Computerized Adaptive Testing that Allows for Response Revision: Design and Asymptotic Theory

Shiyu Wang[1], Georgios Fellouris[2] and Hua-Hua Chang[2]

University of Georgia[1] *and University of Illinois at Urbana Champaign*[2]

Supplementary Material

This note contains the proofs of Theorems in section 2, 3 and 4 as well as the histogram for the item parameters used in the simulation study

S1 Proofs in Section 2

Proof of Lemma 1. The continuity of g^* and g_* follows from the so-called Maximum Theorem (see, e.g., Sundaram, R.K.(1996), p. 239). In order to prove the remaining part of the Lemma, we can assume without loss of generality that $g(x_0; \mathbf{b}) = 0$ for every $\mathbf{b} \in \mathbb{B}$. Indeed, if this is not the case, then we can work with $g(x_n, \mathbf{b}) - g(x_0, \mathbf{b})$. Then, for any given n we have

$$\sup_{\mathbf{b}\in\mathbb{B}} |g(x_n; \mathbf{b})| = \sup_{\mathbf{b}\in\mathbb{B}} \max\{g(x_n; \mathbf{b}), -g(x_n; \mathbf{b})\} \leq \max\{g^*(x_n)\ ,\ -g_*(x_n)\},$$

and consequently

$$\limsup_{n} \sup_{\mathbf{b}\in\mathbb{B}} |g(x_n; \mathbf{b})| \leq \max\{g^*(x_0)\ ,\ -g_*(x_0)\} = 0,$$

which completes the proof. ♢

Proof of Lemma 2. For any θ and $\mathbf{b}$ we have

$$|s(\theta; b, \cdot)| \leq \max_{1\leq k\leq m} |a_k - \bar{a}(\theta; \mathbf{b})| \leq 2a^*(\mathbf{b}) \leq 2 \sup_{\mathbf{b}\in\mathbb{B}} a^*(\mathbf{b}).$$

Moreover,

$$0 < J(\theta; \mathbf{b}) \leq \sum_{k=1}^{m} a_k^2\, p_k(\theta; \mathbf{b}) \leq m\ (a^*(\mathbf{b}))^2 \leq m \sup_{\mathbf{b}\in\mathbb{B}} (a^*(\mathbf{b}))^2\ ,$$

where the first inequality holds because the a_k's cannot be identical due to (2.2). When $\mathbb{B}$ is compact, the upper bounds are finite and do not depend on $\mathbf{b}$ or θ. On the other hand, from Lemma 1 it follows that J_* is continuous, therefore $J_*(\theta) > 0$ for every θ when $\mathbb{B}$ compact. $\diamondsuit$

S2 Proofs in Section 3

Proof of Lemma 3. The final ability estimator, $\hat{\theta}_n$, is not a root of $S_n(\theta)$ on the event $A_n \cup B_n$, where

$$A_n = \{X_i \in k^*(\mathbf{b}_i),\ \forall\ 1 \leq i \leq n\}, \quad B_n = \{X_i \in k_*(\mathbf{b}_i),\ \forall\ 1 \leq i \leq n\}.$$

Thus, it suffices to show that $\mathsf{P}_\theta(\limsup_n A_n) = 0$ and $\mathsf{P}_\theta(\limsup_n B_n) = 0$. We will prove only the first identity, since the second can be shown in a similar way. Indeed, $\mathsf{P}_\theta(A_n) = \mathsf{E}_\theta\left[\mathsf{P}_\theta\left(A_n \mid \mathbf{b}_{1:n}\right)\right]$ and

$$\mathsf{P}_\theta\left(A_n \mid \mathbf{b}_{1:n}\right) = \prod_{i=1}^{n} \mathsf{P}_\theta(X_i \in k^*(\mathbf{b}_i)) = \prod_{i=1}^{n} p^*(\theta; \mathbf{b}_i) \leq \left(p^*(\theta)\right)^n,$$

where the first equality follows the assumption of conditional independence (3.2), whereas the second identity and the inequality follow from the following definitions:

$$p^*(\theta; \mathbf{b}) := \sum_{j \in k^*(\mathbf{b})} p_j(\theta; \mathbf{b}), \quad p^*(\theta) := \sup_{\mathbf{b} \in \mathbb{B}} p^*(\theta; \mathbf{b}).$$

