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A UNIFIED APPROACH TO CAPABILITY INDICES

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Abstract. A new class of capability indices, containing C_p, C_{pk}, C_{pm}, and C_{pmk}, is defined. By varying the parameters of the studied class, indices with different properties can be found. Two estimators of the indices are considered and, assuming that the studied characteristic of the process is normally distributed and that the target value is equal to the mid-point of a two-sided specification interval, their expected values, variances, and mean square errors are derived. It is shown that studying the properties of the class of indices alone, without taking the properties of its estimators into account, might be misleading.

Key words and phrases: Capability indices, estimation, expected values, mean square errors, variances.

1. Introduction

Process capability indices have received much interest in the statistical literature during recent years. For thorough discussions of different capability indices see, for instance, Kane (1986), Chan, Cheng, and Spiring (1988a), Boyles (1991), Pearn, Kotz, and Johnson (1992), henceforth abbreviated as PKJ, Rodriguez (1992), and Kotz and Johnson (1993).

Here we consider the case where there is a two-sided specification interval, $[LSL, USL]$. The four different indices C_p, C_{pk}, C_{pm}, and C_{pmk} described in the next section, have been suggested in the literature in that situation. Here we define a class of capability indices that generalizes the four basic indices mentioned. By varying the parameters of this class we can find indices with different desirable properties. We also consider different estimators of the indices under investigation and derive their expected values, variances, and mean square errors assuming that the studied characteristic of the process is normally distributed and that the target value T is equal to the mid-point M of the specification interval. Furthermore, numerical investigations are made to explore the behavior of these estimators for different values of the parameters.

2. Today's Capability Indices

Among capability indices the earliest form is, Juran (1974),

$$C_p = \frac{USL - LSL}{6\sigma} = \frac{d}{3\sigma}, \tag{1}$$

where USL and LSL are the upper and lower specification limits, respectively, σ the process standard deviation, and $d = (USL - LSL)/2$, that is, half the length of the specification interval. C_p does not take into account that the process mean, μ, may differ from the target value T. To avoid that drawback the so called second generation of capability indices, C_{pk} (Kane (1986)) and C_{pm} (Hsiang and Taguchi (1985) and, independently, Chan, Cheng, and Spiring (1988a)) were defined, with

$$C_{pk} = \frac{\min(USL - \mu, \mu - LSL)}{3\sigma} = \frac{d - |\mu - M|}{3\sigma} \tag{2}$$

and

$$C_{pm} = \frac{USL - LSL}{6\sqrt{E((X-T)^2)}} = \frac{d}{3\sqrt{\sigma^2 + (\mu - T)^2}}, \tag{3}$$

where $M = (USL + LSL)/2$, that is, the mid-point of the specification interval, and T is the target value.

To obtain a capability index which is more sensitive than C_{pk} and C_{pm} with regard to departures of the process mean, μ, from the target value, T, PKJ defined a so called third generation of capability index, C_{pmk}, as

$$C_{pmk} = \frac{\min(USL - \mu, \mu - LSL)}{3\sqrt{\sigma^2 + (\mu - T)^2}} = \frac{d - |\mu - M|}{3\sqrt{\sigma^2 + (\mu - T)^2}}. \tag{4}$$

For a thorough overview of these four basic and several related indices and their properties see the monograph by Kotz and Johnson (1993).

The four above mentioned capability indices are all equal when $\mu = T = M$, but differ in behavior when $\mu \neq T$. By plotting the four indices as surfaces, in analogy with Figure 1, we can get a feeling for the sensitivity with regard to departures of the process mean, μ, from the target value, T, assuming that $T = M$. We then see that, for fixed σ, when μ moves away from T, then C_p does not change, C_{pk} changes, but slowly, C_{pm} changes somewhat more rapidly than C_{pk}, but C_{pmk} is the one that changes most rapidly. (See Vännman (1993).)

The capability indices (1)-(4) are designed to measure the process capability when the studied characteristic of the process is normal. In such a case the index C_p can be interpreted using the probability of non-conformance, that is, the probability of obtaining a value outside the specification limits. Elementary probabilistic arguments show that the probability of non-conformance is equal to

$2\Phi(-3C_p)$, where Φ denotes the standard normal cumulative distribution function.

As PKJ pointed out the value of C_{pk} does not determine the probability of non-conformance, but limits it, and the probability of non-conformance is never more than $2\Phi(-3C_{pk})$. The corresponding is true for both C_{pm} and C_{pmk} and the probability of non-conformance is never more than $2\Phi(-3C_{pm})$ and $2\Phi(-3C_{pmk})$, respectively.

