Statistica Sinica **19** (2009), 71-81

SMOOTHED BLOCK EMPIRICAL LIKELIHOOD FOR QUANTILES OF WEAKLY DEPENDENT PROCESSES

Song Xi Chen[1,2] and Chiu Min Wong[3]

[1]*Iowa State University,* [2]*Peking University and* [3]*Temasek Junior College*

Abstract: Inference on quantiles associated with dependent observation is a common task in risk management. This paper employs empirical likelihood to construct confidence intervals for quantiles of the stationary distribution of a weakly dependent process. To accommodate data dependence and avoid any secondary variance estimation, empirical likelihood is formulated based on blocks of observations. To reduce the length of the confidence intervals, the weighted empirical distribution is smoothed by a kernel function. This shows that a rescaled version of the smoothed block empirical likelihood ratio admits a limiting chi-square distribution with one degree of freedom and facilitates likelihood ratio confidence intervals for quantiles. The practical performance of these confidence intervals is evaluated in a simulation study.

Key words and phrases: *alpha*-mixing, empirical likelihood, kernel smoothing, quantile, Value-at-Risk.

1. Introduction

Let $X_1, \ldots, X_N$ be a sequence of weakly dependent stationary random variables, and F be their common marginal distribution. The interest of this paper is in confidence intervals for $\theta_q =: F^{-1}(q) = \inf\{x|F(x) \geq q\}$, the qth quantile of F for $q \in (0,1)$. In financial risk management, θ_q is called the Value-at-Risk and specifies the level of excessive losses at a confidence level $1-q$. As financial returns are most likely dependent, the proposed confidence intervals for θ_q have direct application.

We propose using empirical likelihood for the construction of confidence intervals for θ_q. Empirical likelihood, introduced by Owen (1988, 1990), is a nonparametric method of inference that enables a likelihood-type inference in a nonparametric setting. Two striking properties of empirical likelihood are Wilks Theorem and the Bartlett correction, mirroring those of a parametric likelihood. Qin and Lawless (1994) established Wilks Theorem for estimating-equations-based empirical likelihood, and Chen and Cui (2006, 2007) showed that this empirical likelihood is Bartlett correctable with or without nuisance parameters. Tsao (2004) studied the effect of the number of constraints on the coverage probability of the empirical likelihood confidence intervals for a mean parameter. Like its parametric counterpart, the empirical likelihood confidence intervals/regions are constructed by contouring the empirical likelihood ratio. This brings two benefits: shape and orientation are determined by data; confidence intervals/regions are obtained without secondary estimation.

These features of the empirical likelihood confidence intervals are the major motivations for our current proposal for quantiles. Indeed, when considering extreme quantiles in risk analysis, the distribution of the sample quantile estimator can be quite skewed. Therefore, it is more appealing to have confidence intervals that are determined by the data rather than forced to be symmetric about a point estimate, as is the case for intervals based on asymptotic normality of the sample quantile estimator. The fact that the empirical likelihood intervals are obtained by contouring the likelihood ratio without a secondary variance estimation is particularly advantageous for dependent data, since data dependence leads to a variance which involves covariances at all lags.

A key ingredient of our proposal is to smooth a weighted empirical distribution function. The purpose of the kernel smoothing is to reduce the length of the confidence intervals, as is clearly demonstrated in our simulation study. Combining empirical likelihood and kernel smoothing for confidence intervals of a quantile with independent and identically distribution was proposed in Chen and Hall (1993). They showed that employing a kernel quantile estimator with the empirical likelihood significantly reduces the coverage errors from $O(N^{-1/2})$ to $O(N^{-1})$ before the Bartlett correction, and to $O(N^{-7/4})$ after Bartlett correction. Further investigations have been carried out by Zhou and Jing (2003a,b) Quantile estimation using empirical likelihood in the context of survey sampling is considered in Chen and Wu (2002).

The paper is organized as follows. We introduce a kernel-smoothed empirical likelihood for a quantile based on blocks of data in Section 2. Section 3 gives the main results. Results from a simulation study are reported in Section 4. All the technical details are relegated in the appendix.