Since $p^*(\theta; \mathbf{b})$ is jointly continuous and $\mathbb{B}$ is compact, from Lemma 1 it follows that $p^*(\theta) < 1$. Therefore, $\sum_{n=1}^{\infty} \mathsf{P}_\theta(A_n) < \infty$, and from the Borel-Cantelli lemma we obtain $\mathsf{P}_\theta(\limsup_n A_n) = 0$, which completes the proof. $\diamondsuit$

Proof of Lemma 4. Fix $n \in \mathbb{N}$. Then, $S_n(\theta) - S_{n-1}(\theta) = s(\theta; \mathbf{b}_n, X_n)$, and from Lemma 2 it follows that $|S_n(\theta) - S_{n-1}(\theta)| \leq K$. Moreover, since $\mathbf{b}_n$ is $\mathcal{F}_{n-1}$-measurable, from representation (2.5) it follows that

$$\mathsf{E}_\theta[S_n(\theta) - S_{n-1}(\theta) | \mathcal{F}_{n-1}] = \mathsf{E}_\theta[s(\theta; \mathbf{b}_n, X_n) | \mathcal{F}_{n-1}] = 0,$$

which proves the martingale property of $S_n(\theta)$. Next, from (2.5)–(2.6) it follows that

$$\mathsf{E}_\theta[(S_n(\theta) - S_{n-1}(\theta))^2|\mathcal{F}_{n-1}] = \mathsf{E}_\theta[s^2(\theta;\mathbf{b}_n, X_n)|\mathcal{F}_{n-1}] = J(\theta;\mathbf{b}_n),$$

which proves that $\langle S(\theta)\rangle_n = \sum_{i=1}^n J(\theta;\mathbf{b}_i)$.

◊

Proof of Theorem 3.1. Let $(\mathbf{b}_n)_{n\in\mathbb{N}}$ be an arbitrary item selection strategy. From Lemma 4 it follows that $S_n(\theta)$ is a P_θ-martingale with mean 0 and predictable variation $I_n(\theta) \geq nJ_*(\theta) \to \infty$, since $J_*(\theta) > 0$. Then, from the Martingale Strong Law of Large Numbers (see, e.g., Williams, D.(1991), p. 124), it follows that as $n \to \infty$

$$\frac{S_n(\theta)}{I_n(\theta)} \to 0 \quad \mathsf{P}_\theta - \text{a.s.} \tag{S2.1}$$

From a Taylor expansion of $S_n(\theta)$ around $\hat{\theta}_n$ it follows that there exists some $\tilde{\theta}_n$ that lies between $\hat{\theta}_n$ and θ such that

$$\begin{aligned} 0 = S_n(\hat{\theta}_n) &= S_n(\theta) + S_n^{'}(\tilde{\theta}_n)(\hat{\theta}_n - \theta) \\ &= S_n(\theta) - I_n(\tilde{\theta}_n)(\hat{\theta}_n - \theta) \quad \mathsf{P}_\theta - \text{a.s.} \end{aligned} \tag{S2.2}$$

where the second equality follows from (3.6). From (**??**) and (**??**) we then obtain

$$\frac{I_n(\tilde{\theta}_n)}{I_n(\theta)} (\hat{\theta}_n - \theta) \to 0 \quad \mathsf{P}_\theta - \text{a.s.}$$

The strong consistency of $\hat{\theta}_n$ will then follow as long as we can guarantee that the fraction in the last relationship remains bounded away from 0 as $n \to \infty$. However, for every n we have

$$\frac{I_n(\tilde{\theta}_n)}{I_n(\theta)} = \frac{\sum_{i=1}^n J(\tilde{\theta}_n;\mathbf{b}_i)}{\sum_{i=1}^n J(\theta;\mathbf{b}_i)} \geq \frac{nJ_*(\tilde{\theta}_n)}{nJ^*(\theta)} = \frac{J_*(\tilde{\theta}_n)}{J^*(\theta)}.$$