Boyles (1991) showed that C_{pk} becomes arbitrarily large as σ approaches 0 and this characteristic makes C_{pk} unsuitable as a measure of process centering. The same is, of course, true for C_p. Boyles (1991) showed also that for fixed μ the index C_{pm} is bounded above when σ tends to 0 and, furthermore, that $C_{pm} < d/(3|\mu - T|)$ and hence $|\mu - T| < d/(3C_{pm})$. This inequality can be interpreted as a C_{pm}-value of 1 implies that the process mean μ lies within the middle third of the specification range, and in general it lies within the middle $1/(3C_{pm})$ of the specification range.

When $T = M$ a similar interpretation can be made for C_{pmk}. It is seen from (4) that for fixed μ the index C_{pmk} is bounded above when σ tends to 0 and that $C_{pmk} < d/(3|\mu - T|) - 1/3$ or, equivalently $|\mu - T| < d/(1 + 3C_{pmk})$. This inequality can be interpreted as a C_{pmk}-value of 1 implies that the process mean μ lies within the middle fourth of the specification range. In general the process mean μ lies within the middle $1/(1 + 3C_{pmk})$ of the specification range, when $T = M$.

3. A New Class of Capability Indices

We now define a new class of capability indices, depending on two non-negative parameters, u and v, as

$$C_p(u, v) = \frac{d - u|\mu - M|}{3\sqrt{\sigma^2 + v(\mu - T)^2}}. \tag{5}$$

All four indices (1)-(4) in the previous section can be considered as special cases of the index in (5) by letting $u = 0$ or 1 and $v = 0$ or 1. We find that

$$C_p(0,0) = C_p; \quad C_p(1,0) = C_{pk}; \quad C_p(0,1) = C_{pm}; \quad C_p(1,1) = C_{pmk}.$$

PKJ pointed out some undesirable properties of C_{pm} when the target value T is between LSL and USL, but not equal to M. Furthermore, when we have a two-sided specification interval the case when $T = M$ is quite common in practical situations. For these reasons we restrict our attention in this paper to the case when $T = M$, that is, study the case when $C_p(u, v)$ in (5) is given by

$$C_p(u, v) = \frac{d - u|\mu - T|}{3\sqrt{\sigma^2 + v(\mu - T)^2}}. \tag{6}$$

For fixed values of $u > 0$ and $v > 0$, the index $C_p(u,v)$ can be interpreted in a similar way as C_{pm} and C_{pmk}. The value of the $C_p(u,v)$-index does not determine the probability of non-conformance, but limits it, and the probability of non-conformance is never more than $2\Phi(-3C_p(u,v))$. Furthermore we see from (6) that for fixed μ the index $C_p(u,v)$ is bounded above when σ tends to 0 and that

$$C_p(u,v) < \frac{d}{3\sqrt{v}|\mu - T|} - \frac{u}{3\sqrt{v}} \quad \text{or, equivalently} \quad |\mu - T| < \frac{d}{u + 3\sqrt{v}C_p(u,v)}.$$

One interpretation of this inequality is that the process mean μ lies within the middle $1/(u + 3v^{1/2}C_p(u,v))$ of the specification range.

If it is of interest to have a capability index that is very sensitive with regard to departures of the process mean, μ, from the target value, T, then the values of u and v in (6) should be large. In Figure 1 some plots of $C_p(u,v)$ are given for some combinations of u and v.

The indices have been expressed in the two variables σ/d and $|\mu - T|/d$ in Figure 1, and the surface describing the index has been cut off at the index value 1 to facilitate comparisons of the indices. From Figure 1 we can see how the sensitivity, with regard to departures of the process mean, μ, from the target value, T, depends on u and v.

Since in most cases the values of the process mean μ and the process standard deviation σ are unknown and have to be estimated, we cannot consider the behavior of the theoretical capability indices in (6) alone. We also have to study the estimators of the indices and their properties. This will be done in the subsequent sections. There we will see that it is not of interest to increase u and v too much since then the estimators will have undesirable properties.

4. Estimation of $C_p(u,v)$

We treat the case when the studied characteristic of the process is normally distributed. Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal distribution with mean μ and variance σ^2 measuring the studied characteristic.

