

QUENCHED CENTRAL LIMIT THEOREMS FOR A STATIONARY LINEAR PROCESS

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Abstract: We establish a sufficient condition under which a central limit theorem for a stationary linear process is quenched. We find a stationary linear process for which the Maxwell-Woodroffe's condition is satisfied, $\sigma_n = \|S_n\|_2 = o(\sqrt{n})$, S_n/σ_n converges to the standard normal law, and the convergence is not quenched; the weak invariance principle does not hold.

Key words and phrases: Hannan condition, martingale differences, Maxwell-Woodroffe condition, quenched central limit theorem, stationary linear process, weak invariance principle.

1. Introduction

Let T be an ergodic automorphism of a probability space $(\Omega, \mathcal{A}, \mu)$. For $h \in L^2$, $Uh = h \circ T$ is a unitary operator; we freely switch from the notation $h \circ T^i$ to $U^i h$, and vice versa.

Let $(\mathcal{F}_i)_i$ be a filtration such that $\mathcal{F}_{i+1} = T^{-1}\mathcal{F}_i$, and $e \in L^2(\mathcal{F}_0) \ominus L^2(\mathcal{F}_{-1})$. For simplicity we suppose $\|e\|_2 = 1$. Let a_i be real numbers with $\sum_{i \in \mathbb{N}} a_i^2 < \infty$ and let

$$f = \sum_{i \leq 0} a_{-i} U^i e.$$

Then $f \in L^2$ and we say that $(f \circ T^i)_i$ is a causal stationary linear process. The stationary linear process is a classical and important case of a (strictly) stationary process and, moreover, any regular stationary process is a sum of stationary linear processes "living" in mutually orthogonal and U -invariant subspaces of L^2 (cf. Volný, Woodroffe and Zhao (2011)). If $e_k \in L^2(\mathcal{F}_0) \ominus L^2(\mathcal{F}_{-1})$, $\|e_k\|_2 = 1$, are mutually orthogonal, $a_{k,i}$ are real numbers with $\sum_{k=1}^{\infty} \sum_{i \in \mathbb{N}} a_{k,i}^2 < \infty$, and if

$$f = \sum_{k=1}^{\infty} \sum_{i \leq 0} a_{k,-i} U^i e_k \tag{1.1}$$

then we say that $(f \circ T^i)_i$ is a causal superlinear process. As shown in Volný, Woodroffe and Zhao (2011), if $f \in L^2$ is \mathcal{F}_0 -measurable and $E(f|\mathcal{F}_{-\infty}) = 0$, then a representation (1.1) exists.

Let $S_n(f) = \sum_{i=0}^{n-1} f \circ T^i$. Recall (Peligrad and Utev (2006)) that if $\sigma_n = \|S_n(f)\|_2 \rightarrow \infty$ then the distributions of $S_n(f)/\sigma_n$ converge weakly to $N(0, 1)$. We will study when this CLT is quenched.

Suppose that the regular conditional probabilities m_ω with respect to the σ -field \mathcal{F}_0 exist. If for μ a.e. ω the distributions $m_\omega(S_n(f)/\sigma_n)^{-1}$ weakly converge to $N(0, 1)$, we say the CLT is quenched. A quenched CLT can be defined using Markov Chains. Suppose that the sigma algebras in $(\mathcal{F}_i)_i$ are countably generated. Any stationary process $(f \circ T^i)_i$ adapted to $(\mathcal{F}_i)_i$ can be expressed, using a homogeneous and stationary Markov Chain $(\xi_i)_i$, as $(g(\xi_i))_i$. A CLT is quenched if it takes place for a.e. starting point. This approach is probably earlier than ours; it has been used in e.g. Derriennic and Lin (2001). For countably generated filtrations the approaches are equivalent. To see this, there exists a function h such that the sigma algebra $\sigma\{h\}$ generated by h equals \mathcal{F}_0 . Then $(h \circ T^i)_i$ is a Markov Chain, and for f there exists a Borel function g such that $f = g(h)$ (cf. e.g. Volný (2010) or Cuny and Volný (2013)).

In the next section, for a stationary linear process, we give a sufficient condition for a quenched CLT. In Section 3, we present a stationary linear processes $(f \circ T^i)_i$ for which the Maxwell-Woodroffe and the Hannan conditions are satisfied but the CLT is not quenched and the weak invariance principle (WIP) does not hold. As noticed in Remarks 3 and 4, under the Maxwell-Woodroffe and Hannan conditions the weak invariance principle for $[S_n(f) - E(S_n(f)|\mathcal{F}_0)]/\sqrt{n}$ is quenched. Because under the Maxwell-Woodroffe and Hannan conditions σ_n/\sqrt{n} converges to a (finite) limit we under the assumptions of Theorem 2 have $\sigma_n/\sqrt{n} \rightarrow 0$. We conjecture that a version of Theorem 2 with $\sigma_n/\sqrt{n} \rightarrow \infty$, and without the Maxwell-Woodroffe and the Hannan conditions, is true.

We study quenched versions of limit theorems for $S_n(f)/\sigma_n$, hence one might call the results “self-normalized quenched central limit theorems”. In Proposition 3 we will see that the results of Theorem 2 are valid not only for norming by σ_n but also for norming by the standard deviation calculated for the measure m_ω (for μ a.e. ω).