Since $J^*(\theta) > 0$, it suffices to show that $\mathsf{P}_\theta(\liminf_n J_*(\tilde{\theta}_n) > 0) = 1$. Since $J_*(\theta)$ is continuous, positive and bounded away from 0 when $|\theta|$ is bounded away from infinity (Lemma 2) and $\tilde{\theta}_n$

lies between $\hat{\theta}_n$ and θ, it suffices to show that

$$\mathsf{P}_\theta(\limsup_n |\hat{\theta}_n| = \infty) = 0. \tag{S2.3}$$

In order to prove (**??**), we observe first of all that since $S_n(\hat{\theta}_n) = 0$ for large n, (**??**) can be rewritten as follows:

$$\frac{S_n(\theta) - S_n(\hat{\theta}_n)}{I_n(\theta)} \to 0 \quad \mathsf{P}_\theta - \text{a.s.} \tag{S2.4}$$

But for every n we have $I_n(\theta) \leq nJ^*(\theta)$ and

$$\begin{aligned} S_n(\theta) - S_n(\hat{\theta}_n) &= \sum_{i=1}^n \Big[s(\theta; \mathbf{b}_i, X_i) - s(\hat{\theta}_n; \mathbf{b}_i, X_i) \Big] \\ &= \sum_{i=1}^n \Big[\bar{a}(\hat{\theta}_n; \mathbf{b}_i) - \bar{a}(\theta; \mathbf{b}_i) \Big] \geq n \inf_{\mathbf{b} \in \mathbb{B}} \Big[\bar{a}(\hat{\theta}_n; \mathbf{b}) - \bar{a}(\theta; \mathbf{b}) \Big], \end{aligned}$$

therefore we obtain

$$\frac{S_n(\theta) - S_n(\hat{\theta}_n)}{I_n(\theta)} \geq \frac{\inf_{\mathbf{b} \in \mathbb{B}} \Big[\bar{a}(\hat{\theta}_n; \mathbf{b}) - \bar{a}(\theta; \mathbf{b}) \Big]}{J^*(\theta)}. \tag{S2.5}$$

On the event $\{\limsup_n \hat{\theta}_n = \infty\}$ there exists a subsequence $(\hat{\theta}_{n_j})$ of $(\hat{\theta}_n)$ such that $\hat{\theta}_{n_j} \to \infty$. Consequently, for any $\mathbf{b} \in \mathbb{B}$ we have

$$\lim_{n_j \to \infty} \Big[\bar{a}(\hat{\theta}_{n_j}; \mathbf{b}) - \bar{a}(\theta; \mathbf{b}) \Big] = a^*(\mathbf{b}) - \bar{a}(\theta; \mathbf{b}) > 0 \tag{S2.6}$$

and from Lemma 1 we obtain

$$\liminf_{n_j \to \infty} \inf_{\mathbf{b} \in \mathbb{B}} \Big[\bar{a}(\hat{\theta}_{n_j}; \mathbf{b}) - \bar{a}(\theta; \mathbf{b}) \Big] \geq \inf_{\mathbf{b} \in \mathbb{B}} [a^*(\mathbf{b}) - \bar{a}(\theta; \mathbf{b})] > 0. \tag{S2.7}$$

From (**??**) and (**??**) it follows that

$$\liminf_{n_j \to \infty} \frac{S_{n_j}(\theta) - S_{n_j}(\hat{\theta}_{n_j})}{I_{n_j}(\theta)} > 0$$

and comparing with (**??**) we conclude that $\mathsf{P}_\theta(\limsup_n \hat{\theta}_n = \infty) = 0$. In an identical way we can show that $\mathsf{P}_\theta(\liminf_n \hat{\theta}_n = -\infty) = 0$, which establishes (**??**) and completes the proof of the strong consistency of $\hat{\theta}_n$. In order to prove (3.7), we observe that

$$\frac{|I_n(\hat{\theta}_n) - I_n(\theta)|}{I_n(\theta)} \leq \frac{1}{nJ_*(\theta)} \sum_{i=1}^{n} |J(\hat{\theta}_n; \mathbf{b}_i) - J(\theta; \mathbf{b}_i)|$$
$$\leq \frac{1}{J_*(\theta)} \sup_{\mathbf{b} \in \mathbb{B}} |J(\hat{\theta}_n; \mathbf{b}) - J(\theta; \mathbf{b})|.$$