2. Block Empirical Likelihood for Quantiles

let $\mathcal{F}_k^l$ be the σ-algebra of events generated by $\{X_t, k \le t \le l\}$ for $l \ge k$. The α-mixing coefficient introduced by Rosenblatt (1956) is

$$\alpha(k) = \sup_{\mathcal{A}\in\mathcal{F}_1^i, \mathcal{B}\in\mathcal{F}_{i+k}^\infty} |P(AB) - P(A)P(B)|.$$

The series is said to be α-mixing if $\lim_{k\to\infty} \alpha(k) = 0$. The dependence described by α-mixing is the weakest, as it is implied by other types of mixing; see Doukhan (1994) for a comprehensive discussion on mixing.

Let $F_n(x) = n^{-1}\sum_{i=1}^{n} I(X_i \le x)$ be the empirical distribution of the weakly dependent data $\{X_i\}_{i=1}^{N}$, where $I(\cdot)$ is the indicator function. We first smooth the empirical distribution with a kernel K and a smoothing bandwidth h; then invert it to obtain a kernel estimator for the quantile function which is smoother than the conventional sample quantile estimator.

Let K be an rth order kernel that satisfies

$$\int u^j K(u)du = \begin{cases} 1, & \text{if } j = 0 \\ 0, & \text{if } 1 \le j \le r-1 \\ \kappa, & \text{if } j = r \end{cases} \tag{2.1}$$

for some integer $r \ge 2$ and some $\kappa \ne 0$. Let $G_h(x) = \int_{-\infty}^{x/h} K(y)dy$ where $h \to 0$ as $N \to \infty$.

A kernel estimator of $F(x)$ is $\hat{F}_{n,h}(x) = n^{-1}\sum_{i=1}^{n} G_h(x - X_i)$ and the kernel quantile estimator $\hat{\theta}_{q,h}$ is the solution of

$$\hat{F}_{n,h}(x) = q.$$

Kernel estimators have been applied to estimation and testing for time series data; see Robinson (1989), Hjellvik and Tjøstheim (1995), Hjellvik, Chen and Tjøstheim (2004) and the book of Fan and Yao (2003)

Chen and Tang (2005) studied the statistical properties of $\hat{\theta}_{q,h}$ and its variance estimation. Unlike estimation of a regression or a probability density function for weakly dependent observations, the data dependence contributes to the leading order variance of the kernel quantile estimator. In particular, for each $h > 0$ let $\gamma_h(k) = Cov\{G_h(\theta_q - X_1), G_h(\theta_q - X_{k+1})\}$. The leading variance term of $\hat{\theta}_{q,h}$ is

$$\sigma_{n,h}^2 = q(1-q) + 2\sum_{k=1}^{n-1}\Big(1 - \frac{k}{n}\Big)\gamma_h(k), \tag{2.2}$$

which indicates clearly the first order effect of dependence. Chen and Tang (2005) proposed estimating the variance via a kernel estimation of the spectral density of a derived sequence. The variance estimator together with the asymptotic normality of $\hat{\theta}_{q,h}$ can be used to obtain confidence intervals for θ_q. The simulation study reported in Section 4 compares confidence interval of this type with the proposed empirical likelihood intervals.

There are two limitations of confidence intervals based on asymptotic normality. One is that the intervals are always symmetric while, for the extreme quantiles commonly used in risk analysis, the finite sample distribution of the quantile estimator can be quite skewed; it is more appealing to have asymmetric confidence intervals to reflect the skewness of the underlying distribution. Another limitation is that a secondary variance estimation is required for (2.2). In

Chen and Tang (2005), the estimation is via estimation of the spectral density function. For spectral density estimation, see Brockwell and Davies (1991).

The proposed empirical likelihood intervals for θ_q are not only asymmetric but free of any secondary variance. The latter is due to empirical likelihood's ability to standardize internally via its built-in algorithm.

Let $\{p_i\}_{i=1}^N$ be probability weights adding to one. A weighted kernel estimator for the distribution function F is

$$\hat{F}_{p,h}(x) = \sum_{i=1}^{N} p_i G_h(x - X_i). \tag{2.3}$$

If the observations were independent and identically distributed, we could formulate the empirical likelihood for the quantile θ_q as

$$L_h(\theta_q) = \sup \prod_{i=1}^{N} p_i$$

subject to $\hat{F}_{p,h}(\theta_q) = q$ and $\sum_{i=1}^N p_i = 1$. This is the formulation of Chen and Hall (1993). However, as pointed out by Kitamura (1997) in the context of estimating equations, the above empirical likelihood ignores data dependency and can cause the empirical likelihood ratio to lose its limiting chi-square distribution. The latter has been a major attraction of the likelihood ratio statistics.

