SUPPLEMENTARY MATERIALS: LOCAL BUCKLEY-JAMES ESTIMATION FOR HETEROSCEDASTIC ACCELERATED FAILURE TIME MODEL

Lei Pang, Wenbin Lu*, and Huixia Judy Wang

Merck & Co., Inc, North Carolina State University and George Washington University

S1. Derivation of the Asymptotic Results

To establish the asymptotic results given in Theorem 1, as in Jin et al. (2006), we assume that the tail modification considered in Lai and Ying (1991, p. 1376) is used in the construction of the estimating function. Define the linear indices $\mu_i = X_i^T\beta$ and $\mu_{ib} = X_i^T b$, and the residual of censoring time $c_i(b) = C_i - X_i^T b$. Note that $\tilde{e}_i(b) = \min\{e_i(b), c_i(b)\}$. Throughout, when $b = \beta$, β will be omitted for notational brevity whenever there is no confusion. For example, we use $\tilde{e}_i = Y_i - X_i^T\beta$ instead of $\tilde{e}_i(\beta)$. Define $Y_n(t, b) = \sum_{i=1}^n (X_i - \bar{X}_n) I(\tilde{e}_i(b) \geq t)$. For the local Kaplan-Meier estimate, define the local empirical-type processes:

$$\begin{aligned} Y_n(t, b|v) &= \sum_{i=1}^n I(\tilde{e}_i(b) \geq t)\frac{1}{h}K(\frac{X_i^T b - v}{h}), \\ N_n(t, b|v) &= \sum_{i=1}^n I(\tilde{e}_i(b) \leq t, \delta_i = 1)\frac{1}{h}K(\frac{X_i^T b - v}{h}). \end{aligned}$$

We have

$$\log\{1 - \hat{F}_b(t|v)\} = -\int_{-\infty}^t \frac{dN_n(s, b|v)}{Y_n(s, b|v)} + o_p(1).$$

Denote the limit of $\hat{F}_b(t|v)$ by

$$F_b^*(t|v) = 1 - \exp\Big\{-\int_{-\infty}^t \frac{dN^*(s, b|v)}{Y^*(s, b|v)}\Big\}, \tag{A.1}$$

where $N^*(s, b|v) = \lim_{n\to\infty} \frac{1}{n} N_n(s, b|v)$ and $Y^*(s, b|v) = \lim_{n\to\infty} \frac{1}{n} Y_n(s, b|v)$. If $b = \beta$, we have $dN^*(s, \beta|v) = G(s|v) f_\mu(v) dF_\beta(s|v)$ and $Y^*(s, \beta|v) = G(s|v) f_\mu(v)\{1 - F_\beta(s|v)\}$, where $G(s|v) = E\{P(C - X^T\beta > s|X)|X^T\beta = v\}$. It follows that

$$F_\beta^*(t|v) = 1 - \exp\Big\{-\int_{-\infty}^t \frac{dF_\beta(s|v)}{1 - F_\beta(s|v)}\Big\} = F_\beta(t|v).$$

Let $F_i(t) = F_\beta(t|\mu_i)$, $\hat{F}_i(t) = \hat{F}_\beta(t|\mu_i)$, and $F_{ib}(t) = F_b^*(t|\mu_{ib})$.

*Corresponding Author

Under conditions A1-A5, almost surely for sufficiently large n,

$$U_n(b) = \sum_{i=1}^{n} \left\{ \int_{-n^\lambda}^{n^\lambda} t dY_i^x(t,b) + \int_{-n^\lambda}^{n^\lambda} \int_{t}^{n^\lambda} \frac{1-\hat{F}_{ib}(s)}{1-\hat{F}_{ib}(t)} ds dJ_i^x(t,b) \right\},$$

where λ is a constant between zero and one, and

$$V_n(b) = \epsilon_n(b) + \sum_{i=1}^{n} \left\{ \int_{-n^\lambda}^{n^\lambda} t dEY_i^x(t,b) + \int_{-n^\lambda}^{n^\lambda} \int_{t}^{n^\lambda} \frac{1-F_{ib}(s)}{1-F_{ib}(t)} ds dEJ_i^x(t,b) \right\},$$

with $\sup_{\|b\|\le\rho} \|\epsilon_n(b)\| \to 0$. To prove Theorem 1, we need the following lemmas. In particular, Lemma 2-5 are used directly to prove Theorem 1, while Lemma 1 is used to establish Lemma 3 and 5.

Lemma 1. *Assume that conditions A1-A5 hold. For arbitrarily small $\varepsilon > 0$, $0 < \gamma < 1$, $\|a\| \le \rho$ and $\|b\| \le \rho$, we have*

$$\sup \left\{ |\hat{F}_{ib}(t) - F_{ib}(t)| : i = 1, \ldots, n, t \le n^\lambda \right\} = O(n^{-1/3+\varepsilon}), \ a.s.$$

$$\sup \left\{ \left| \frac{1-\hat{F}_{ib}(s)}{1-\hat{F}_{ib}(t)} - \frac{1-F_{ib}(s)}{1-F_{ib}(t)} - \frac{1-\hat{F}_{ia}(s)}{1-\hat{F}_{ia}(t)} + \frac{1-F_{ia}(s)}{1-F_{ia}(t)} \right| : i = 1, \ldots, n, \|a-b\| \le n^{-\gamma}, t \le \min(s,\nu_1), s \le n^\lambda \right\} = O(n^{-1/3-\gamma/2+\varepsilon}), \ a.s. \tag{A.2}$$