But since $J(\theta; \mathbf{b})$ is jointly continuous and $\hat{\theta}_n$ strongly consistent, from Lemma 1 it follows that

$$\sup_{\mathbf{b} \in \mathbb{B}} |J(\hat{\theta}_n; \mathbf{b}) - J(\theta; \mathbf{b})| \to 0 \quad \mathsf{P}_\theta - \text{a.s.} \tag{S2.8}$$

which completes the proof, since from Lemma 2 we know that $J_*(\theta) > 0$. ♢

S3 Proofs in Section 4

Proof of Lemma 5. (i) After $t-1$ responses, the examinee either proceeds to a new item or revises a previous item. Therefore, the difference $S_t(\theta) - S_{t-1}(\theta)$ admits the following decomposition:

$$s\left(\theta; \mathbf{b}_{f_t}, X_1^{f_t}\right) \mathbb{1}_{\{d_{t-1}=0\}} + \sum_{i \in C_{t-1}} s\left(\theta; \mathbf{b}_i, X_{g_t^i}^i | X_{1:g_t^i-1}^i\right) \mathbb{1}_{\{d_{t-1}=i\}}, \tag{S3.1}$$

where the sum in the second term is understood to be 0 when C_{t-1} is the empty set. Since d_{t-1}, C_{t-1} are $\mathcal{F}_{t-1}$-measurable, taking conditional expectations with respect to $\mathcal{F}_{t-1}$ we obtain

$$\mathsf{E}_\theta[S_t(\theta) - S_{t-1}(\theta)|\mathcal{F}_{t-1}] = \mathsf{E}_\theta\left[s\left(\theta; \mathbf{b}_{f_t}, X_1^{f_t}\right) \Big| \mathcal{F}_{t-1}\right] \mathbb{1}_{\{d_{t-1}=0\}}$$
$$+ \sum_{i \in C_{t-1}} \mathsf{E}_\theta\left[s\left(\theta; \mathbf{b}_i, X_{g_t^i}^i | X_{1:g_t^i-1}^i\right) \Big| \mathcal{F}_{t-1}\right] \mathbb{1}_{\{d_{t-1}=i\}}.$$

Since f_t and g_t^i are $\mathcal{F}_{t-1}$-measurable, it follows that

$$\mathsf{E}_\theta\left[s\left(\theta; \mathbf{b}_{f_t}, X_1^{f_t}\right) \Big| \mathcal{F}_{t-1}\right] = 0 = \mathsf{E}_\theta\left[s\left(\theta; \mathbf{b}_i, X_{g_t^i}^i | X_{1:g_t^i-1}^i\right) \Big| \mathcal{F}_{t-1}\right],$$

which proves that $S_t(\theta)$ is a zero-mean $\mathcal{F}_t$-martingale under P_θ. From (**??**) we also have

$$\begin{aligned}&\mathsf{E}_\theta[(S_t(\theta)-S_{t-1}(\theta))^2\,|\mathcal{F}_{t-1}]\\&=J(\theta;\mathbf{b}_{f_t})\,\mathbb{1}_{\{d_{t-1}=0\}}+\sum_{i\in C_{t-1}}J\left(\theta;\mathbf{b}_i|X^i_{1:g^i_t-1}\right)\,\mathbb{1}_{\{d_{t-1}=i\}}\end{aligned}$$

and, consequently, the predictable variation of $S_t(\theta)$ will be

$$\begin{aligned}\langle S(\theta)\rangle_t &:= \sum_{s=1}^{t}\mathsf{E}_\theta\left[(S_s(\theta)-S_{s-1}(\theta))^2\,|\mathcal{F}_{s-1}\right]\\&=\sum_{s=1}^{t}\left[J(\theta;\mathbf{b}_{f_s})\,\mathbb{1}_{\{d_{s-1}=0\}}+\sum_{j\in C_{s-1}}J\left(\theta;\mathbf{b}_j|X^j_{1:g^j_{s-1}}\right)\,\mathbb{1}_{\{d_{s-1}=j\}}\right]=I_t.\end{aligned}$$

(ii) This follows from the Optional Sampling Theorem and the fact that $(\tau_n)_{n\in\mathbb{N}}$ is a strictly increasing sequence of $\{\mathcal{F}_t\}_{t\in\mathbb{N}}$-stopping times that are bounded, since $\tau_n\le(m-1)n$ for every $n\in\mathbb{N}$.