2. A Sufficient Condition

Let $(e \circ T^i)$ be the martingale difference sequence

$$f = \sum_{i=0}^{\infty} a_i e \circ T^{-i},$$

where $\sum_{i=0}^{\infty} a_i^2 < \infty$, and let $S_n = \sum_{i=0}^{n-1} f \circ T^i$, $n = 1, 2, \dots$. By definition,

$$\begin{aligned} S_n &= \sum_{j=0}^{n-1} f \circ T^j = \sum_{i=0}^{\infty} \sum_{j=0}^{n-1} a_i e \circ T^{j-i} = \sum_{k=-\infty}^{n-1} \sum_{j=k \vee 0}^{n-1} a_{j-k} e \circ T^k \\ &= \sum_{k=1}^{n-1} b_{n-k} e \circ T^k + \sum_{k=0}^{n-1} (b_{n-k} - b_{-k}) e \circ T^k, \end{aligned}$$

where $u \vee v = \max\{u, v\}$, $b_0 = 0$, $b_j = \sum_{i=0}^{j-1} a_i$, $j \geq 1$.

We take $\bar{\sigma}_n^2 = E[(S_n - E(S_n|\mathcal{F}_0))^2] = \sum_{k=1}^{n-1} b_{n-k}^2$.

Theorem 1. *Let $\bar{\sigma}_n^2 \rightarrow \infty$. If*

(i) *$e \circ T^i$ are iid, or*

(ii)
$$\sup_{n \geq 1} \max_{k \leq n} \frac{nb_k^2}{\bar{\sigma}_n^2} = c < \infty, \tag{2.1}$$

then for $(1/\sigma_n)[S_n - E(S_n|\mathcal{F}_0)]$, a quenched CLT holds true.

REMARK 1. If the sums b_k converge to a limit b such that $\sigma_n^2/n \rightarrow b^2$ then the Heyde condition (cf. e.g. Hall and Heyde (1980, Chap. 5)) is satisfied and we get a CLT. As proved in Volný and Woodroffe (2014), in general, for $S_n - E(S_n|\mathcal{F}_0)$ the CLT is not quenched. Our theorem shows that it is quenched in the particular case that $(f \circ T^i)$ is a stationary linear process with martingale differences innovation.

REMARK 2. Theorem 1 implies a quenched CLT for $S_n - E(S_n|\mathcal{F}_0)$ as soon as $\sum_{k=1}^{\infty} a_k^2 < \infty$, $\liminf_{n \rightarrow \infty} \bar{\sigma}_n^2/n > 0$, and the sequence of $b_k = \sum_{i=0}^k a_i$ is bounded.

Proof of Theorem 1. We have to prove a quenched CLT for the triangular array of random variables $b_{n-k}e \circ T^k/\bar{\sigma}_n$, $k = 1 \dots, n$, $n = 1, 2, \dots$.

The $e \circ T^k$ are iid and they remain iid for the conditional probabilities m_ω as well. From $\bar{\sigma}_n^2 = \sum_{k=1}^{n-1} b_k^2 \rightarrow \infty$ we get the CLT.

Let $e \circ T^k$ be martingale differences and let (2.1) hold. To prove the CLT we use Lachout’s refinement (Lachout (1985)) of the McLeish central limit theorem (McLeish (1974)), applied to regular conditional probabilities with respect to the σ -algebra \mathcal{F}_0 . We thus will prove

(a)
$$E(\max_{k \leq n-1} |b_{n-k}e \circ T^k|/\bar{\sigma}_n \mid \mathcal{F}_0) \rightarrow 0 \text{ a.s.},$$

(b) $\sum_{k=1}^{n-1} b_{n-k}^2 e^2 \circ T^k / \bar{\sigma}_n^2$ converge to a constant a.s..

By (2.1),

$$\frac{b_{n-k}^2}{\bar{\sigma}_n^2} \leq \frac{c}{n}$$

for all n , $1 \leq k \leq n-1$, hence (a) follows in the same way as in Volný and Woodroffe (2014).

To prove (b), let

$$T_n f = \frac{1}{\bar{\sigma}_n^2} \sum_{k=1}^{n-1} b_{n-k}^2 f \circ T^k, \quad f \in L^1.$$

Recall the Banach principle (cf. Krengel (1985, Thm. 7.2)): If

- (i) $T_n : L^1 \rightarrow L^1$ are continuous for every $n \in \mathbb{N}$,
- (ii) for every $f \in L^1$, $\sup_n |T_n f| < \infty$ a.e.,
- (iii) there is a dense subset of $h \in L^1$ for which $(T_n h)_n$ converges a.s.,

then for all $f \in L^1$, $T_n f$ converge a.s..

We verify (i)-(iii). (i) follows from the definition. For (ii),

$$|T_n f| \leq \frac{1}{\bar{\sigma}_n^2} \sum_{k=1}^{n-1} b_{n-k}^2 |f| \circ T^k \leq \frac{c}{n} \sum_{k=1}^{n-1} |f| \circ T^k$$