$$\sup \left\{ \left| \frac{1-F_{ia}(s)}{1-F_{ia}(t)} - \frac{1-F_{ib}(s)}{1-F_{ib}(t)} \right| : i = 1, \ldots, n, \|a-b\| \le n^{-\gamma}, t \le \min(s,\nu_1), s \le n^\lambda \right\} = O(n^{-\gamma}),$$

$$\sup \left\{ \left| \frac{1-\hat{F}_{ib}(s)}{1-\hat{F}_{ib}(t)} - \frac{1-F_{ib}(s)}{1-F_{ib}(t)} \right| : i = 1, \ldots, n, t \le \min(s,\nu_1), s \le n^\lambda \right\} = O(n^{-1/3+\varepsilon}), \ a.s.$$

$$\sup \left\{ \left| \frac{1-\hat{F}_{ib}(s-\zeta)}{1-\hat{F}_{ib}(t-\zeta)} - \frac{1-F_{ib}(s-\zeta)}{1-F_{ib}(t-\zeta)} - \frac{1-\hat{F}_{ib}(s)}{1-\hat{F}_{ib}(t)} + \frac{1-F_{ib}(s)}{1-F_{ib}(t)} \right| : i = 1, \ldots, n, |\zeta| \le n^{-\gamma}, t \le \min(s,\nu_1), s \le n^\lambda \right\} = O(n^{-1/3-\gamma/2+\varepsilon}), \ a.s. \tag{A.3}$$

where ν_1 and λ are the constants used in conditions A5 and A6, respectively.

Lemma 1 states the consistency of the local Kaplan-Meier estimates, and it follows from the consistency of the local empirical-type processes (Theorem 2.1 of González-Manteiga and Cadarso-Suarez (1994)). Here we omit the lengthy proof of Lemma 1.

Lemma 2. *Assume condition A7, $V_n(b)$ satisfies:*

(i). $V_n(\beta) = 0$;

(ii). $V_n(b) = \mathbf{\Gamma} n(b-\beta) + o(n\|b-\beta\|)$ *uniformly in* $\|b-\beta\| \le n^{-\lambda}$.

Proof. We want to show

$$V_n(\beta) = \sum_{i=1}^n \left\{ \int_{-\infty}^{\infty} t dEY_i^x(t,\beta) + \int_{-\infty}^{\infty}\int_t^{\infty} \frac{1-F_i(s)}{1-F_i(t)} ds\, dEJ_i^x(t,\beta) \right\} = 0.$$

Denote $Y_i(t) = I(\tilde{e}_i \ge t)$ and $J_i(t) = I(\tilde{e}_i \ge t, \delta_i = 0)$. It suffices to show that for any i,

$$\int_{-\infty}^{\infty} t dE\{Y_i(t)|X_i\} + \int_{-\infty}^{\infty}\int_t^{\infty} \frac{1-F_i(s)}{1-F_i(t)} ds\, dE\{J_i(t)|X_i\} = \int_{-\infty}^{\infty} t d\{1-F_i(t)\},$$

which is a constant across i under our model assumption.

Let $G(t|X_i) = P(C_i \ge t|X_i)$. It follows that $E\{Y_i(t)|X_i\} = \{1-F_i(t)\}G(t|X_i)$ and $dE\{J_i(t)|X_i\} = \{1-F_i(t)\}dG(t|X_i)$. Integration by parts gives

$$\begin{aligned} & \int_u^{\infty} td[\{1-F_i(t)\}G(t|X_i)] + \int_u^{\infty}\int_t^{\infty}\{1-F_i(s)\}dsdG(t|X_i) \\ = \quad & -G(u|X_i)u\{1-F_i(t)\} - \int_u^{\infty} G(t|X_i)\{1-F_i(t)\}dt \\ & -G(u|X_i)\int_u^{\infty}\{1-F_i(t)\}ds + \int_u^{\infty} G(t|X_i)\{1-F_i(t)\}dt \\ = \quad & G(u|X_i)\int_u^{\infty} sd\{1-F_i(t)\}. \end{aligned}$$

Let $u \to -\infty$, we have the desired conclusion that $V_n(\beta) = 0$.

Since $V_n(b)$ is a smooth function of b, by Taylor's expansion,

$$\begin{aligned} V_n(b) - V_n(\beta) \quad = \quad & \sum_{i=1}^n \Big[\int_{-\infty}^{\infty} td\{EY_i^x(t,b) - EY_i^x(t,\beta)\} \\ + \quad & \int_{-\infty}^{\infty}\int_t^{\infty}\left\{ \frac{1-F_{ib}(s)}{1-F_{ib}(t)} dsdEJ_i^x(t,b) - \frac{1-F_i(s)}{1-F_i(t)} dsdEJ_i^x(t,\beta) \right\} \Big]. \end{aligned}$$