Here we introduce a smoothing bandwidth into the estimating equation for the quantile that makes the estimating equation dependent on the sample size. In order to capture data dependence, we employ the blocking technique first applied to the bootstrap method (Carlstein (1986) and Künsch (1989)) and then to empirical likelihood (Kitamura (1997)). The data blocking divides the entire sample into a sequence of data blocks. The block length is taken to be sufficiently large so that the data dependence can be captured. At the same time, the weakly dependence allows us to treat the blocks as independent if the gap between successive blocks becomes large, although this gap will be much smaller than the block length.

Let M be a positive integer representing the block length, L be the gap between the beginnings of two consecutive blocks, and Q be the total number of blocks, so that $Q = [(N-M)/L] + 1$. Assumptions on M and L will be specified in Condition C4 in the next section.

For $i = 1, \ldots, Q$, define $g_h(X_i, \theta_q) = G_h(\theta_q - X_i) - q$ and let $T_i(\theta_q) = (1/M)\sum_{j=1}^M g_h(X_{(i-1)L+j}, \theta_q)$ be the ith block average.

Let $p_1, \ldots, p_Q$ be empirical likelihood weights allocated to the Q blocks respectively. The block empirical likelihood for θ_q is

$$L_h(\theta_q) = \sup \prod_{i=1}^{Q} p_i, \tag{2.4}$$

subject to $\sum_{i=1}^{Q} p_i = 1$ and $\sum_{i=1}^{Q} p_i T_i(\theta_q) = 0$. From the standard algorithm of empirical likelihood, the p_i that maximize the profile likelihood (2.4) are

$$p_i = \frac{1}{Q[1 + \lambda(\theta_q)T_i(\theta_q)]}, \tag{2.5}$$

$i = 1, \ldots, Q$, where $\lambda(\theta_q)$ is a Lagrange multiplier satisfying

$$\sum_{i=1}^{Q} \frac{T_i(\theta_q)}{1 + \lambda(\theta_q)T_i(\theta_q)} = 0. \tag{2.6}$$

Since $L_h(\theta_q)$ attains its maximum at $p_i = Q^{-1}$ for all $i \in \{1, \ldots, Q\}$, we can take

$$\ell_h(\theta_q) = -2 \log \left\{ \frac{L_h(\theta_q)}{Q^{-Q}} \right\} \tag{2.7}$$

to be the log empirical log-likelihood ratio for θ_q. From (2.5),

$$\ell_h(\theta_q) = 2 \sum_{i=1}^{Q} log[1 + \lambda(\theta_q)T_i(\theta_q)], \tag{2.8}$$

where $\lambda(\theta_q)$ is the solution of (2.6).

If we choose $h = 0$ in the above formulation, then $g_h(X_i, \theta_q) = G_h(\theta_q - X_i) - q$ is $I(X_i \leq \theta_q) - q$, so the estimating equation is free of N. Then the results in Kitamura (1997) are applicable to this unsmoothed empirical likelihood formulation for the quantile.

3. Main Results

We make the following assumptions.

C1: $\{X_i\}_{i=1}^{N}$ is a strictly stationary α-mixing sequence. The mixing coefficients $\alpha(k)$ satisfy $\sum_{k=1}^{\infty} k\alpha^{1/p}(k) < \infty$ for some $p > 1$. The spectral density function ϕ of $\{I(X_k < \theta_q)\}_{k=1}^{N}$ has $\phi(0) > 0$.

C2: K is a bounded and compactly supported rth order kernel satisfying (2.1); the smoothing bandwidth h satisfies $Nh^{2r} \to 0$ with $Nh \to \infty$ as $N \to \infty$.

C3: The distribution function F of X_i is absolutely continuous with a density f which has continuous $(r-1)$th derivatives in a neighbourhood of θ_q, with $f(\theta_q) > 0$.

C4: The block length satisfies $M \to \infty$ and $M = o(N^{1/2})$ as $N \to \infty$ and the gap L between the starting points of two adjacent blocks satisfies $kL \leq M$ and $(k+1)L > M$ for some $k > 1$.