hence, by the Birkhoff ergodic theorem (cf. Krengel (1985, Thm. 2.3)),

$$\sup |T_n f| < \infty \quad a.s. \quad \forall f \in L^1.$$

We prove (iii). Let $f = g - g \circ T$, $g \in L^\infty$. Then

$$\begin{aligned} T_n f &= \frac{1}{\bar{\sigma}_n^2} \sum_{k=1}^{n-1} b_{n-k}^2 [g \circ T^k - g \circ T^{k+1}] \\ &= \frac{1}{\bar{\sigma}_n^2} \sum_{k=1}^n [b_{n-k}^2 - b_{n-k+1}^2] g \circ T^k + \frac{b_n^2}{\bar{\sigma}_n^2} g \circ T \\ &\leq \frac{1}{\bar{\sigma}_n^2} \sqrt{\sum_{k=1}^n (b_{n-k} + b_{n-k+1})^2} \sqrt{\sum_{k=1}^n a_{n-k+1}^2} \|g\|_\infty + \frac{c}{n} \|g\|_\infty \\ &\leq \frac{2}{\bar{\sigma}_n} \sqrt{1 + \frac{c}{n} A} \|g\|_\infty + \frac{c}{n} \|g\|_\infty, \end{aligned} \tag{2.2}$$

where $A^2 = \sum_{k=1}^\infty a_k^2 < \infty$, hence $T_n f \rightarrow 0$ a.s..

The set of functions $c + g - g \circ T$, $c \in \mathbb{R}$, $g \in L^\infty$, is dense in L^1 (recall that T is ergodic). For $f' = g - g \circ T$ we have $T_n f' \rightarrow 0$ a.s. by the calculation above, for $f'' = c$ we have $T_n f'' = f'' \equiv c$ hence the convergence towards c takes place for $f = f' + f''$.

By the Banach principle we conclude that

$$T_n e^2 = \frac{1}{\bar{\sigma}_n^2} \sum_{k=1}^n b_{n-k}^2 e^2 \circ T^k \tag{2.3}$$

converges almost surely for every $e \in L^2$. Let f^* be the limit in (2.3). Using a similar calculation as in (2.2) we can see that $T_n e^2 - (T_n e^2) \circ T \rightarrow 0$ in L^1 hence $f^* = f^* \circ T$. By ergodicity, f^* is a constant a.s..

3. A Non-quenched CLT

If the process $(f \circ T^i)$ is adapted to the filtration $(\mathcal{F}_i)_i$ and if

$$\sum_{n=1}^{\infty} \frac{\|E(S_n(f) | \mathcal{F}_0)\|_2}{n^{3/2}} < \infty,$$

we say that the Maxwell-Woodroffe condition takes place. Let $P_i f = E(f | \mathcal{F}_i) - E(f | \mathcal{F}_{i-1})$, $i \in \mathbb{Z}$, and suppose that the process $(f \circ T^i)_i$ is adapted and that $f = \sum_{i \leq 0} P_i f$ (f is regular). If, moreover,

$$\sum_{i=0}^{\infty} \|P_0 U^i f\|_2 < \infty,$$

then we say that the Hannan condition takes place.

Theorem 2. *There exists a regular causal stationary linear process $(f \circ T^i)$ with martingale difference innovations such that*

- (i) *the Maxwell-Woodroffe and the Hannan conditions are satisfied;*
- (ii) *for $\sigma_n = \|S_n(f)\|_2$, $\sigma_n \rightarrow \infty$, $\sigma_n/\sqrt{n} \rightarrow 0$, $\|E(S_n(f) | \mathcal{F}_0)\|_2/\sigma_n \rightarrow 0$, so $\bar{\sigma}_n/\sigma_n \rightarrow 1$;*
- (iii) *$S_n(f)/\sigma_n$ converge in distribution to $\mathcal{N}(0, 1)$;*
- (iv) *the convergence is not quenched for $S_n(f)/\sigma_n$ or for $(S_n(f) - E(S_n(f) | \mathcal{F}_0))/\sigma_n$;*
- (v) *the WIP does not hold.*

REMARK 3. The Hannan condition implies the WIP for $S_n(f)/\sqrt{n}$ (cf. Dedecker, Merlevède and Volný (2007)). For $[S_n(f) - E(S_n(f) | \mathcal{F}_0)]/\sqrt{n}$ the invariance principle is quenched (cf. Cuny and Volný (2013)), for $S_n(f)/\sqrt{n}$ the CLT is not quenched in general (cf. Volný and Woodroffe (2010)).

REMARK 4. As shown in Cuny and Merlevède (2014), the Maxwell-Woodroffe

condition implies a quenched CLT and WIP for $S_n(f)/\sqrt{n}$ (cf. also Peligrad and Utev (2005)).

Proof. We will find a filtration $(\mathcal{F}_i)_i$ such that $\mathcal{F}_{i+1} = T^{-1}\mathcal{F}_i$ and $e \in L^2(\mathcal{F}_0) \ominus L^2(\mathcal{F}_{-1})$, $\|e\|_2 = 1$. The construction of e and $(\mathcal{F}_i)_i$ will be presented later; it is needed for the proof of (iv) only.

We define a function f by

$$f = e + \sum_{k=1}^{\infty} \frac{-\gamma_k}{V_k} \sum_{i=1}^{V_k} U^{-i} e,$$

where $\gamma_k > 0$, $\sum_{k=1}^{\infty} \gamma_k = 1$, $V_k \nearrow \infty$, are such that

$$\|S_n(f)\|_2 \rightarrow \infty, \quad \frac{\|S_n(f)\|_2}{\sqrt{n}} \rightarrow 0, \quad \frac{\|E(S_n(f) | \mathcal{F}_0)\|_2}{\|S_n(f)\|_2} \rightarrow 0,$$

and $\frac{S_n(f)}{\|S_n(f)\|_2} \rightarrow N(0, 1)$.