Define $\mathbf{\Gamma}_i(b-\beta)$ as the linear majorant of

$$\int_{-\infty}^{\infty} td\{EY_i^x(t,b) - EY_i^x(t,\beta)\} + \int_{-\infty}^{\infty}\int_t^{\infty}\left\{ \frac{1-F_{ib}(s)}{1-F_{ib}(t)} dsdEJ_i^x(t,b) - \frac{1-F_i(s)}{1-F_i(t)} dsdEJ_i^x(t,\beta) \right\},$$

we have

$$\begin{aligned} V_n(b) - V_n(\beta) \quad &= \quad n^{-1}\sum_{i=1}^n \mathbf{\Gamma}_i n(b-\beta) + o_p(n\|b-\beta\|) \\ &= \quad \mathbf{\Gamma}_n n(b-\beta) + o_p(n\|b-\beta\|). \end{aligned}$$

Define $\mathbf{\Gamma} = \lim_{n\to\infty} \mathbf{\Gamma}_n$ and by assumption A7, $\mathbf{\Gamma}$ is positive definite. It follows that almost surely for large n, when $\|b-\beta\| \le n^{-\lambda}$, $V_n(b) - V_n(\beta) = \mathbf{\Gamma} n(b-\beta) + o_p(n\|b-\beta\|)$. $\diamond$ $\square$

Lemma 3. *For* $0 < \lambda < \frac{1}{12}$, *we have*

$$\sup_{\|b-\beta\|\le n^{-\lambda}} \frac{1}{\sqrt{n}} \|U_n(b) - V_n(b) - U_n(\beta) + V_n(\beta)\| = o_p(\max\{1, \sqrt{n}(\|b-\beta\|)\}). \tag{A.4}$$

Proof. To prove Lemma 3, we write $U_n(b) - V_n(b) - U_n(\beta) + V_n(\beta)$ as summation of three parts and bound each part by Lemma 1. Recall that $Y_n(t,b) = \sum_{i=1}^n (X_i - \bar{X}_n) I(\tilde{e}_i(b) \ge t)$. We have

$$\begin{aligned}
& U_n(b) - V_n(b) - U_n(\beta) + V_n(\beta) \\
= & \int_{-n^\lambda}^{n^\lambda} t\{dY_n(t,b) - dEY_n(t,b) - dY_n(t,\beta) + dEY_n(t,\beta)\} \\
& + \sum_{i=1}^n \{ \int_{-n^\lambda}^{n^\lambda} \int_t^{n^\lambda} \frac{1-\hat{F}_{ib}(s)}{1-\hat{F}_{ib}(t)} ds dJ_i^x(t,b) - \int_{-n^\lambda}^{n^\lambda} \int_t^{n^\lambda} \frac{1-F_{ib}(s)}{1-F_{ib}(t)} ds dEJ_i^x(t,b) \\
& - \int_{-n^\lambda}^{n^\lambda} \int_t^{n^\lambda} \frac{1-\hat{F}_i(s)}{1-\hat{F}_i(t)} ds dJ_i^x(t,\beta) + \int_{-n^\lambda}^{n^\lambda} \int_t^{n^\lambda} \frac{1-F_i(s)}{1-F_i(t)} ds dEJ_i^x(t,\beta) \}.
\end{aligned}$$

After some algebraic manipulations, we can divide $U_n(b) - V_n(b) - U_n(\beta) + V_n(\beta)$ further into the sum of the following components:

$$D_1(b) = \int_{-n^\lambda}^{n^\lambda} t\{dY_n(t,b) - dEY_n(t,b) - dY_n(t,\beta) + dEY_n(t,\beta)\},$$

$$\begin{aligned}
D_2(b) &= \sum_{i=1}^n \Big[\int_{-n^\lambda}^{n^\lambda} \int_t^{n^\lambda} \frac{1-F_{ib}(s)}{1-F_{ib}(t)} ds \{dJ_i^x(t,b) - dEJ_i^x(t,b)\} \\
&\quad - \int_{-n^\lambda}^{n^\lambda} \int_t^{n^\lambda} \frac{1-F_i(s)}{1-F_i(t)} ds \{dJ_i^x(t,\beta) - dEJ_i^x(t,\beta)\} \Big],
\end{aligned}$$

$$\begin{aligned}
D_3(b) &= \sum_{i=1}^n \Big[\int_{-n^\lambda}^{n^\lambda} \int_t^{n^\lambda} \{ \frac{1-\hat{F}_{ib}(s)}{1-\hat{F}_{ib}(t)} - \frac{1-F_{ib}(s)}{1-F_{ib}(t)} \} ds dJ_i^x(t,b) \\
&\quad - \int_{-n^\lambda}^{n^\lambda} \int_t^{n^\lambda} \{ \frac{1-\hat{F}_i(s)}{1-\hat{F}_i(t)} - \frac{1-F_i(s)}{1-F_i(t)} \} ds dJ_i^x(t,\beta) \Big].
\end{aligned}$$

Integration by parts gives that for sufficiently large n,

$$\begin{aligned}
D_1(b) &= n^\lambda \{Y_n^x(-n^\lambda, b) - EY_n^x(-n^\lambda, b) - Y_n^x(-n^\lambda, \beta) + EY_n^x(-n^\lambda, \beta)\} \\
&\quad - \int_{-n^\lambda}^{n^\lambda} \{Y_n^x(t,b) - EY_n^x(t,b) - Y_n^x(t,\beta) + EY_n^x(t,\beta)\} dt, \quad a.s.
\end{aligned}$$

By Theorem 1 in Lai and Ying (1988), for arbitrarily small $\varepsilon > 0$ and $0 < \gamma < 1$,

$$\sup_{s,\,\|b-\beta\|\le n^{-\gamma}} \|Y_n(s,b) - EY_n(s,b) - Y_n(s,\beta) + EY_n(s,\beta)\| = O_p(n^{(1-\gamma)/2+\varepsilon}).$$

It follows that $\sup_{\|b-\beta\|\le n^{-\gamma}} \|D_1(b)\| = O_p(n^{(1-\gamma)/2+\varepsilon+\lambda})$.