We consider two different estimators of $C_p(u,v)$, differing in the way the variance σ^2 is estimated. In analogy with the estimator of C_{pmk} studied by PKJ and the estimator of C_{pm} studied by Boyles (1991) we will define the estimator $C^*_{p,n}(u,v)$ of $C_p(u,v)$ as

$$C^*_{p,n}(u,v) = \frac{d - u|\bar{X} - T|}{3\sqrt{\sigma^{2*} + v(\bar{X} - T)^2}}, \tag{7}$$

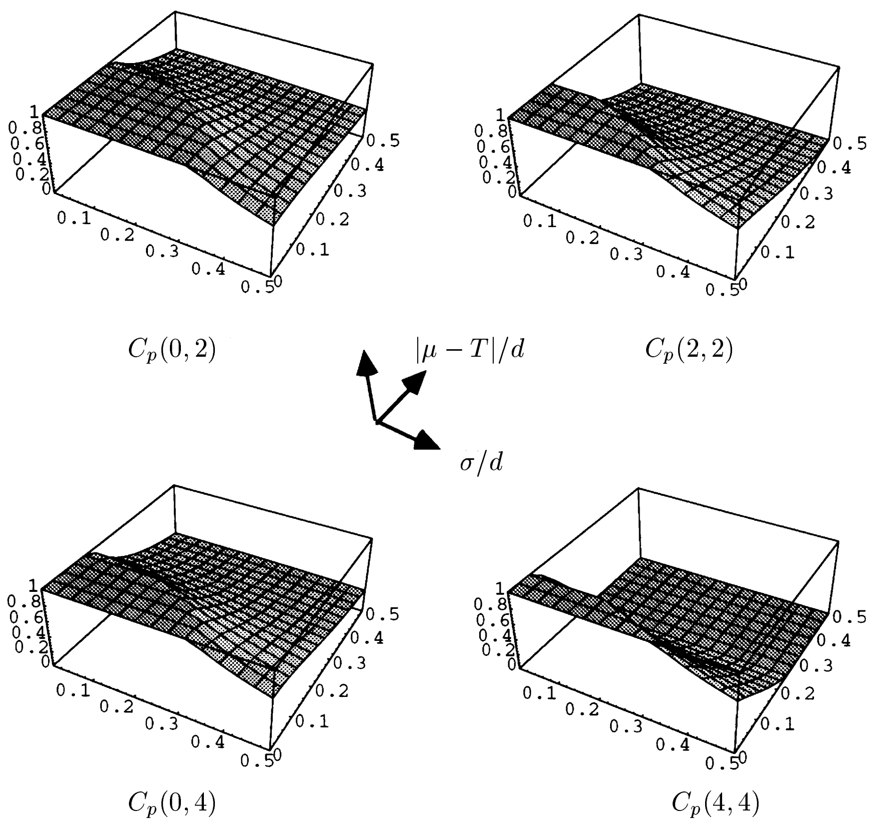

Figure 1. The capability indices $C_p(0,2)$, $C_p(2,2)$, $C_p(0,4)$, and $C_p(4,4)$ as surface plots with $x = \sigma/d$ and $y = |\mu - T|/d$.

where the mean μ is estimated by the sample mean, $\bar{X}$, and the variance σ^2 is estimated by the maximum likelihood estimator

$$\sigma^{2*} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2.$$

From (7) we find that $C^*_{p,n}(1,1)$ is equal to $\hat{C}_{pmk}$ studied by PKJ and that $C^*_{p,n}(0,1)$ is equal to $\hat{C}_{pm}$ studied by Boyles (1991) and by PKJ. We also find that the estimator $C^*_{p,n}(0,1)$ is equal to $(n/(n-1))^{1/2}\hat{C}_{pm}$, where $\hat{C}_{pm}$ is studied by Chan, Cheng, and Spiring (1988a,b).

The second estimator, $C^*_{p,n-1}(u,v)$, studied is obtained by estimating the variance σ^2 by the sample variance, and hence we get

$$C^*_{p,n-1}(u,v) = \frac{d - u|\bar{X} - T|}{3\sqrt{s^2 + v(\bar{X} - T)^2}}, \tag{8}$$

where

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2.$$

From (8) we find that $C^*_{p,n-1}(0,0)$ is equal to $\hat{C}_p$ and $C^*_{p,n-1}(1,0)$ is equal to $\hat{C}_{pk}$, discussed by several authors, among others, Chan, Cheng, and Spiring (1988a,b), Bissell (1990), PKJ, Kotz, Pearn, and Johnson (1993). Discussions of the above mentioned estimators, corresponding to the cases when $u = 0$ or 1 and $v = 0$ or 1, are also found in Kotz and Johnson (1993).