To do so, we define

$$\gamma_k = \frac{2}{k+2} \prod_{j=1}^k \left(1 - \frac{1}{j+1}\right) = 2\left(\frac{1}{k+1} - \frac{1}{k+2}\right), \quad k = 1, 2, \dots$$

Here,

$$\sum_{k=1}^{\infty} \gamma_k = 1, \quad 1 - \sum_{j=1}^{k-1} \gamma_j = 2 \prod_{j=1}^k \left(1 - \frac{1}{j+1}\right) = (k+2)\gamma_k.$$

We suppose that $V_1 = 1$ and for all $k \geq 1$, $V_{k+1}/V_k \geq \lambda$, $\lim_{k \rightarrow \infty} V_{k+1}/V_k = \lambda > 1$. We have

$$\left\| \frac{\gamma_k}{V_k} \sum_{i=1}^{V_k} U^{-i} e \right\|_2 = \frac{\gamma_k}{\sqrt{V_k}}$$

which guarantees that

$$f = \sum_{k=1}^{\infty} \gamma_k \left(e - \frac{1}{V_k} \sum_{i=1}^{V_k} U^{-i} e \right) = \sum_{k=1}^{\infty} \gamma_k f_k \in L^2,$$

where

$$f_k = e - \frac{1}{V_k} \sum_{i=1}^{V_k} U^{-i} e = g_k - U g_k, \quad g_k = -\frac{1}{V_k} \sum_{j=1}^{V_k} j U^{-V_k-1+j} e.$$

For $h = \sum_{i=0}^{\infty} c_i U^{-i} e$ we have

$$S_n(h) = \sum_{j=0}^{n-1} \sum_{i=0}^{\infty} c_i U^{j-i} e = \sum_{u=-\infty}^{n-1} \sum_{j=0}^{n-1} c_{j-u} U^u e,$$

$$S_n(h) - E(S_n(h) | \mathcal{F}_0) = \sum_{u=1}^{n-1} \sum_{j=u}^{n-1} c_{j-u} U^u e \tag{3.1}$$

(where $0 \vee u = \max\{0, u\}$). If

$$f_k = e - \frac{1}{V_k} \sum_{i=1}^{V_k} U^{-i} e = \sum_{i=0}^{\infty} c_{k,i} U^{-i} e,$$

we have

$$\left| \sum_{j=0 \vee u}^{n-1} c_{k,j-u} \right| \leq 1 \text{ for every } u; \tag{3.2a}$$

$$\left| \sum_{j=0 \vee u}^{n-1} c_{k,j-u} \right| \leq n/V_k \text{ for } -1 \geq u \geq -V_k; \tag{3.2b}$$

$$\left| \sum_{j=0 \vee u}^{n-1} c_{k,j-u} \right| = 0 \text{ for } u < -V_k; \tag{3.2c}$$

$$\sum_{j=0 \vee u}^{n-1} c_{k,j-u} \geq 0 \vee \left(1 - \frac{n}{V_k}\right) \text{ for } u \geq 0. \tag{3.2d}$$

From (3.2a), (3.2b), and (3.2c) we deduce that for every choice of V_k ,

$$\|S_n(f_k)\|_2 = \left\| S_n\left(e - \frac{1}{V_k} \sum_{i=1}^{V_k} U^{-i} e\right) \right\|_2 \leq \sqrt{2n}.$$

From this, the Lebesgue Dominated Convergence Theorem, and the fact that each f_k is a coboundary with an L^2 cobounding function g_k we deduce that

$$\frac{\|S_n(f)\|_2}{\sqrt{n}} \leq \sum_{k=1}^{\infty} \gamma_k \frac{\|S_n(f_k)\|_2}{\sqrt{n}} \rightarrow 0. \tag{3.3}$$

Recall that $1 - \sum_{j=1}^{k+1} \gamma_j = \sum_{j=k+2}^{\infty} \gamma_j = (k+4)\gamma_{k+2}$ and $\|S_n(f)\|_2 \geq \|S_n(f) - E(S_n(f) | \mathcal{F}_0)\|_2$. By (3.1),

$$S_n(f) - E(S_n(f) | \mathcal{F}_0) = \sum_{k=1}^{\infty} \sum_{u=1}^{n-1} \sum_{j=u}^{n-1} \gamma_k c_{k,j-u} U^u e,$$

where $f_k = \sum_{i=0}^{\infty} c_{k,i} U^{-i} e$.

Let $V_k \leq n < V_{k+1}$. From (3.2d), $V_{j+1}/V_j \geq \lambda > 1$, and properties of the numbers γ_j we deduce that for a constant $C > 0$

$$\|S_n(f) - E(S_n(f) | \mathcal{F}_0)\|_2 \geq \sqrt{n-1} \sum_{j=k+1}^{\infty} \gamma_j \left(1 - \frac{n}{V_j}\right)$$

$$\begin{aligned} &\geq \sqrt{n-1} \sum_{j=k+2}^{\infty} \gamma_j \left(1 - \frac{V_{k+1}}{V_j}\right) \geq \sqrt{n-1} C \sum_{j=k+2}^{\infty} \gamma_j \\ &= \sqrt{n-1} C \left(1 - \sum_{j=1}^{k+1} \gamma_j\right) \geq C(k+4) \gamma_{k+2} \sqrt{n-1}, \end{aligned}$$

and hence

$$\|S_n(f)\|_2 \geq C(k+4) \gamma_{k+2} \sqrt{n-1} \quad (3.4)$$

for some constant $C > 0$. In the following text we denote other constants by the same letter C .