For the second component $D_2(b)$, its ith summand $d_{2i}(b)$ is given by

$$\int_{-n^\lambda}^{n^\lambda}\int_t^{n^\lambda} \frac{1-F_{ib}(s)}{1-F_{ib}(t)} ds\{dJ_i^x(t,b) - dEJ_i^x(t,b)\} - \int_{-n^\lambda}^{n^\lambda}\int_t^{n^\lambda} \frac{1-F_i(s)}{1-F_i(t)} ds\{dJ_i^x(t,\beta) - dEJ_i^x(t,\beta)\}.$$

Taking the expectation of $d_{2i}(b)$, we have

$$\begin{aligned} & E\int_{-n^\lambda}^{n^\lambda}\int_t^{n^\lambda} \left[\frac{1-F_{ib}(s)}{1-F_{ib}(t)} ds\{dJ_i^x(t,b) - dEJ_i^x(t,b)\} - \frac{1-F_i(s)}{1-F_i(t)} ds\{dJ_i^x(t,\beta) - dEJ_i^x(t,\beta)\}\right] \\ = & \int_{-n^\lambda}^{n^\lambda}\int_t^{n^\lambda} \left[\frac{1-F_{ib}(s)}{1-F_{ib}(t)} ds\{dEJ_i^x(t,b) - dEJ_i^x(t,b)\} - \frac{1-F_i(s)}{1-F_i(t)} ds\{dEJ_i^x(t,\beta) - dEJ_i^x(t,\beta)\}\right] \\ = & \ 0. \end{aligned}$$

Under our assumptions, $d_{2i}(b), i = 1, \ldots, n$ are bounded i.i.d. random variables. In addition,

$$\begin{aligned} d_{2i}(b) \ = & \int_{c_i(b)}^{n^\lambda} \frac{1-F_{ib}(s)}{1-F_{ib}(c_i(b))} ds - \int_{c_i(\beta)}^{n^\lambda} \frac{1-F_i(s)}{1-F_i(c_i(\beta))} ds \\ & + \int_{-n^\lambda}^{n^\lambda}\int_t^{n^\lambda} \frac{1-F_i(s)}{1-F_i(t)} ds dEJ_i^x(t,\beta) - \int_{-n^\lambda}^{n^\lambda}\int_t^{n^\lambda} \frac{1-F_{ib}(s)}{1-F_{ib}(t)} ds dEJ_i^x(t,b), \end{aligned}$$

where the first part

$$\begin{aligned} & \int_{c_i(b)}^{n^\lambda} \frac{1-F_{ib}(s)}{1-F_{ib}(c_i(b))} ds - \int_{c_i(\beta)}^{n^\lambda} \frac{1-F_i(s)}{1-F_i(c_i(\beta))} ds \\ = & \int_{c_i(b)}^{n^\lambda} \left\{\frac{1-F_{ib}(s)}{1-F_{ib}\{c_i(b)\}} - \frac{1-F_i(s)}{1-F_i\{c_i(b)\}}\right\} ds \\ & + \int_{c_i(\beta)}^{n^\lambda} \{1-F_i(s)\} ds \left\{\frac{1}{1-F_i\{c_i(b)\}} - \frac{1}{1-F_i\{c_i(\beta)\}}\right\} + \int_{c_i(b)}^{c_i(\beta)} \frac{1-F_i(s)}{1-F_i\{c_i(b)\}} ds. \end{aligned}$$

By Lemma 1, we can bound the first part of $d_{2i}(b)$ as

$$\sup_{\|b-\beta\|\le n^{-\gamma}} \left| \int_{c_i(b)}^{n^\lambda} \frac{1-F_{ib}(s)}{1-F_{ib}\{c_i(b)\}} ds - \int_{c_i(\beta)}^{n^\lambda} \frac{1-F_i(s)}{1-F_i\{c_i(\beta)\}} ds \right| = O_p(n^{-\gamma+\lambda}).$$

The second part $d_{2i}(b)$ is a smooth function of b. In view of the proof of Lemma 2, by Taylor's expansion, it follows that

$$\sup_{\|b-\beta\|\le n^{-\gamma}} \left\| \int_{-n^\lambda}^{n^\lambda} \int_t^{n^\lambda} \frac{1-F_i(s)}{1-F_i(t)} dsdEJ_i^x(t,\beta) - \int_{-n^\lambda}^{n^\lambda} \int_t^{n^\lambda} \frac{1-F_{ib}(s)}{1-F_{ib}(t)} dsdEJ_i^x(t,b) \right\| = O_p(n^{-\gamma+\lambda}).$$

Thus, $\sup_{\|b-\beta\|\le n^{-\gamma}} \|d_{2i}(b)\| = O_p(n^{-\gamma+\lambda})$. It follows that $\sup_{\|b-\beta\|\le n^{-\gamma}} \|D_2(b)\| = O_p(n^{1/2-\gamma+\lambda+\varepsilon})$.