One reason for using σ^{2*} as an estimator of σ^2 is that for $v = 1$ the expression under the radical sign in the denominator in (7) will be an unbiased estimator of the expression under the radical sign in the denominator in (6). See Boyles (1991) and PKJ.

By comparing the expressions in (7) and (8) we see that the two studied estimators are related as

$$C^*_{p,n-1}(u,v) = \sqrt{\frac{n-1}{n}}C^*_{p,n}\Big(u, \frac{n-1}{n}v\Big). \tag{9}$$

In the next section we derive the expected value, variance, and mean square error of the estimator $C^*_{p,n}(u,v)$. Using (9) the properties of the estimator $C^*_{p,n-1}(u,v)$ can easily be derived once the properties of $C^*_{p,n}(u,v)$ are derived.

5. Expected Value, Variance, and Mean Square Error

To derive the expected value, variance and mean square error of the estimator in (7) we use a reasoning inspired by PKJ, when they derive the expected value of $\hat{C}_{pmk}$. The derivations are given in the appendix from which we have the expected value of $C^*_{p,n}(u,v)$

$$\begin{aligned} &E(C^*_{p,n}(u,v)) \\ &= \frac{e^{-\lambda/2}}{3}\left[\frac{d\sqrt{n}}{\sigma\sqrt{2}}\sum_{j=0}^{\infty}\frac{\left(\frac{\lambda}{2}\right)^j}{j!}\frac{\Gamma\left(\frac{n-1}{2}+j\right)}{\Gamma\left(\frac{n}{2}+j\right)}\cdot {}_2F_1\left(\frac{1}{2}, j+\frac{1}{2}; j+\frac{n}{2}; 1-v\right)\right. \\ &\qquad \left. -u\sum_{j=0}^{\infty}\frac{\left(\frac{\lambda}{2}\right)^j}{\Gamma\left(\frac{1}{2}+j\right)}\frac{\Gamma\left(\frac{n}{2}+j\right)}{\Gamma\left(\frac{n+1}{2}+j\right)}\cdot {}_2F_1\left(\frac{1}{2}, j+1; j+\frac{n+1}{2}; 1-v\right)\right], \end{aligned} \tag{10}$$

where ${}_2F_1$ is the hypergeometric function (see, for instance, Abramowitz and Stegun (1965)) and

$$\lambda = \frac{n(\mu - T)^2}{\sigma^2}. \tag{11}$$

The formula for the expected value might act as a deterrent but with today's computer software like Mathematica (Wolfram (1991)) the expression in (10) can be calculated for given values of u, v, μ, σ, and n. Doing so we find that the estimator $C^*_{p,n}(u,v)$ is biased and that the bias is larger for certain choices of the parameters. This will be discussed in more detail in Section 6.

From the appendix we have

$$\begin{aligned} &E((C^*_{p,n}(u,v))^2) \\ &= \frac{e^{-\lambda/2}}{9}\Bigg[\left(\frac{d}{\sigma}\right)^2 \frac{n}{2}\sum_{j=0}^{\infty}\frac{\left(\frac{\lambda}{2}\right)^j}{j!}\cdot\frac{2}{n+2j-2}\cdot {}_2F_1\left(1, j+\frac{1}{2}; j+\frac{n}{2}; 1-v\right) \\ &\qquad - 2u\frac{d\sqrt{n}}{\sigma\sqrt{2}}\sum_{j=0}^{\infty}\frac{\left(\frac{\lambda}{2}\right)^j}{\Gamma\left(\frac{1}{2}+j\right)}\cdot\frac{2}{n+2j-1}\cdot {}_2F_1\left(1, j+1; j+\frac{n+1}{2}; 1-v\right) \\ &\qquad + u^2\sum_{j=0}^{\infty}\frac{\left(\frac{\lambda}{2}\right)^j}{j!}\cdot\frac{2j+1}{n+2j}\cdot {}_2F_1\left(1, j+\frac{3}{2}; j+\frac{n+2}{2}; 1-v\right)\Bigg] \end{aligned} \tag{12}$$

where, as above, ${}_2F_1$ is the hypergeometric function and λ is given in (11).