Because the V_k grow exponentially fast,

$$\sqrt{V_k} \left(1 - \sum_{j=1}^{k+1} \gamma_j\right) = \sqrt{V_k} (k+4) \gamma_{k+2} \rightarrow \infty$$

hence

$$\|S_n(f)\|_2 \rightarrow \infty. \quad (3.5)$$

Using exponential growth of the V_k again we get, for $V_k \leq n < V_{k+1}$,

$$\|E(S_n(f) | \mathcal{F}_0)\|_2 \sim \gamma_k \sqrt{n}. \quad (3.6)$$

To (3.6), we have

$$P_0 S_n(f) = E(S_n(f) | \mathcal{F}_0) - E(S_n(f) | \mathcal{F}_{-1}) = \sum_{k=1}^{\infty} \gamma_k \sum_{j=0}^{n-1} c_{k,j} e_j,$$

and hence, by (3.2a), $\|P_0 S_n(f)\|_2 \leq 1$ for all $n \geq 1$. It is thus sufficient to prove (3.6) for $\|E(S_n(f) | \mathcal{F}_{-1})\|_2$.

Suppose that $1 \leq l \leq k$. By a direct calculation we prove that

$$\|E(S_n(f_l) | \mathcal{F}_{-1})\|_2^2 = \frac{\gamma_l^2}{V_l^2} \sum_{j=1}^{V_l-1} j^2 \leq c \gamma_l^2 V_l$$

for a constant c not depending on l . Because the V_l grow exponentially fast, we deduce

$$\frac{1}{\gamma_k \sqrt{n}} \sum_{l=1}^{k-1} \|E(S_n(f_l) | \mathcal{F}_{-1})\|_2 \leq \sqrt{c} \sum_{l=1}^{k-1} \gamma_l \sqrt{\frac{V_l}{V_k}} \leq C$$

for some constant C not depending on k . For $l \geq k$ we have, by (3.2b),

$$\|E(S_n(f_l) | \mathcal{F}_{-1})\|_2^2 \leq \gamma_l^2 V_l \left(\frac{n}{V_l}\right)^2 = \gamma_l^2 \frac{n^2}{V_l},$$

hence

$$\frac{1}{\gamma_k \sqrt{n}} \sum_{l=k+1}^{\infty} \|E(S_n(f_l) | \mathcal{F}_{-1})\|_2 \leq \sum_{l=k+1}^{\infty} \frac{\gamma_l}{\gamma_k} \left(\frac{n}{V_l}\right)^{1/2} \leq C$$

for some constant C not depending on k (recall that $V_k \leq n < V_{k+1}$). This together with

$$\|E(S_n(f_k) | \mathcal{F}_{-1})\|_2^2 = n\gamma_k^2 \frac{V_k}{n} \frac{1}{V_k^3} \sum_{j=1}^{V_k-1} j^2$$

finishes the proof of (3.6).

By definition, $\gamma_k = 1/(k + 2)(1 - \sum_{j=1}^{k-1} \gamma_j)$, and hence by (3.4) and (3.6) we have

$$\frac{\|E(S_n(f) | \mathcal{F}_0)\|_2}{\|S_n(f)\|_2} \rightarrow 0. \tag{3.7}$$

From (3.5), (3.3), and (3.7) we get (ii). By Peligrad and Utev (2006) and $\sigma_n^2 \rightarrow \infty$ we get the central limit theorem (iii).

The Hannan and Maxwell-Woodroffe Conditions

We have $\|P_0 U^i f\|_2 = |a_i|$, $i \geq 0$, where $f = \sum_{i=0}^{\infty} a_i U^{-i} e$. From the definition of f we deduce that the process is regular, $a_0 = 1$, and $a_i < 0$ for $i \geq 1$, $\sum_{i=1}^{\infty} a_i = -1$. This implies the Hannan condition.

Let

$$h_k = \frac{-\gamma_k}{V_k} \sum_{i=1}^{V_k} U^{-i} e, \quad f' = \sum_{k=1}^{\infty} h_k,$$

so $f = e - f'$. In the same way as we deduced (3.2a)–(3.2d), we deduce, for all $k \geq 1$,

$$\begin{aligned} \|E(S_n(h_k) | \mathcal{F}_0)\|_2^2 &\leq \gamma_k^2 V_k \left(\frac{n}{V_k}\right)^2 = \gamma_k^2 \frac{n^2}{V_k}, \quad n = 1, 2, \dots, V_k, \\ \|E(S_n(h_k) | \mathcal{F}_0)\|_2 &= \|E(S_{V_k}(h_k) | \mathcal{F}_0)\|_2 \leq \gamma_k \sqrt{V_k}, \quad n \geq V_k. \end{aligned}$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{\|E(S_n(h_k) | \mathcal{F}_0)\|_2}{n^{3/2}} \leq \gamma_k \frac{1}{\sqrt{V_k}} \sum_{n=1}^{V_k} \frac{1}{\sqrt{n}} + \gamma_k \sqrt{V_k} \sum_{n=V_k+1}^{\infty} \frac{1}{n^{3/2}} \leq C\gamma_k$$

for some constant C . We thus have

$$\sum_{n=1}^{\infty} \frac{\|E(S_n(f') | \mathcal{F}_0)\|_2}{n^{3/2}} \leq \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{\|E(S_n(h_k) | \mathcal{F}_0)\|_2}{n^{3/2}} \leq C \sum_{k=1}^{\infty} \gamma_k = C < \infty$$

and (i) follows.