The ith summand of $D_3(b)$ is

$$\begin{aligned}
&\int_{-n^\lambda}^{n^\lambda} \int_t^{n^\lambda} \left[\left\{ \frac{1-\hat F_{ib}(s)}{1-\hat F_{ib}(t)} - \frac{1-F_{ib}(s)}{1-F_{ib}(t)} \right\} - \left\{ \frac{1-\hat F_i(s)}{1-\hat F_i(t)} - \frac{1-F_i(s)}{1-F_i(t)} \right\} \right] dsdJ_i^x(t,\beta) \\
= \;& (X_i - \bar X_n)\delta_i \left[\int_{c_i(b)}^{n^\lambda} \left\{ \frac{1-\hat F_{ib}(s)}{1-\hat F_{ib}\{c_i(b)\}} - \frac{1-F_{ib}(s)}{1-F_{ib}\{c_i(b)\}} \right\} ds \right. \\
&\left. - \int_{c_i(\beta)}^{n^\lambda} \left\{ \frac{1-\hat F_i(s)}{1-\hat F_i\{c_i(\beta)\}} - \frac{1-F_i(s)}{1-F_i\{c_i(\beta)\}} \right\} ds \right] \\
= \;& (X_i - \bar X_n)\delta_i \left[\int_{c_i(b)}^{n^\lambda} \left\{ \frac{1-\hat F_{ib}(s)}{1-\hat F_{ib}\{c_i(b)\}} - \frac{1-F_{ib}(s)}{1-F_{ib}\{c_i(b)\}} - \frac{1-\hat F_i(s)}{1-\hat F_i\{c_i(b)\}} + \frac{1-F_i(s)}{1-F_i\{c_i(b)\}} \right\} ds \right. \\
&\left. + \int_{c_i(b)}^{n^\lambda} \left\{ \frac{1-\hat F_i(s)}{1-\hat F_i\{c_i(b)\}} - \frac{1-F_i(s)}{1-F_i\{c_i(b)\}} \right\} ds - \int_{c_i(\beta)}^{n^\lambda} \left\{ \frac{1-\hat F_i(s)}{1-\hat F_i\{c_i(\beta)\}} - \frac{1-F_i(s)}{1-F_i\{c_i(\beta)\}} \right\} ds \right],
\end{aligned}$$

where

$$\begin{aligned}
&\int_{c_i(b)}^{n^\lambda} \left\{ \frac{1-\hat F_i(s)}{1-\hat F_i(c_i(b))} - \frac{1-F_i(s)}{1-F_i(c_i(b))} \right\} ds \\
= \;& \int_{c_i(\beta)}^{n^\lambda} \left\{ \frac{1-\hat F_i(s - X_i^T(b-\beta))}{1-\hat F_i(c_i(\beta) - X_i^T(b-\beta))} - \frac{1-F_i(s - X_i^T(b-\beta))}{1-F_i(c_i(\beta) - X_i^T(b-\beta))} \right\} ds \\
+ \;& \int_{n^\lambda - X_i^T(b-\beta)}^{n^\lambda} \left\{ \frac{1-\hat F_i(s)}{1-\hat F_i(c_i(b))} - \frac{1-F_i(s)}{1-F_i(c_i(b))} \right\} ds,
\end{aligned}$$

and the last term above is $o_p(n^{-1/3+\varepsilon})$ by Lemma 1. Thus,

$$\begin{aligned}
& d_{3i}(b) \\
= \;& (X_i - \bar X_n)\delta_i \left[\int_{c_i(b)}^{n^\lambda} \left\{ \frac{1-\hat F_{ib}(s)}{1-\hat F_{ib}(c_i(b))} - \frac{1-F_{ib}(s)}{1-F_{ib}(c_i(b))} - \frac{1-\hat F_i(s)}{1-\hat F_i(c_i(b))} + \frac{1-F_i(s)}{1-F_i(c_i(b))} \right\} ds \right. \\
&+ \int_{c_i(\beta)}^{n^\lambda} \left\{ \frac{1-\hat F_i(s - X_i^T(b-\beta))}{1-\hat F_i(c_i(\beta) - X_i^T(b-\beta))} - \frac{1-F_i(s - X_i^T(b-\beta))}{1-F_i(c_i(\beta) - X_i^T(b-\beta))} \right. \\
&\left.\left. - \frac{1-\hat F_i(s)}{1-\hat F_i(c_i(\beta))} + \frac{1-F_i(s)}{1-F_i(c_i(\beta))} \right\} ds \right] + o_p(n^{-1/3+\varepsilon}).
\end{aligned}$$

In view of (A.2) and (A.3), it follows that $\|d_{3i}(b)\|$ is of order $O_p(n^{-1/3-\gamma/2+\lambda+\varepsilon})$ and we have $\sup_{\|b-\beta\|\le n^{-\gamma}} \|D_3(b)\| = O_p(n^{2/3-\gamma/2+\lambda+\varepsilon})$. Therefore,

$$\begin{aligned}\sup_{\|b-\beta\|\le n^{-\gamma}} \frac{1}{\sqrt{n}}\|U_n(b)-V_n(b)-U_n(\beta)+V_n(\beta)\| &= O_p(n^{1/6-\gamma/2+\lambda+\varepsilon}+n^{-\gamma/2+\varepsilon}+n^{-\gamma/2+\varepsilon+\lambda}) \\ &= O_p(n^{1/6-\gamma/2+\lambda+\varepsilon}).\end{aligned}$$

