2\rho} \left\{ \sum_{i=1}^{n} A_{iu} A_{iv} (I_{iu} - \kappa_u) \kappa_v X_i^2 \right\} \\
& - n^{-1-2\rho} \left\{ \sum_{i=1}^{n} A_{iu} A_{iv} (I_{iv} - \kappa_v) \kappa_u X_i^2 \right\} \Bigg] \{1 + o_p(1)\} \\
= \;& \Bigg[- n^{-1} \left\{ A_{0u} A_{0v} \sum_{i=1}^{n} (i/n)^{2\rho} \kappa_u \kappa_v E(X_i^2) \right\} - n^{-1-2\rho} \left\{ \sum_{i=1}^{n} A_{iu} A_{iv} (I_{iu} - \kappa_u) \kappa_v X_i^2 \right\} \\
& - n^{-1-2\rho} \left\{ \sum_{i=1}^{n} A_{iu} A_{iv} (I_{iv} - \kappa_v) \kappa_u X_i^2 \right\} \Bigg] \{1 + o_p(1)\} \\
= \;& -(1 + 2\rho)^{-1} A_{0u} A_{0v} \kappa_u \kappa_v E(X_i^2) + o_p(1).
\end{aligned}$$

The second equality holds because $I_{iu} I_{iv} = 0$ for $u \neq v$. The third equality holds because

$$\begin{aligned}
& \lim_{n\to\infty} \Pr \left\{ n^{-1-2\rho} \left| \sum_{i=1}^{n} A_{iu} A_{iv} X_i^2 - E(X_i^2) \sum_{i=1}^{n} A_{iu} A_{iv} \right| > \xi \right\} \\
\leq \;& \lim_{n\to\infty} n^{-2-4\rho} \sum_{i=1}^{n} A_{iu}^2 A_{iv}^2 \mathrm{var}(X_i^2) \xi^{-2} \\
\leq \;& \lim_{n\to\infty} (n^{-2-4\rho} \sum_{i=1}^{n} A_{iu}^4 + n^{-1-2\rho} \sum_{i=1}^{n} A_{iv}^4) \mathrm{var}(X_i^2) \xi^{-2}/2,
\end{aligned}$$

which goes to 0 by (S.9). The fourth equality holds because

$$\frac{n^{-1-2\rho}\sum_{i=1}^{n} A_{iu}A_{iv}}{n^{-1-2\rho}A_{0u}A_{0v}\sum_{i=1}^{n} i^{2\rho}} \to 1$$

in probability, and

$$n^{-1}\sum_{i=1}^{n}(i/n)^{2\rho} \to \int_0^1 x^{2\rho}dx = (1+2\rho)^{-1}.$$

Finally, the last equality holds by Remark 1 and the fact that A_{iu}/n^{ρ} is bounded due to Lemma 3.

Now we proceed to show the convergence of s_{nuv}. But note that if $n^{-1-2\rho}\sum_{i=1}^{n}(\Delta M_{iu})^2$ and $n^{-1-2\rho}\sum_{i=1}^{n}\Delta M_{iu}\Delta M_{iv}$ are dominated by integrable functions, then the asymptotic properties of s_{nuu} and $s_{nuv}, u \neq v$, can be derived easily by using the dominated convergence theorem. Further, since

$$\begin{aligned}\left|n^{-1-2\rho}\sum_{i=1}^{n}\Delta M_{iu}\Delta M_{iv}\right| &\leq n^{-1-2\rho}\sum_{i=1}^{n}\left|\Delta M_{iu}\Delta M_{iv}\right| \\ &\leq 1/2n^{-1-2\rho}\left\{\sum_{i=1}^{n}(\Delta M_{iu})^2+\sum_{i=1}^{n}(\Delta M_{iv})^2\right\},\end{aligned}$$

we need to show the boundedness of $n^{-1-2\rho}\sum_{i=1}^{n}(\Delta M_{iu})^2$ for obtaining the convergence result. Thus, we evaluate the upper bound of $n^{-1-2\rho}\sum_{i=1}^{n}(\Delta M_{iu})^2$ as follows. Because there exists a constant