The filtration and e

Let $\mathcal{B}'_k, \mathcal{B}''_l \subset \mathcal{A}$, $k, l = 1, 2, \dots$, be mutually independent σ -algebras with $\mathcal{B}'_k \subset T^{-1}\mathcal{B}'_k$, $\mathcal{B}''_l \subset T^{-1}\mathcal{B}''_l$, $\bigcap_{j=1}^\infty T^j \mathcal{B}'_k = \{\Omega, \emptyset\}$, $\bigcap_{j=1}^\infty T^j \mathcal{B}''_l = \{\Omega, \emptyset\}$ (modulo sets of measure 0 or 1) for every k, l ; $\xi_k \circ T^i$ are iid \mathcal{B}'_k -measurable random variables, $\mu(\xi_k = 1) = 1/2 = \mu(\xi_k = -1)$ for all i .

All these objects can be constructed by taking finite alphabets \mathbb{A}'_k and \mathbb{A}''_l , $k, l = 1, 2, \dots$, $\Omega'_k = \prod_{i \in \mathbb{Z}} \mathbb{A}'_{k,i}$ where $\mathbb{A}'_{k,i}$ are identical copies of \mathbb{A}'_k , similarly we define Ω''_l , $k, l = 1, 2, \dots$. On the sets Ω'_k and Ω''_l we define product σ -algebras, product measures, and left shift transformations T'_k, T''_l . Ω is the product of all Ω'_k and Ω''_l equipped with the product σ -algebra \mathcal{A} , the product (probability) measure μ , and the product transformation T . For projections ξ_k and ζ_l of Ω onto $\mathbb{A}'_{k,0}$ and $\mathbb{A}''_{l,0}$ we thus get mutually independent processes of iid $(\xi_k \circ T^i)_i, (\zeta_l \circ T^i)_i$. We suppose that $\mathbb{A}'_k = \mathbb{A}''_k = \{-1, 1\}$, $k = 1, 2, \dots$, and $\mu(\xi_k = 1) = 1/2 = \mu(\xi_k = -1) = \mu(\zeta_k = 1) = \mu(\zeta_k = -1)$. For \mathcal{B}'_k we take the past σ -algebras $\sigma\{\xi_k \circ T^i : i \leq 0\}$ and for \mathcal{B}''_l we take the past σ -algebras $\sigma\{\zeta_l \circ T^i : i \leq 0\}$. The properties above can be easily verified, the latter follow from Kolmogorov's 0-1 law.

We thus have that $\xi_k \circ T^i$ are iid \mathcal{B}'_k -measurable random variables, $\mu(\xi_k = 1) = 1/2 = \mu(\xi_k = -1)$ for all i .

By (3.7), $\|E(S_n(f) | \mathcal{F}_0)\|_2 = o(\sigma_n)$, hence (by Wu and Woodroffe (2004)) $\sigma_n = h(n)\sqrt{n}$ where $h(n)$ is a slowly varying function in the sense of Karamata. By (3.3), $\sigma_n/\sqrt{n} \rightarrow 0$. We deduce that there exists a sequence $N_k \nearrow \infty$ of positive integers such that for all $k \geq 1$ odd and $d = 2(\sum_{k=1}^\infty 1/k^3)^{-1/2}$,

$$2^k \sigma_{N_k} \leq d \frac{\sqrt{N_k}}{k^{3/2}}, \quad \sigma_{4N_k} \leq 2\sigma_{N_k}, \quad \sum_{k=1}^\infty \frac{1}{4N_k} < \frac{1}{2}; \tag{3.8}$$

we define $N_{k+1} = 4N_k$ (k odd).

For $k = 1, 2, \dots$, let $A_k \in \mathcal{B}''_k$ be sets such that $T^{-i}A_k, i = 0, \dots, 3N_k$, are mutually disjoint (hence $\{T^{-i}A_k : i = 0, \dots, 3N_k\}$ are Rokhlin towers) and that $\mu(A_k) = 1/(4N_k)$ (existence of Rokhlin Towers is proved e.g. in Cornfeld, Fomin and Sinai (1982, p.242)). From (3.8) it follows

$$\sum_{k=1}^\infty \mu(A_k) < \frac{1}{2}. \tag{3.9}$$

By \mathcal{B}'' we define the σ -algebra generated by all $T^{-i}\mathcal{B}''_k, i \in \mathbb{Z}, k = 1, 2, \dots$; we thus have $T^{-1}\mathcal{B}'' = \mathcal{B}''$, all Rokhlin towers defined above are \mathcal{B}'' -measurable. By \mathcal{F}_j we denote the σ -algebra generated by \mathcal{B}'' and all $\xi_k \circ T^i, i \leq j, k = 1, 2, \dots$;

notice that $T^{-1}\mathcal{F}_j = \mathcal{F}_{j+1}$.

For $d = 2(\sum_{k=1}^\infty 1/k^3)^{-1/2}$ we define

$$e_k = d\xi_k \frac{\sqrt{N_k}}{k^{3/2}} 1_{A_k}, \quad e = \sum_{k=1}^\infty e_k.$$

Here $\|e_k\|_2 = d/2k^{3/2}$, hence $e \in L^2$. By definition, e is \mathcal{F}_0 -measurable. By definition, the e_k are mutually independent and hence $\|e\|_2^2 = (d^2/4) \sum_{k=1}^\infty 1/k^3$; we thus have $\|e\|_2 = 1$.