For $\gamma \ge 1/2$ and $0<\lambda+\varepsilon<1/12$, we have $1/6-\gamma/2+\lambda+\varepsilon<0$. This implies

$$\sup_{\|b-\beta\|\le n^{-1/2}} \frac{1}{\sqrt{n}}\|U_n(b)-V_n(b)-U_n(\beta)+V_n(\beta)\| = o_p(1).$$

For $\lambda \le \gamma < 1/2$, suppose $n^{-\pi-\varsigma} \le \|b-\beta\| \le n^{-\pi}$, where $\lambda \le \pi < 1/2$ and $\varsigma > 0$ is an arbitrarily small number. We have

$$\sup_{n^{-\pi-\varsigma}\le\|b-\beta\|\le n^{-\pi}} \frac{1}{\sqrt{n}}\|U_n(b)-V_n(b)-U_n(\beta)+V_n(\beta)\| = O_p(n^{1/6-\pi/2+\lambda+\epsilon}).$$

Note that $1/6-\pi/2+\lambda+\varepsilon < 1/2-\pi-\varsigma$ for $0<\lambda+\varepsilon<1/12$ since ς and ε can be arbitrarily small. Therefore,

$$\sup_{n^{-\pi-\varsigma}\le\|b-\beta\|\le n^{-\pi}} \frac{1}{\sqrt{n}}\|U_n(b)-V_n(b)-U_n(\beta)+V_n(\beta)\| = o_p(\sqrt{n}\|b-\beta\|).$$

It follows that for $0<\lambda<\frac{1}{12}$,

$$\sup_{\|b-\beta\|\le n^{-\lambda}} \frac{1}{\sqrt{n}}\|U_n(b)-V_n(b)-U_n(\beta)+V_n(\beta)\| = o_p(\max\{1,\sqrt{n}\|b-\beta\|\}). \quad \diamond$$

□

Lemma 4. *There exists a positive definite matrix* $\mathbf{\Sigma}$ *such that*

$$\frac{1}{\sqrt{n}}U_n(\beta) \xrightarrow{d} N(\mathbf{0},\mathbf{\Sigma}). \tag{A.5}$$

Proof. We prove Lemma 4 by expressing $U_n(\beta)$ as the sum of three parts and showing that $U_n(\beta)$ is a summation of i.i.d. random vectors with mean 0. The i.i.d. representation of $\hat{F}_i(t)-F_i(t)$ (González-Manteiga and Cadarso-Suarez, 1994) is essential in the proof of Lemma 4. Define

$$u_{n1}(\beta) = \frac{1}{\sqrt{n}}\sum_{i=1}^n (X_i-\bar{X}_n)\Big\{\int_{-\infty}^{\infty} t dY_i(t) + \int_{-\infty}^{\infty}\int_t^{\infty} \frac{1-F_i(s)}{1-F_i(t)} ds dJ_i(t)\Big\},$$

$$u_{n2}(\beta) = \frac{1}{\sqrt{n}}\sum_{i=1}^n (X_i-\bar{X}_n)\int_{-\infty}^{\infty}\int_t^{\infty} \frac{1-F_i(s)}{\{1-F_i(t)\}^2}\{\hat{F}_i(t)-F_i(t)\} ds dJ_i(t),$$

$$u_{n3}(\beta) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i - \bar{X}_n) \int_{-\infty}^{\infty} \int_{t}^{\infty} \frac{\hat{F}_i(s) - F_i(s)}{1 - F_i(t)} ds dJ_i(t).$$

Based on the consistency of $\hat{F}_i(s)$, we have

$$\frac{1}{\sqrt{n}} U_n(\beta) = u_{n1}(\beta) + u_{n2}(\beta) + u_{n3}(\beta) + o_p(1).$$

Define

$$A_i = \int_{-\infty}^{\infty} t dY_i(t) + \int_{-\infty}^{\infty} \int_{t}^{\infty} \frac{1 - F_i(s)}{1 - F_i(t)} ds dJ_i(t).$$

We have shown that conditional on X_i, A_i's have common conditional expectation denoted by μ_A. It follows that

$$\begin{aligned} u_{n1}(\beta) &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i - \bar{X}_n)(A_i - \mu_A) \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i - \mu_X)(A_i - \mu_A) - \sqrt{n}(\bar{X}_n - \mu_X)(\bar{A} - \mu_A) \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i - \mu_X)(A_i - \mu_A) + o_p(1). \end{aligned}$$

Thus, $u_{n1}(\beta)$ has a limiting normal distribution. Similarly, we can replace $\bar{X}_n$ with μ_X in $u_{n2}(\beta)$ and $u_{n3}(\beta)$ without changing their limiting distributions.