♢

Proof of Theorem 4.1. From Lemma 5 we have that $S_{\tau_n}(\theta)$ is a $\{\mathcal{F}_{\tau_n}\}$-martingale with predictable variation $I_{\tau_n}(\theta)$. Moreover, from (4.10) we have $I_{\tau_n}(\theta)\ge nJ_*(\theta)\to\infty$ and from the Martingale Strong Law of Large Numbers (Williams, D. (1991), p. 124) it follows that

$$\frac{S_{\tau_n}(\theta)}{I_{\tau_n}(\theta)}\to 0\quad \mathsf{P}_\theta-\text{a.s.} \tag{S3.2}$$

Since $S_{\tau_n}(\hat\theta_{\tau_n})=0$ for large enough n with probability 1, with a Taylor expansion around θ we have

$$\begin{aligned}0=S_{\tau_n}(\hat\theta_{\tau_n})&=S_{\tau_n}(\theta)+S^{'}_{\tau_n}(\tilde\theta_{\tau_n})(\hat\theta_{\tau_n}-\theta)\\&=S_{\tau_n}(\theta)-I_{\tau_n}(\tilde\theta_{\tau_n})(\hat\theta_{\tau_n}-\theta)\quad \mathsf{P}_\theta-\text{a.s.}\end{aligned} \tag{S3.3}$$

where $\tilde\theta_{\tau_n}$ lies between $\hat\theta_{\tau_n}$ and θ, and (**??**) takes the form

$$\frac{I_{\tau_n}(\tilde\theta_{\tau_n})}{I_{\tau_n}(\theta)}\,(\hat\theta_{\tau_n}-\theta)\to 0\quad \mathsf{P}_\theta-\text{a.s.}$$

However, since $\tau_n \le (m-1)n$ and $J_*(\theta) f_t \le I_t(\theta) \le Kt$ for every t, we have

$$\frac{I_{\tau_n}(\tilde{\theta}_{\tau_n})}{I_{\tau_n}(\theta)} \ge \frac{n J_*(\tilde{\theta}_{\tau_n})}{\tau_n K} \ge \frac{1}{(m-1)K} J_*(\tilde{\theta}_{\tau_n})$$

and it suffices to show that

$$\limsup_n |\hat{\theta}_{\tau_n}| < \infty \quad \mathsf{P}_\theta - \text{a.s.} \tag{S3.4}$$

For large n we have $S_{\tau_n}(\hat{\theta}_{\tau_n}) = 0$ and (**??**) can be rewritten as follows

$$\frac{S_{\tau_n}(\theta) - S_{\tau_n}(\hat{\theta}_{\tau_n})}{I_{\tau_n}(\theta)} \to 0 \quad \mathsf{P}_\theta - \text{a.s.} \tag{S3.5}$$

But from the definition of the score function in (4.8) it follows that

$$\begin{aligned}
&S_{\tau_n}(\theta) - S_{\tau_n}(\hat{\theta}_{\tau_n}) \\
&= \sum_{i=1}^{n} \left[\Big(s(\theta; \mathbf{b}_i) - s(\hat{\theta}_{\tau_n}; \mathbf{b}_i) \Big) + \sum_{j=2}^{g^i_{\tau_n}} \Big(s(\theta; \mathbf{b}_i, X^i_j | X^i_{1:j-1}) - s(\hat{\theta}_{\tau_n}; \mathbf{b}_i, X^i_j | X^i_{1:j-1}) \Big) \right] \\
&= \sum_{i=1}^{n} \left[\Big(\bar{\alpha}(\hat{\theta}_{\tau_n}; \mathbf{b}_i) - \bar{\alpha}(\theta; \mathbf{b}_i) \Big) + \sum_{j=2}^{g^i_{\tau_n}} \Big(\bar{\alpha}(\hat{\theta}_{\tau_n}; \mathbf{b}_i | X^i_{1:j-1}) - \bar{\alpha}(\theta; \mathbf{b}_i | X^i_{1:j-1}) \Big) \right] \\
&\ge n \inf_{\mathbf{b} \in \mathbb{B}} \Big[\bar{\alpha}(\hat{\theta}_{\tau_n}; \mathbf{b}) - \bar{\alpha}(\theta; \mathbf{b}) \Big] \\
&+ (\tau_n - n) \min_{2 \le j \le m-1} \min_{X_{1:j-1}} \inf_{\mathbf{b} \in \mathbb{B}} \Big[\bar{\alpha}(\hat{\theta}_{\tau_n}; \mathbf{b} \,|\, X_{1:j-1}) - \bar{\alpha}(\theta; \mathbf{b} | X_{1:j-1}) \Big],
\end{aligned}$$