Because $A_k \in \mathcal{B}''$ and ξ_k is independent of \mathcal{F}_{-1} , we have $E(e_k | \mathcal{F}_{-1}) = 0$ for every k , so $E(e | \mathcal{F}_{-1}) = 0$. $(U^i e)_i$ is thus a martingale difference sequence adapted to the filtration (\mathcal{F}_i) .

We have

$$f = e + \sum_{k=1}^\infty \frac{-\gamma_k}{V_k} \sum_{i=1}^{V_k} U^{-i} e = a_0 e - \sum_{i=1}^\infty a_i U^{-i} e,$$

where $a_0 = 1$, $a_i > 0$ for all $i \geq 1$, and $\sum_{i=1}^\infty a_i = 1$.

By m_ω we denote regular conditional probabilities w.r.t. \mathcal{F}_0 (\mathcal{A} is a Borel σ -algebra of a Polish space hence the regular conditional probabilities exist). Notice that all sets $T^{-i}A_k$, $k = 1, 2, \dots$, $i \in \mathbb{Z}$, are \mathcal{F}_0 -measurable, hence $m_\omega(T^{-i}A_k) = 0$ (if $\omega \notin T^{-i}A_k$) or $m_\omega(T^{-i}A_k) = 1$ (if $\omega \in T^{-i}A_k$).

Fix a $k \geq 1$ such that $\sum_{i=N_k}^\infty a_i < 1/2$ and take $A'_k = A_k \setminus \bigcup_{j \neq k} A_j$. By (3.9) and independence, $\mu(A'_k) \geq \mu(A_k)/2$. The sets $A'_k, \dots, T^{-3N_k+1}A'_k$ are mutually disjoint and $\mu(A_k) \geq 1/(4N_k)$, so

$$\mu\left(\bigcup_{N=0}^{N_k-1} T^{-N+1}A'_k\right) \geq \frac{1}{8}. \tag{3.10}$$

We have

$$\begin{aligned} S_N(f) &= \sum_{j=0}^{N-1} U^j \left(e - \sum_{i=1}^\infty a_i U^{-i} e \right) \\ &= U^{N-1} e + \sum_{j=1}^{N-2} U^j e - \sum_{i=0}^\infty \sum_{j=0 \vee 1-i}^{N-1} a_{i+j} U^{-i} e + e - \sum_{i=2-N}^{-1} \sum_{j=1-i}^{N-1} a_{i+j} U^{-i} e \\ &= U^{N-1} e + I - II + III - IV. \end{aligned} \tag{3.11}$$

Suppose that for a $1 \leq k$ odd, $N_k \leq N < N_{k+1} = 4N_k$. For $\omega \in T^{-N+1}A'_k$ we have

$$m_\omega(U^{N-1} e = d \frac{\sqrt{N_k}}{k^{3/2}}) = m_\omega(U^{N-1} e = -d \frac{\sqrt{N_k}}{k^{3/2}}) = \frac{1}{2}.$$

$-II + III = E(S_N(f)|\mathcal{F}_0)$ hence it is a constant m_ω almost surely. $U^{N-1}e + I - IV$ is (an infinite) linear combination of products of $U^i\xi_l$ with \mathcal{F}_0 -measurable functions, $1 \leq i \leq N - 1, l \geq 1$. Because $\mathcal{F}_0, U^i\xi_l, 1 \leq i \leq N - 1$, are mutually independent, $U^i\xi_l, 1 \leq i \leq N - 1$, are iid with respect to the measure m_ω and $m_\omega(\xi_l = \pm 1) = 1/2$ (for μ a.e. ω). Therefore, $I - IV$ is a symmetric random variable independent of $U^{N-1}e$ w.r.t. m_ω ; $U^{N-1}e + I - IV = S_N(f) - E(S_N(f)|\mathcal{F}_0)$ is a symmetric random variable as well. We thus have

$$m_\omega(|S_N(f) - E(S_N(f)|\mathcal{F}_0)| \geq d\frac{\sqrt{N_k}}{k^{3/2}}) = m_\omega(|U^{N-1}e + I - IV| \geq d\frac{\sqrt{N_k}}{k^{3/2}}) \geq \frac{1}{2}.$$

From (3.11) and $\sum_{i=1}^\infty a_i = 1, \sigma_n \leq \sigma_{n+1}$ for all $n \geq 1$. Using (3.8) we get

$$\begin{aligned} & m_\omega\left(\frac{|S_N(f) - E(S_N(f)|\mathcal{F}_0)|}{\sigma_N} \geq 2^{k-1}\right) \\ & \geq m_\omega\left(\frac{|S_N(f) - E(S_N(f)|\mathcal{F}_0)|}{\sigma_{N_k}} \geq 2^k\right) \geq \frac{1}{2}. \end{aligned}$$

Because $E(S_N(f)|\mathcal{F}_0)$ is m_ω a.s. a constant and $S_N(f) - E(S_N(f)|\mathcal{F}_0)$ is a symmetric random variable, we get

$$m_\omega\left(\frac{|S_N(f)|}{\sigma_N} \geq 2^{k-1}\right) \geq \frac{1}{4}.$$