For $u_{n2}(\beta)$ and $u_{n3}(\beta)$, we replace $\hat{F}_i(t) - F_i(t)$ with its i.i.d. expansion. It follows from Theorem 2.3 of Gonzalez-Manteiga and Cadarso- Suarez (1994) that under our assumptions,

$$\hat{F}_i(t) - F_i(t) = \frac{1}{nh} \sum_{j=1}^{n} K(\frac{\mu_j - \mu_i}{h}) \xi(Y_j, \delta_j, t, \mu_i) + O_p(n^{-\frac{1}{2}+\varepsilon}),$$

where $\xi(Y_j, \delta_j, t, \mu_i)$, $j = 1 \ldots, n$, are (conditional on X_i) independent random variables with zero mean and finite variance for any t . Plugging the i.i.d. representation of $\hat{F}_i(t) - F_i(t)$ into $u_{n2}(\beta)$ and $u_{n3}(\beta)$, we have

$$\begin{aligned} & u_{n2}(\beta) \\ = \ & \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i - \mu_X) \int_{-\infty}^{\infty} \frac{\sum_{j=1}^{n} K(\frac{\mu_i - \mu_j}{h}) \xi(Y_j, \delta_j, t, \mu_i)}{nh\{1 - F_i(t)\}^2} \int_{t}^{\infty} \{1 - F_i(s)\} ds dJ_i(t) + o_p(1) \\ = \ & \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \int_{-\infty}^{\infty} \frac{1}{n} \sum_{i=1}^{n} (X_i - \mu_X) \frac{K(\frac{\mu_i - \mu_j}{h}) \xi(Y_j, \delta_j, t, \mu_i)}{h\{1 - F_i(t)\}^2} \int_{t}^{\infty} \{1 - F_i(s)\} ds dJ_i(t) + o_p(1) \\ = \ & \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \left[\frac{1}{n} \sum_{i=1}^{n} (X_i - \mu_X) \frac{1}{h} K(\frac{\mu_i - \mu_j}{h}) \int_{-\infty}^{\infty} \frac{\xi(Y_j, \delta_j, t, \mu_i)}{\{1 - F(t|\mu_i)\}^2} \int_{t}^{\infty} \{1 - F(s|\mu_i)\} ds dJ_i(t) \right] \\ & + o_p(1). \end{aligned}$$

Since $dE\{J_i(t)|X_i\} = \{1 - F_\beta(t|X_i^T\beta)\}dG(t|X_i)$, we have

$$
\begin{aligned}
& \frac{1}{n}\sum_{i=1}^{n}(X_i-\mu_X)\frac{1}{h}K(\frac{\mu_i-\mu_j}{h})\int_{-\infty}^{\infty}\frac{\xi(Y_j,\delta_j,t,\mu_i)}{\{1-F(t|\mu_i)\}^2}\int_t^{\infty}\{1-F(s|\mu_i)\}dsdJ_i(t) \\
= \; & E\left[(X-\mu_X)\int_{-\infty}^{\infty}\frac{1}{h}K(\frac{\mu-\mu_j}{h})\frac{\xi(Y_j,\delta_j,t,\mu)}{1-F(t|\mu)}\int_t^{\infty}\{1-F(s|\mu)\}dsdG(t|X)\right]+o_p(n^{-1/2}) \\
= \; & E_X\left[(X-\mu_X)f_\mu(\mu_j)\int_{-\infty}^{\infty}\frac{\xi(Y_j,\delta_j,t,\mu_j)}{1-F(t|\mu_j)}\int_t^{\infty}\{1-F(s|\mu_j)\}dsdG(t|X)|X^T\beta=\mu_j\right]+o_p(n^{-1/2}) \\
\equiv \; & \eta(Y_j,\delta_j,\mu_j)+o_p(n^{-1/2}),
\end{aligned}
$$

with $E\{\eta(Y_j,\delta_j,\mu_j)\} = 0$. Then we have

$$u_{n2}(\beta) = \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\eta(Y_j,\delta_j,\mu_j) + o_p(1).$$

Similarly, we can show that

$$
\begin{aligned}
u_{n3}(\beta) &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(X_i-\mu_X)\int_{-\infty}^{\infty}\int_t^{\infty}\frac{\frac{1}{nh}\sum_{j=1}^{n}K(\frac{\mu_i-\mu_j}{h})\xi(Y_j,\delta_j,s,\mu_i)}{1-F_i(t)}dsdJ_i(t)+o_p(1) \\
&= \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\frac{1}{n}\sum_{j=1}^{n}(X_i-\mu_X)\frac{1}{h}K(\frac{\mu_i-\mu_j}{h})\int_{-\infty}^{\infty}\frac{\int_t^{\infty}\xi(Y_j,\delta_j,s,\mu_i)}{1-F(t|\mu_i)}dsdJ_i(t)+o_p(1) \\
&= \frac{1}{\sqrt{n}}\sum_{j=1}^{n}E_X\left\{(X-\mu_X)f_\mu(\mu_j)\int_{-\infty}^{\infty}\xi(Y_j,\delta_j,s,\mu_j)dsdG(t|X)|X^T\beta=\mu_j\right\}+o_p(1) \\
&\equiv \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\theta(Y_j,\delta_j,\mu_j)+o_p(1),
\end{aligned}
$$

with $E\{\theta(Y_j,\delta_j,\mu_j)\} = 0$.