$C_1 < \infty$,

$$
\begin{aligned}
& n^{-1-2\rho}\sum_{i=1}^{n}(\Delta M_{iu})^2 \\
= \;& n^{-1-2\rho}\left\{\sum_{i\le n_0}(\Delta M_{iu})^2 + \sum_{i>n_0} A_{iu}^2 (A_{iu}^{-1}\Delta M_{iu})^2\right\} \\
\le \;& \left[n_0 \max_{i\le n_0} X_i^2 + C_1 n^{-1}\sum_{i>n_0}^{n}(i/n)^{2\rho}\left\{(I_{iu}-\kappa_u)X_i + \rho D_{i-1u}/(i-1) - \beta_{(i-1)u}\right\}^2\right] \\
\le \;& \left[n_0 \max_{i\le n_0} X_i^2 + C_1 n^{-1}\sum_{i>n_0}^{n}\left\{(I_{iu}-\kappa_u)X_i + \rho D_{i-1u}/(i-1) - \beta_{(i-1)u}\right\}^2\right], \quad \text{(S.11)}
\end{aligned}
$$

we first show the boundedness of $\rho D_{i-1u}/(i-1)$ and $\beta_{n-1u} = \gamma_{1(n-1)u} - \rho\gamma_{2(n-1)u}$.

Clearly $|\rho D_{i-1u}/(i-1)| \le |\rho| \max_{i<n}|X_i|$. Further, since $|U_{nu}(X_{n+1})| \le 1$ and π_u'' is bounded by Condition (A2), there exists a constant $C_2 < \infty$ so that

$$
\gamma_{1(n-1)u} = E\left[1/2\{\mathbf{U}_{n-1}(X_n) - \boldsymbol{\kappa}\}^{\mathrm{T}}\pi_u''(\boldsymbol{\kappa}^*)\{\mathbf{U}_{n-1}(X_n) - \boldsymbol{\kappa}\}X_n \middle| \mathcal{F}_n\right] \le C_2 m \max_{i\le n}|X_i|,
$$

where $\boldsymbol{\kappa}^* = (\kappa_1^*, \ldots, \kappa_m^*)$ with κ_u^* defined as a point on the line connecting κ_u and $U_{nu}(X_{n+1})$. In addition,

$$
\begin{aligned}
\gamma_{2(n-1)u} \;=\; & \int \frac{\sum_{i=1}^{n} K_{h_n}(X_i - X_n)(I_{iu}-\kappa_u)}{\sum_{i=1}^{n} K_{h_n}(X_i - X_n)} X_n f(X_n)\, dX_n \\
& - \int \frac{\sum_{i=1}^{n} K_{h_n}(X_i - X_n)(I_{iu}-\kappa_u)X_n}{n f(X_n)} f(X_n)\, dX_n \\
\le \;& E(|X_n|) + \max_{i\le n}|X_i|.
\end{aligned}
$$

Therefore, (S.11) implies that there exist constants $C_3, C_4 < \infty$ such that

$$n^{-1-2\rho}\sum_{i=1}^{n}(\Delta M_{iu})^2 \quad \le \quad C_3 \max_{i\le n} X_i^2 + C_4,$$

almost surely. Since $C_3 \max_{i\le n} X_i^2 + C_4$ is an integrable function, by the dominated convergence theorem, we have

$$\begin{aligned}
n^{-1-2\rho} s_{nuu} &= n^{-1-2\rho}\sum_{i=1}^{n} E\left\{(\Delta M_{iu})^2\right\} \\
&\to (1+2\rho)^{-1}(1-\kappa_u)\kappa_u A_{0u}^2 Q, \\
n^{-1-2\rho} s_{nuv} &= n^{-1-2\rho}\sum_{i=1}^{n} E\left(\Delta M_{iu}\Delta M_{iu}\right) \\
&\to -(1+2\rho)^{-1}\kappa_u\kappa_v A_{0u}A_{0v}Q. \qquad \text{(S.12)}
\end{aligned}$$

These give the limiting form of $\mathbf{s}_n$ in (S.3).