where $X_{1:j-1} := (X_1, \ldots, X_{j-1})$ is a vector of $j-1$ distinct responses on an item with parameter $\mathbf{b}$. On the other hand, $I_{\tau_n}(\theta) \le \tau_n K$, which implies that

$$\begin{aligned}
\frac{S_{\tau_n}(\theta) - S_{\tau_n}(\hat{\theta}_{\tau_n})}{I_{\tau_n}(\theta)} &\ge \frac{1}{K} \inf_{\mathbf{b} \in \mathbb{B}} [\bar{\alpha}(\hat{\theta}_{\tau_n}; \mathbf{b}) - \bar{\alpha}(\theta; \mathbf{b})] \\
&+ \frac{1}{K} \min_{2 \le j \le m-1} \min_{X_{1:j-1}} \inf_{\mathbf{b} \in \mathbb{B}} \Big[\bar{\alpha}(\hat{\theta}_{\tau_n}; \mathbf{b} \,|\, X_{1:j-1}) - \bar{\alpha}(\theta; \mathbf{b} | X_{1:j-1}) \Big].
\end{aligned}$$

On the event $\{\limsup_n \hat{\theta}_{\tau_n} \to \infty\}$ there is a subsequence $(\hat{\theta}_{\tau_{n_j}})$ of $(\hat{\theta}_{\tau_n})$ such that $\hat{\theta}_{\tau_{n_j}} \to \infty$ and from (**??**) we have

$$\liminf_{n_j \to \infty} \inf_{\mathbf{b} \in \mathbb{B}} \Big[\bar{\alpha}(\hat{\theta}_{\tau_{n_j}}; \mathbf{b}) - \bar{\alpha}(\theta; \mathbf{b}) \Big] > 0.$$

Similarly, due to Lemma 6 (ii), for any $2 \leq j \leq m-1$ and $X_{1:j-1}$ we have

$$\liminf_{n_j \to \infty} \inf_{\mathbf{b} \in \mathbb{B}} \left[\bar{\alpha}(\hat{\theta}_{\tau_{n_j}}; \mathbf{b} \,|\, X_{1:j-1}) - \bar{\alpha}(\theta; \mathbf{b} \,|\, X_{1:j-1}) \right] \geq 0.$$

Therefore,

$$\liminf_{n_j} \frac{S_{\tau_{n_j}}(\theta) - S_{\tau_{n_j}}(\hat{\theta}_{\tau_{n_j}})}{I_{\tau_{n_j}}(\theta)} > 0,$$

and comparing with (**??**) we conclude that $\mathsf{P}(\limsup_n \hat{\theta}_{\tau_n} = \infty) = 0$. Similarly we can show that $\mathsf{P}(\limsup_n \hat{\theta}_{\tau_n} = -\infty) = 0$, which proves (**??**) and, consequently, the strong consistency of $\hat{\theta}_{\tau_n}$. In order to prove the second claim of the theorem, we need to show that

$$\frac{|I_{\tau_n}(\hat{\theta}_{\tau_n}) - I_{\tau_n}(\theta)|}{I_{\tau_n}(\theta)} \to 0 \quad \mathsf{P}_\theta - \text{a.s.} \tag{S3.6}$$