Therefore, we have

$$\frac{1}{\sqrt{n}}U_n(\beta) = \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\left\{(X_j-\mu_X)(A_j-\mu_A)+\theta(Y_j,\delta_j,\mu_j)+\eta(Y_j,\delta_j,\mu_j)\right\}+o_p(1)$$

Denote the covariance matrix of $(X_j-\mu_X)(A_j-\mu_A)+\theta(Y_j,\delta_j,\mu_j)+\eta(Y_j,\delta_j,\mu_j)$ by $\mathbf{\Sigma}$. By the central limit theorem, Lemma 4 holds. $\diamond$

□

Lemma 5. *With* $0 < \lambda < 1$ *and* $\varepsilon > 0$*,*

$$\sup_{|b|\le\rho}\|U_n(b)-V_n(b)\| = O(n^{2/3+\lambda+\varepsilon}). \tag{A.6}$$

Proof. Lemma 5 gives the bound of the difference between $U_n(b)$ and its smooth approximate

$V_n(b)$. We prove Lemma 5 using similar techniques as those for proving Lemma 1. Note that

$$\begin{aligned} U_n(b) - V_n(b) &= \int_{-n^\lambda}^{n^\lambda} t\{dY_n(t,b) - dEY_n(t,b)\} \\ &+ \sum_{i=1}^n \left\{ \int_{-n^\lambda}^{n^\lambda} \int_t^{n^\lambda} \frac{1-\hat{F}_{ib}(s)}{1-\hat{F}_{ib}(t)} ds dJ_i^x(t,b) - \int_{-n^\lambda}^{n^\lambda} \int_t^{n^\lambda} \frac{1-F_{ib}(s)}{1-F_{ib}(t)} ds dEJ_i^x(t,b) \right\}. \end{aligned}$$

The first term can be written as

$$E_1(b) = n^\lambda\{Y_n(-n^\lambda, b) - EY_n(-n^\lambda, b)\} - \int_{-n^\lambda}^{n^\lambda} \{Y_n(t,b) - EY_n(t,b)\} dt.$$

Similar to the proof of Lemma 3, we can show that $\sup_{|b|<\rho} \|E_1(b)\| = O(n^{1/2+\lambda+\varepsilon})$. In addition, the second term can be written as

$$\begin{aligned} E_2(b) &= \sum_{i=1}^n \int_{-n^\lambda}^{n^\lambda} \int_t^{n^\lambda} \frac{1-F_{ib}(s)}{1-F_{ib}(t)} ds \{dJ_i^x(t,b) - dEJ_i^x(t,b)\} \\ &+ \sum_{i=1}^n \int_{-n^\lambda}^{n^\lambda} \int_t^{n^\lambda} \left\{ \frac{1-\hat{F}_{ib}(s)}{1-\hat{F}_{ib}(t)} - \frac{1-F_{ib}(s)}{1-F_{ib}(t)} \right\} ds dJ_i^x(t,b) \end{aligned}$$

Using the same technique, we can show that $\sup_{|b|<\rho} \|E_2(b)\| = O(n^{2/3+\lambda+\varepsilon})$. Thus, we have $\sup_{|b|\le\rho} \|U_n(b) - V_n(b)\| = O(n^{2/3+\lambda+\varepsilon})$, which is $o(n^{3/4})$ when $\lambda + \varepsilon < 1/12$. ⋄ □

Proof of Theorem 1. For $0 < \lambda < 1/12$ and arbitrarily small $\varepsilon > 0$ with $\lambda + \varepsilon < 1/12$, by Lemma 5, we have

$$\sup_{|b|\le\rho} \|U_n(b) - V_n(b)\| = O(n^{2/3+\lambda+\varepsilon}) = o(n^{3/4}). \tag{A.7}$$

In view of condition A6 and (A.7), it follows that

$$P\big\{U_n(b) \text{ does not have a zero-crossing for large } n \text{ on } \|b-\beta\| > n^{-\lambda}, |b| \le \rho\big\} = 1.$$

This implies $\hat{\beta}_{LBJ} = \beta + o(n^{-\lambda})$.

In addition, by Lemma 3, we have

$$\frac{1}{\sqrt{n}}\{U_n(\hat{\beta}_{LBJ}) - V_n(\hat{\beta}_{LBJ}) - U_n(\beta) + V_n(\beta)\} = o(\max\{1, \sqrt{n}(\hat{\beta}_{LBJ} - \beta)\}).$$

It follows by Lemma 2 that

$$\mathbf{\Gamma}\sqrt{n}(\hat{\beta}_{LBJ} - \beta) = \frac{1}{\sqrt{n}} U_n(\beta) + o(\max\{1, \sqrt{n}(\hat{\beta}_{LBJ} - \beta)\}).$$

Thus, $\sqrt{n}(\hat{\beta}_{LBJ} - \beta)$ converges in distribution to $N(0, \mathbf{\Gamma}^{-1}\mathbf{\Sigma}\mathbf{\Gamma}^{-1})$ by Lemma 4.

References

González-Manteiga, W. and C. Cadarso-Suarez (1994). Asymptotic properties of a generalized Kaplan-Meier estimator with some applications. *Journal of Nonparametric Statistics* **4**, 65–78.

Jin, Z., D. Y. Lin, and Z. Ying (2006). On least-squares regression with censored data. *Biometrika* **93**, 147.

Lai, T. and Z. Ying (1988). Stochastic integrals of empirical-type processes with applications to censored regression. *Journal of Multivariate Analysis* **27**, 334–358.

Lai, T. L. and Z. Ying (1991). Large sample theory of a modified Buckley-James estimator for regression analysis with censored data. *The Annals of Statistics* **19**, 1370–1402.