For any $K < \infty$ there thus exists a k_0 such that $2^{k_0-1} \geq K$, and for any integer $k \geq k_0$ there exists a set $B_k = \cup_{N_k}^{N_{k+1}-1} T^{-N+1} A'_k$ of measure bigger than $1/16$ (cf. (3.10)) such that, for $\omega \in B_k$ and the probability m_ω , there exists an $N_k \leq N < N_{k+1}$ for which

$$m_\omega\left(\frac{|S_N(f) - E(S_N(f)|\mathcal{F}_0)|}{\sigma_N} \geq K\right) \geq \frac{1}{2}, \quad m_\omega\left(\frac{|S_N(f)|}{\sigma_N} \geq K\right) \geq \frac{1}{4}. \quad (3.12)$$

We conclude that there exists a set B of positive measure such that for $\omega \in B$ there is an infinite sequence of k odd (say $k = k(j), j = 1, 2, \dots$) and $N_k \leq N \leq N_{k+1}$ (say $N = N(j)$) such that for the probability m_ω , the laws of $(S_N(f) - E(S_N(f)|\mathcal{F}_0))/\sigma_N$ and of $S_N(f)/\sigma_N$ do not weakly converge to $N(0, 1)$. This proves that the CLT for $(S_n - E(S_n|\mathcal{F}_0))/\sigma_n$ and for $S_N(f)/\sigma_N$ are not quenched.

This finishes the proof of (iv).

We have proved (cf. (3.13)) that for any $K < \infty, k$ sufficiently big (with $2^{k-1} \geq K$), and $\omega \in B_k = \cup_{N_k}^{N_{k+1}-1} T^{-N+1} A'_k$ (B_k of measure bigger than $1/16$),

$$m_\omega\left(\bigcup_{N=N_k}^{N_{k+1}-1} \left\{\frac{|S_N(f) - E(S_N(f)|\mathcal{F}_0)|}{\sigma_{N_{k+1}}} \geq K\right\}\right) \geq \frac{1}{2}$$

and therefore

$$\begin{aligned} & \mu\left(\bigcup_{N=N_k}^{N_{k+1}-1} \left\{ \frac{|S_N(f) - E(S_N(f)|\mathcal{F}_0)|}{\sigma_{N_{k+1}}} \geq K \right\}\right) \\ &= \mu\left(\max_{N_k \leq N \leq N_{k+1}-1} \left\{ \frac{|S_N(f) - E(S_N(f)|\mathcal{F}_0)|}{\sigma_{N_{k+1}}} \geq K \right\}\right) \geq \frac{1}{32}. \end{aligned}$$

Similarly we get

$$\mu\left(\max_{N_k \leq N \leq N_{k+1}-1} \left\{ \frac{|S_N(f)|}{\sigma_{N_{k+1}}} \geq K \right\}\right) \geq \frac{1}{64}.$$

The sequence of distributions of

$$s_n(t) = \left(\frac{1}{\sqrt{n}}\right) S_{[(n-1)t]}(f) + \frac{f \circ T^{n-1} - f \circ T^{(n-1)t - [(n-1)t]}}{\sqrt{n}}$$

(s_n being a random variable with values in $C([0, 1])$ and $[x]$ the integer part of x) is thus not tight (cf. (Billingsley, 1968, Chap. 2)). Similarly we get non tightness for $s'_n(t) = s_n(t) - E(s_n(t) | \mathcal{F}_0)$. This proves (v).

Theorem 2 can be given a stronger version. Again, m_ω denotes the regular conditional probabilities given \mathcal{F}_0 , and for a measurable function h , we write $\|h\|_{2,\omega}^2 = \int h^2 dm_\omega$ with $\|S_n(f)\|_{2,\omega}$ denoted by $\sigma_{n,\omega}$.

Proposition 1. *There exists a regular causal stationary linear process ($f \circ T^i$) with martingale difference innovations such that (i), (ii), and (iii) of Theorem 2 are satisfied and for almost all (μ) ω , $S_n(f)/\sigma_{n,\omega}$ do not converge in distribution.*

Proof. In the proof of Theorem 2 we have, for any $K < \infty$ and $k \in \mathbb{N}$ sufficiently big ($k \geq k(K)$), found a set B_k , $\mu(B_k) \geq 1/16$, such that, for $\omega \in B_k$ and the probability m_ω , there exists an $N_k \leq N < N_{k+1}$ for which

$$m_\omega\left(\frac{|S_N(f) - E(S_N(f)|\mathcal{F}_0)|}{\sigma_N} \geq K\right) \geq \frac{1}{2}, \quad m_\omega\left(\frac{|S_N(f)|}{\sigma_N} \geq K\right) \geq \frac{1}{4}. \tag{3.13}$$

As $\sigma_N^2 = \int \sigma_{N,\omega}^2 \mu(d\omega)$,

$$\mu(\sigma_{N,\omega} > \sqrt{K}\sigma_n) \leq \frac{1}{K}.$$

On a set of measure at least $1/16 - 1/K$ we thus have

$$m_\omega\left(\frac{|S_N(f) - E(S_N(f)|\mathcal{F}_0)|}{\sigma_N} \geq \sqrt{K}\right) \geq \frac{1}{2}, \quad m_\omega\left(\frac{|S_N(f)|}{\sigma_N} \geq \sqrt{K}\right) \geq \frac{1}{4} \tag{3.13'}$$

and we conclude in the same way as in the proof of Theorem 2.

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