One assumes $h = o(N^{-1/(2r)})$ in C2 is to reduce the effect of the bias that results from kernel smoothing; it is satisfied by the optimal bandwidth for estimating the quantile function. Indeed, the bandwidth that minimizes the mean square error for quantile estimation with a rth order kernel is $h = O(N^{-1/(2r-1)})$ (Chen and Tang (2005)), provided the underlying distribution function is sufficiently smooth. Hence, the standard bandwidth for quantile estimation can be employed for the empirical likelihood intervals. Condition C4 is standard when data blocking for dependent data (Künsch (1989) and Kitamura (1997)).

We first establish the order of magnitude for the Lagrange multiplier $\lambda(\theta_q)$, which is a key result in establishing stochastic expansions for $l_h(\theta_q)$.

Theorem 1. *Under Conditions* C1−C4, $\lambda(\theta_q) = O_p\{M(N^{-1/2} + h^r)\}$.

The next theorem shows that a scaled version of the empirical likelihood ratio converges to the χ_1^2 distribution.

Theorem 2. *Under Conditions* C1−C4 *and as* $N \to \infty$,

$$\frac{N}{MQ} l_h(\theta_q) \xrightarrow{d} \chi_1^2.$$

Theorem 2 readily leads to an empirical likelihood confidence interval for θ_q at $1-\alpha$ level of confidence:

$$I_{\alpha,h} = \left\{\theta_q \middle| \frac{N}{MQ}\ell_h(\theta_q) \le c_\alpha\right\},$$

where c_α is the upper α-quantile of χ_1^2 such that $P(\chi_1^2 > c_\alpha) = \alpha$. Theorem 2 ensures that $I_{\alpha,h}$ will attain the nominal coverage level $1-\alpha$ asymptotically. A major attraction of the proposed confidence interval is that it avoids any secondary estimation of the variance of the kernel quantile estimator $\hat{\theta}_{q,h}$ given by (2.2).

If we choose not to carry out the kernel smoothing in the empirical likelihood formulation, which effectively assigns $h = 0$ as discussed at the end of last section, then Theorem 2 is still valid as a special case of Kitamura (1997). Let $l_0(\theta_q)$ be the unsmoothed empirical likelihood ratio. Then, a $1-\alpha$ confidence unsmoothed empirical likelihood confidence interval for θ_q is

$$I_{\alpha,0} = \left\{\theta_q \middle| \frac{N}{MQ}\ell_0(\theta_q) \le c_\alpha\right\}.$$

We expect that the smoothed confidence intervals $I_{\alpha,h}$ have shorter length than $I_{\alpha,0}$. This is based on the fact (Chen and Tang (2005)) that the kernel estimator for θ_q reduces the variance of the unsmoothed sample quantile estimator to the second order. This is confirmed by the simulation study reported in the next section.

4. Simulation Results

We report results from a simulation study that is designed to evaluate the performance of the empirical likelihood confidence intervals for the quantile θ_q. For comparison purposes, we carried out simulation for both the kernel smoothed intervals $I_{\alpha,h}$ and the unsmoothed intervals $I_{\alpha,0}$. We are interested in the lengths and coverage levels of the confidence intervals.

We considered two time series models in the simulation: an AR(1) model

$$X_t = 0.5X_{t-1} + \varepsilon_t,$$

and an AR(2) model

$$X_t = \frac{5}{6}X_{t-1} - \frac{1}{6}X_{t-2} + \varepsilon_t.$$

In both models, the ε_t are independent and identically distributed $N(0,1)$ random variables. Clearly, both models are strictly stationary and α-mixing. In the simulations, the initial value X_0 was generated from the stationary normal distribution.

Two levels of quantiles were considered: the 5% and 50% (median) quantiles; the former is a level commonly used in risk assessment. The sample sizes considered were $N = 300$ and 500. The block length M was 12 for $N = 300$ and 16 for $N = 500$. We set the gap between two successive blocks L to be $M/2$ in all cases. We employed the second order ($r = 2$) Epanechnikov kernel throughout. Three bandwidths were used for the kernel smoothed interval: $h_1 = 1.50N^{-1/4}$, $h_2 = N^{-1/4}$ and $h_3 = 0.50N^{-1/4}$. We also considered confidence intervals based on the asymptotic normality of the kernel smoothed quantile estimator as given in Chen and Tang (2005) with the asymptotic variance estimated based on the spectral density estimation approach proposed there.