To show (S.4), we first note that (S.6), (S.12) and Lemma 3 yield

$$\begin{aligned}
& |(\mathbf{z}^{\mathrm{T}}\mathbf{s}_n\mathbf{z})^{-1/2}\mathbf{z}^{\mathrm{T}}\Delta\mathbf{M}_n|^2 \\
= \;& |(\mathbf{z}^{\mathrm{T}}\mathbf{s}_n\mathbf{z})^{-1}||\mathbf{z}^{\mathrm{T}}\Delta\mathbf{M}_n\Delta\mathbf{M}_n^{\mathrm{T}}\mathbf{z}| \\
= \;& \left|\left(\sum_{u=1}^{m-1}\sum_{v=1}^{m-1} z_u z_v s_{nuv}\right)^{-1}\right| \sum_{u=1}^{m-1}\sum_{v=1}^{m-1}\left| z_u z_v A_{nu}(I_{nu}-\kappa_u)A_{nv}(I_{nv}-\kappa_v)X_n^2\{1+o_p(1)\}\right| \\
= \;& O_p(n^{-1})\sum_{u=1}^{m-1}\sum_{v=1}^{m-1}\left| z_u z_v A_{nu}/n^{\rho}(I_{nu}-\kappa_u)A_{nv}/n^{\rho}(I_{nv}-\kappa_v)X_n^2\{1+o_p(1)\}\right| \\
\le \;& O_p\left\{n^{-1}\max_{u\in 1,\ldots,m-1}(I_{nu}-\kappa_u)^2 X_n^2\right\}.
\end{aligned}$$

By Condition (A6), this implies

$$(\mathbf{z}^{\mathrm{T}}\mathbf{s}_n\mathbf{z})^{-1/2}\mathbf{z}^{\mathrm{T}}\Delta\mathbf{M}_n = o_p(1).$$

Further,

$$
\begin{aligned}
& E[\mathbf{z}^{\mathrm{T}}\Delta\mathbf{M}_i(\Delta\mathbf{M}_i)^{\mathrm{T}}\mathbf{z}I\{|\mathbf{z}^{\mathrm{T}}\Delta\mathbf{M}_i| > \epsilon(\mathbf{z}^{\mathrm{T}}\mathbf{s}_n\mathbf{z})^{1/2}\}] \\
\leq \;& E\{(\mathbf{z}^{\mathrm{T}}\Delta\mathbf{M}_i(\Delta\mathbf{M}_i)^{\mathrm{T}}\mathbf{z})^2\}^{1/2} E\left[I\{|\mathbf{z}^{\mathrm{T}}\Delta\mathbf{M}_i| > \epsilon(\mathbf{z}^{\mathrm{T}}\mathbf{s}_n\mathbf{z})^{1/2}\}\right]^{1/2} \\
= \;& E\{(\mathbf{z}^{\mathrm{T}}\Delta\mathbf{M}_i(\Delta\mathbf{M}_i)^{\mathrm{T}}\mathbf{z})^2\}^{1/2} Pr\{(\mathbf{z}^{\mathrm{T}}\mathbf{s}_n\mathbf{z})^{-1/2}|\mathbf{z}^{\mathrm{T}}\Delta\mathbf{M}_i| > \epsilon\}^{1/2} \\
\to \;& 0,
\end{aligned}
$$

by Condition (A6) and because $(\mathbf{z}^{\mathrm{T}}\mathbf{s}_n\mathbf{z})^{-1/2}\mathbf{z}^{\mathrm{T}}\Delta\mathbf{M}_n$ is $o_p(1)$. This result along with the fact that $\mathbf{s}_n = O(n^{1+2\rho})$ proves (S.4). So far, we have proven that $(\mathbf{z}^{\mathrm{T}}\mathbf{s}_n\mathbf{z})^{-1/2}\mathbf{z}^{\mathrm{T}}\mathbf{M}_n$ converges to a standard normal random variable. Since $\mathbf{z}$ is an arbitrary vector, we conclude that $\mathbf{s}_n^{-1/2}\mathbf{M}_n$ converges to a multivariate standard normal vector. Next, in order to use the martingale results to show the asymptotic property of $\mathbf{D}_n$, we first show that for each u,

$$n^{-1/2}|A_{nu}^{-1}B_{nu}| \xrightarrow{p} 0.$$

Note that there exist constants $C_5, C_6, C_7 < \infty$, such that

$$
\begin{aligned}
|B_{nu}/A_{nu}| &\leq \sum_{i=1}^{n} A_{iu}A_{nu}^{-1}|\beta_{i-1}| \\
&\leq \sum_{i=1}^{n} A_{iu}A_{nu}^{-1}C_5(ih_n)^{-1} \\
&\leq C_6 n^{-1}\sum_{i=1}^{n}(i/n)^{\rho-1}h_n^{-1} \\
&\leq C_7 h_n^{-1}
\end{aligned}
$$