But $I_{\tau_n}(\theta) \geq n\, J_*(\theta)$, whereas $|I_{\tau_n}(\hat{\theta}_{\tau_n}) - I_{\tau_n}(\theta)|$ is bounded above by

$$\begin{aligned} &\sum_{i=1}^{n} |J(\hat{\theta}_{\tau_n}; \mathbf{b}_i) - J(\theta; \mathbf{b}_i)| + \sum_{i=1}^{n} \sum_{j=2}^{g^i_{\tau_n}} \left| J(\hat{\theta}_{\tau_n}; \mathbf{b}_i | X^i_{1:j-1}) - J(\theta; \mathbf{b}_i | X^i_{1:j-1}) \right| \\ &\leq n \sup_{\mathbf{b} \in \mathbb{B}} |J(\hat{\theta}_{\tau_n}; \mathbf{b}) - J(\theta; \mathbf{b})| \\ &+ (\tau_n - n) \max_{2 \leq j \leq m-1} \max_{X_{1:j-1}} \sup_{\mathbf{b} \in \mathbb{B}} \left| J(\hat{\theta}_{\tau_n}; \mathbf{b} | X_{1:j-1}) - J(\theta; \mathbf{b} | X_{1:j-1}) \right|, \end{aligned}$$

where again $X_{1:j-1} := (X_1, \ldots, X_{j-1})$ is a vector of $j-1$ distinct responses on an item with parameter $\mathbf{b}$. Therefore, the ratio in (**??**) is bounded above by

$$\begin{aligned} &\frac{1}{J_*(\theta)} \sup_{\mathbf{b} \in \mathbb{B}} |J(\hat{\theta}_{\tau_n}; \mathbf{b}) - J(\theta; \mathbf{b})| \\ &+ \frac{m-2}{J_*(\theta)} \max_{2 \leq j \leq m-1} \max_{X_{1:j-1}} \sup_{\mathbf{b} \in \mathbb{B}} \left| J(\hat{\theta}_{\tau_n}; \mathbf{b} | X_{1:j-1}) - J(\theta; \mathbf{b} | X_{1:j-1}) \right|. \end{aligned}$$

But similarly to (**??**) we can show that

$$\sup_{\mathbf{b} \in \mathbb{B}} |J(\hat{\theta}_{\tau_n}; \mathbf{b}) - J(\theta; \mathbf{b})| \to 0 \quad \mathsf{P}_\theta - \text{a.s.}$$

as well as that for every $2 \leq j \leq m-1$ and $X_{1:j-1}$ we have

$$\sup_{\mathbf{b} \in \mathbb{B}} |J(\hat{\theta}_{\tau_n}; \mathbf{b} \,|\, X_{1:j-1}) - J(\theta; \mathbf{b} \,|\, X_{1:j-1})| \to 0 \quad \mathsf{P}_\theta - \text{a.s.}$$

which completes the proof. ◊

S4 The histogram of item parameters in the discrete item pool

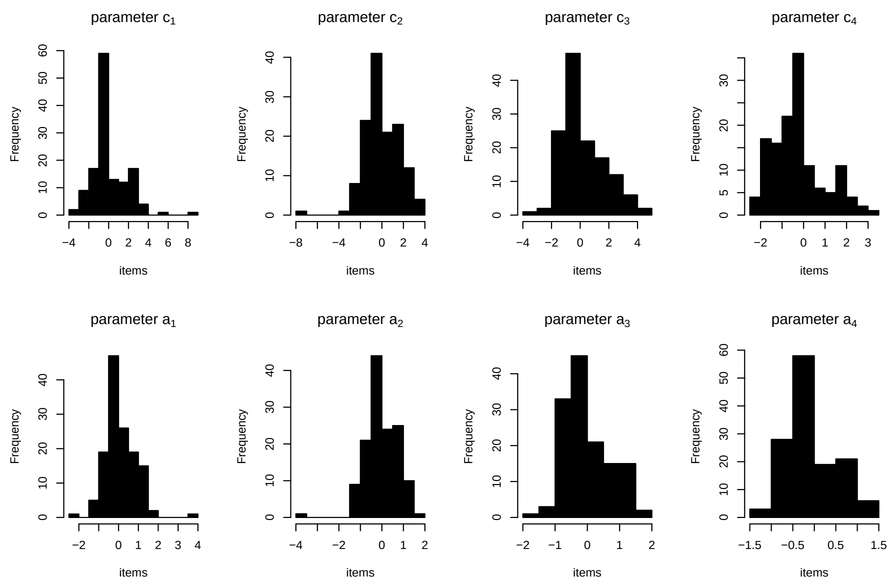

Figure 1: Calibrated item parameters of the nominal response model in a pool with 134 items, each having $m = 4$ categories.