The confidence intervals for the 5% and 50% quantiles with confidence levels 0.95 and 0.99 are reported in Table 1 for the AR(1) model, and in Table 2 for the AR(2) model. We observe from Tables 1 and 2 that both smoothed and unsmoothed confidence intervals had satisfactory coverage in all cases. The observed empirical coverage was not sensitive to the choice of the smoothing bandwidth h. From our discussion in the previous section, we anticipated the kernel-smoothed confidence intervals to be substantially shorter than the unsmoothed counterpart; this turned out to be the case. Our simulation clearly exhibited the usefulness of kernel smoothing in the context of interval estimation for dependent observations. The empirical likelihood confidence intervals were

Table 1. Average coverage levels and lengths (in parentheses) of the empirical likelihood confidence intervals and the confidence intervals based on Chen and Tang (2005) for quantiles of the $AR(1)$ process.

(a) $N = 300$, $M = 12$, $l = 6$ for $AR(1)$ process.

q	Nominal Coverage	Unsmoothed	Bandwidth $1.50N^{-1/4}$	$N^{-1/4}$	$0.50N^{-1/4}$	Chen-Tang
0.05	0.95	0.963	0.940	0.942	0.942	0.954
		(0.622)	(0.502)	(0.518)	(0.526)	(0.864)
	0.99	0.992	0.987	0.987	0.987	0.988
		(0.820)	(0.672)	(0.694)	(0.706)	(1.137)
0.50	0.95	0.952	0.948	0.947	0.947	0.960
		(0.337)	(0.308)	(0.314)	(0.317)	(0.590)
	0.99	0.985	0.982	0.982	0.982	0.992
		(0.450)	(0.413)	(0.421)	(0.426)	(0.776)

(b) $N = 500$, $M = 16$, $l = 8$ for $AR(1)$ process.

q	Nominal Coverage	Unsmoothed	Bandwidth $1.50N^{-1/4}$	$N^{-1/4}$	$0.50N^{-1/4}$	Chen-Tang
0.05	0.95	0.963	0.944	0.947	0.948	0.956
		(0.461)	(0.394)	(0.404)	(0.411)	(0.642)
	0.99	0.994	0.988	0.990	0.990	0.992
		(0.614)	(0.527)	(0.541)	(0.549)	(0.776)
0.50	0.95	0.952	0.948	0.949	0.949	0.952
		(0.259)	(0.240)	(0.244)	(0.246)	(0.450)
	0.99	0.989	0.986	0.986	0.987	0.994
		(0.345)	(0.321)	(0.327)	(0.330)	(0.592)

substantially shorter than the explicit confidence intervals based on the asymptotic normality, although there were not much difference between the coverage levels. The intervals based on asymptotic normality were symmetric, and this led to the intervals being too wide, while the distribution of $\hat{\theta}_{q,h}$ was quite skewed in the case for extreme quantiles.

Acknowledgements

We thanks two referees and Cheng Yong Tang for constructive comments and suggestions. Much of this research was carried out while both authors were associated with Department of Statistics and Applied Probability of National University of Singapore. We thank DSAP of NUS for generous support. Chen's research was partially supported by National Science foundation grants SES-0518904 and DMS-0604563, and Wong's by a Singapore Ministry of Education

Table 2. Average coverage levels and lengths (in parentheses) of the empirical likelihood confidence intervals and the confidence intervals based on Chen and Tang (2005) for quantiles of the $AR(2)$ process.

(a) $N = 300$, $M = 12$, $l = 6$ for $AR(2)$ process.

q	Nominal Coverage	Unsmoothed	Bandwidth $1.50N^{-1/4}$	Bandwidth $N^{-1/4}$	Bandwidth $0.50N^{-1/4}$	Chen-Tang
0.05	0.95	0.963	0.941	0.942	0.942	0.946
		(0.781)	(0.642)	(0.658)	(0.667)	(1.082)
	0.99	0.992	0.987	0.988	0.987	0.984
		(1.029)	(0.860)	(0.883)	(0.895)	(1.424)
0.50	0.95	0.952	0.948	0.947	0.946	
		(0.423)	(0.391)	(0.397)	(0.401)	(0.804)
	0.99	0.985	0.982	0.982	0.983	0.987
		(0.565)	(0.525)	(0.533)	(0.538)	(1.058)