in probability. Here, we use the definition of B_{nu} to obtain the first inequality, the definition of β_{lu} and the results on the orders of $\gamma_{1nu}, \gamma_{2nu}$ lead to the second inequality, Lemma 3 yields the third inequality, and replacing average with integration we can obtain the last inequality. Therefore, together with Condition (A4) we have

$$n^{-1/2}|A_{nu}^{-1}B_{nu}| \xrightarrow{p} 0$$

by Condition (A4), and this gives

$$n^{-1/2}|A_{nu}^{-1}M_{nu} - D_{nu}| = o_p(1).$$

This convergence in probability result for a single $u, u = 1, \ldots, m-1$ implies the joint convergence in probability of the vector constructed by these elements. So, $n^{-1/2}\mathbf{D}_n$ and $n^{-1/2}\mathbf{A}_n^{-1}\mathbf{M}_n$ converge equivalently to the same limit. Also, we have shown that $(\mathbf{z}^{\mathrm{T}}\mathbf{s}_n\mathbf{z})^{-1/2}\mathbf{z}^{\mathrm{T}}\mathbf{M}_n$ converges to a standard normal random variable for an arbitrary $\mathbf{z}$. Therefore,

$$\mathbf{s}_n^{-1/2}\mathbf{M}_n \xrightarrow{d} N(0, I).$$

Further, $\mathbf{A}_n \to \mathrm{diag}(A_{01}n^{\rho}, \ldots, A_{0m}n^{\rho})$ implies $|n^{-1}\mathbf{A}_n^{-1}\mathbf{s}_n\mathbf{A}_n^{-1} - \Omega| = o(1)$ by (S.12), where Ω is defined in the statement of Lemma 1. Hence,

$$n^{-1/2}\mathbf{A}_n^{-1}\mathbf{s}_n^{1/2}\mathbf{s}_n^{-1/2}\mathbf{M}_n \xrightarrow{d} N(0, \Omega).$$

As a result, we have

$$n^{-1/2}\Omega^{-1/2}\mathbf{D}_n \xrightarrow{d} N(0, I).$$

□

S.3 Necessary Lemmas for Theorem 1

Lemma 4. *For a new data point* $\mathbf{X}_{n+1}$*, we have* $E\{(\sum_{i=1}^n \prod_{k=1}^p K_{h_n}(X_{ik} - X_{(n+1)k})(I_{iu} - \kappa_u)\mathbf{X}_{n+1}^{\mathrm{T}}\mathbf{z})^2\} = O\left(nh_n^{-p}\right)$.

Corollary 2. $|\sum_{i=1}^n (I_{iu} - \kappa_u)\mathbf{X}_i^{\mathrm{T}}\mathbf{z}| = O_p\left\{nh_n^2 + n^{1/2}h_n^{-p/2}\right\}$.

Lemma 5. *Let* $f(\mathbf{X}_i)$ *be the density function of* $\mathbf{X}_i$*. We have*

$$\begin{aligned}
&\Bigg|\int \frac{\sum_{i=1}^n \prod_{k=1}^p K_{h_n}(X_{ik} - X_{(n+1)k})(I_{iu} - \kappa_u)}{\sum_{i=1}^n \prod_{k=1}^p K_{h_n}(X_{ik} - X_{(n+1)k})}\mathbf{X}_{n+1}^{\mathrm{T}}\mathbf{z} f(\mathbf{X}_{n+1}) d\mathbf{X}_{n+1} \\
&\quad - \int \frac{\sum_{i=1}^n \prod_{k=1}^p K_{h_n}(X_{ik} - X_{(n+1)k})(I_{iu} - \kappa_u)\mathbf{X}_{n+1}^{\mathrm{T}}\mathbf{z}}{nf(\mathbf{X}_{n+1})} f(\mathbf{X}_{n+1}) d\mathbf{X}_{n+1}\Bigg| \\
= &\ O_p\left(n^{-1}h_n^{-p}\right).
\end{aligned}$$

S.4 Proof of Theorem 1

Following Lemma 4, Corollary 2 and Lemma 5, Theorem 1 holds by using the same arguments as those leading to Lemma 1.

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