(b) $N = 500$, $M = 16$, $l = 8$ for $AR(2)$ process.

q	Nominal Coverage	Unsmoothed	Bandwidth $1.50N^{-1/4}$	Bandwidth $N^{-1/4}$	Bandwidth $0.50N^{-1/4}$	Chen-Tang
0.05	0.95	0.963	0.946	0.947	0.948	0.950
		(0.579)	(0.502)	(0.514)	(0.520)	(0.773)
	0.99	0.994	0.989	0.990	0.990	0.992
		(0.768)	(0.672)	(0.687)	(0.695)	(1.018)
0.50	0.95	0.952	0.949	0.949	0.949	0.952
		(0.325)	(0.305)	(0.309)	(0.311)	(0.624)
	0.99	0.989	0.987	0.987	0.987	0.992
		(0.434)	(0.408)	(0.414)	(0.417)	(0.821)

scholarship.

Appendix: Outline of Proofs

Let $T_i(\theta_q) = M^{-1}\sum_{j=1}^{M} g_h(X_{(i-1)L+j}, \theta_q)$, $\tilde{T}_\beta(\theta) = Q^{-1}\sum_{i=1}^{Q}[T_i(\theta_q)]^\beta$ for a positive integer β, $g_h^o(X_j, \theta_q) = g_h(X_j, \theta_q) - E\{g_h(X_j, \theta_q)\}$, and $\tau_i(\theta_q) = M^{-1} \sum_{j=1}^{M} g_h^o(X_{(i-1)L+j}, \theta_q)$. Furthermore, let $S_Q = \tilde{T}_2(\theta_q)$, and let ϕ to be the spectral density function of $\{I(X_k < \theta_q)\}_{k=1}^{N}$ We need the following lemmas whose proofs, along with the proof of Theorem 2, are available in the full version of the paper.

Lemma A.1. *Under Conditions* C1–C4, $\tilde{T}_\beta(\theta_q) = O_p((M^{-1/2} + h^r)^\beta)$ *for any integer* $\beta \geq 2$.

Lemma A.2. *Under Conditions* C1–C4, $MS_Q \xrightarrow{p} 2\pi\phi(0) > 0$.

Lemma A.3. *Under Conditions* C1−C4, $\tilde{T}_1(\theta_q) = O_p(N^{-1/2} + h^r)$.

Lemma A.4. *Take* $\xi_Q = Q^{-1}\sum_{i=1}^{Q}\tau_i^2(\theta_q)$. *Under Conditions* C1−C4, $M\xi_Q \xrightarrow{p} 2\pi\phi(0)$.

Proof of Theorem 1. By following the standard procedure in empirical likelihood, for instance that outlined in Owen (1990),

$$\begin{aligned} 0 &= |g(\lambda(\theta_q))| \\ &\geq \frac{|\lambda(\theta_q)|}{Q}\sum_{i=1}^{Q}\frac{T_i^2(\theta_q)}{1+\lambda(\theta_q)T_i(\theta_q)} - \frac{1}{Q}\Big|\sum_{i=1}^{Q}T_i(\theta_q)\Big| \\ &\geq \frac{|\lambda(\theta_q)|}{1+|\lambda(\theta_q)|\max_{1\leq i\leq Q}|T_i(\theta_q)|}MS_Q - |M\tilde{T}_1(\theta_q)|. \end{aligned}$$

From Lemma A.2, $MS_Q = 2\pi\phi(0) + o_p(1)$. Recall also Lemma A.3, that $|M\tilde{T}_1(\theta_q)| = O_p(MN^{-1/2})$. This means that

$$\frac{|\lambda(\theta_q)|}{1+|\lambda(\theta_q)|\max_{1\leq i\leq Q}|T_i(\theta_q)|} = O_p\Big(MN^{-\frac{1}{2}}\Big).$$

As $MN^{-1/2} = o(1)$, $h^r = o(N^{1/2}M^{-1})$, where $N^{1/2}M^{-1} \to \infty$ as $N \to \infty$. From a result in Künsch (1989), we see that

$$\begin{aligned} \max_{1\leq i\leq Q}|T_i(\theta_q)| &= \max_{1\leq i\leq Q}|\tau_i(\theta_q) + c_o h^r + o(h^r)| \\ &= o\Big(N^{\frac{1}{2}}M^{-1}\Big) + O(h^r) = o\Big(N^{\frac{1}{2}}M^{-1}\Big). \end{aligned}$$

Then we conclude that $|\lambda(\theta_q)| = O_p(MN^{-1/2})$ because

$$1 + |\lambda(\theta_q)|\max_{1\leq i\leq Q}|T_i(\theta_q)| = 1 + o_p(1).$$

References

Brockwell, P. J. and Davies, R. A. (1991). *Time Series: Theory and Methods.* Springer-Verlag, New York.

Carlstein, E. (1986). The use of subseries values for estimating the variance of a general statistic from a stationary sequence. *Ann. Statist.* **14**, 1171-1179.

Chen, J. and Wu, C. (2002). Estimation of distribution function and quantiles using the model-calibrated pseudo empirical likelihood method. *Statist. Sinica* **12**, 1223-1239.

Chen, S. X. and Cui, H. J. (2006). On Bartlett correction of empirical likelihood in the presence of nuisance parameters. *Biometrika* **93**, 215-220.

Chen, S. X. and Cui, H. J. (2007). On the Second Order Properties of Empirical Likelihood for Generalized Estimation Equations. *J. Econometrics* **141**, 492-516.

Chen, S. X. and Hall, P. (1993). Smoothed empirical likelihood confidence intervals for quantiles. *Ann. Statist.* **21**, 1166-1181.

Chen, S. X. and Tang, C. Y. (2005). Nonparametric inference of Value at Risk for dependent financial returns. *J. Finan. Econom.* **3**, 227-255.

Doukhan, P. (1994). *Mixing: Properties and Examples.* Springer-Verlag, New-York.

Fan, J. and Yao, Q. (2003). *Nonlinear Time Series. Nonparametric and Parametric Methods.* Springer-Verlag, New York.

Hjellvik, V., Chen, R. and Tjøstheim, D. (2004). Nonparametric estimation and testing in panels of intercorrelated time series. *J. Time Ser. Anal.* **25**, 831-872.

Hjellvik, V. and Tjøstheim, D. (1995). Nonparametric tests of linearity for time series. *Biometrika* **82**, 351-368.

Kitamura, Y. (1997). Empirical likelihood methods with weakly dependent processes, *Ann. Statist.* **25**, 2084-2102.

Künsch, H. R. (1989). The Jackknife and the Bootstrap for general stationary observations, *Ann. Statist.* **17**, 1217-1241.

Owen, A. (1988). Empirical likelihood ratio confidence intervals for a single functional. *Biometrika* **75**, 237-249.

Owen, A. (1990). Empirical likelihood ratio confidence regions. *Ann. Statist.* **18**, 90-120.

Qin, J. and Lawless, J. (1994). Empirical likelihood and general estimation equations. *Ann. Statist.* **22**, 300-325.

Robinson, P. (1989). Hypothesis testing in semiparametric and nonparametric models for econometric time series. *Rev. Econom. Stud.* **56**, 511-534.

Rosenblatt, M. (1956). A central limit theorem and the α-mixing condition. *Proc. Natl. Acad. Sci. USA* **42**, 43-47.

Tsao, Min (2004). Bounds on coverage probabilities of the empirical likelihood ratio confidence regions. *Ann. Statist.* **32**, 1215-1221.

Zhou, W. and Jing, B.-Y. (2003a). Smoothed empirical likelihood confidence intervals for the difference of quantiles. *Statist. Sinica* **13**, 83-95.

Zhou, W. and Jing, B.-Y. (2003b). Adjusted empirical likelihood method for quantiles. *Ann. Inst. Statist. Math.* **55**, 689-703.

Department of Statistics, Iowa State University, Ames, Iowa 50011-1210, U.S.A.; and Department of Business Statistics and Econometrics, Guanghua School of Management, Peking University, Beijing 100871, China.

E-mail: songchen@iastate.edu

Department of Mathematics, Temasek Junior College, Singapore.

E-mail: wcmy77@yahoo.com.sg

(Received August 2006; accepted July 2007)