The variance of $C^*_{p,n}(u,v)$ is now obtained as

$$V(C^*_{p,n}(u,v)) = E((C^*_{p,n}(u,v))^2) - (E(C^*_{p,n}(u,v)))^2,$$

where $E(C^*_{p,n}(u,v))$ is given in (10) and $E((C^*_{p,n}(u,v))^2)$ in (12). Since the estimator $C^*_{p,n}(u,v)$ is biased the mean square error of the estimator might be more relevant to study than the variance. The mean square error of $C^*_{p,n}(u,v)$ is obtained as

$$\mathrm{MSE}(C^*_{p,n}(u,v)) = E((C^*_{p,n}(u,v))^2) + (C_p(u,v))^2 - 2C_p(u,v)E(C^*_{p,n}(u,v)),$$

where $C_p(u,v)$ is given in (6), $E(C^*_{p,n}(u,v))$ in (10), and $E((C^*_{p,n}(u,v))^2)$ in (12). Numerical results of the mean square error are discussed in the next section.

In the special case when $v = 0$ the expressions in (10) and (12) reduce to

$$E(C^*_{p,n}(u,0)) = \frac{1}{3}\frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}\left[\frac{d\sqrt{n}}{\sigma\sqrt{2}} - u\left(\frac{e^{-\lambda/2}}{\sqrt{\pi}} + \frac{\sqrt{\lambda}}{\sqrt{2}}\left(1 - 2\Phi(-\sqrt{\lambda})\right)\right)\right], \tag{13}$$

$$E((C^*_{p,n}(u,0))^2)$$
$$= \frac{1}{9}\cdot\frac{n}{n-3}\left[\left(\frac{d}{\sigma}\right)^2 + u^2\frac{\lambda+1}{n} - 2\frac{u}{\sqrt{n}}\frac{d}{\sigma}\left(\frac{\sqrt{2}}{\sqrt{\pi}}e^{-\lambda/2} + \sqrt{\lambda}\left(1-2\Phi(-\sqrt{\lambda})\right)\right)\right]. \quad (14)$$

When, furthermore, $u = 0$ the results in (13) and (14) are in accordance with the results for $\hat{C}_p$ in formula (12) and (20), respectively, in PKJ. When $u = 1$ the result in (13) is in accordance with $E(\hat{C}_{pk})$ in formula (14) in PKJ and with $E(\hat{C}_{pk})$ in formula (5a) in Kotz, Pearn and Johnson (1993), both of which, however, contain misprints. When $u = 1$ the result in (14) is in accordance with the result in formula (21) in PKJ. (See also Kotz and Johnson (1993).)

In the special case when $v = 1$ the expressions in (10) and (12) are simplified since then all the hypergeometric functions in (10) and (12) equal 1. Furthermore, for $u = 0$, $v = 1$ the results in (10) and (12) are in accordance with the results for $\hat{C}_{pk}$ in formulas (17) and (22), respectively, in PKJ. For $u = 1$, $v = 1$ the results in (10) and (12) are the same as the results for $\hat{C}_{pmk}$ given in formulas (25) and (27), respectively, in PKJ. (See also Kotz and Johnson (1993).)

When $v > 1$, we use formula 15.3.4 in Abramowitz and Stegun (1965) to rewrite the formulas of the hypergeometric functions in (10) and (12) in order to avoid problems with convergence. For $v > 1$ the hypergeometric functions were calculated using a recursive formula derived from formula 15.2.27 in Abramowitz and Stegun (1965) to avoid numerical problems in Mathematica.

When the process is on target, that is $\mu = T$, which is equivalent to $\lambda = 0$, the expressions in (10) and (12) reduce to

$$E(C^*_{p,n}(u,v))$$
$$= \frac{1}{3}\left[\frac{d\sqrt{n}}{\sigma R(n)\sqrt{2}}\cdot {}_2F_1(\frac{1}{2},\frac{1}{2};\frac{n}{2};1-v) - \frac{2uR(n)}{(n-1)\sqrt{\pi}}\cdot {}_2F_1(\frac{1}{2},1;\frac{n+1}{2};1-v)\right], \quad (15)$$

$$E((C^*_{p,n}(u,v))^2)$$
$$= \frac{1}{9}\left[\left(\frac{d}{\sigma}\right)^2\frac{n}{n-2}\cdot {}_2F_1(1,\frac{1}{2};\frac{n}{2};1-v) - \frac{4ud\sqrt{n}}{\sigma(n-1)\sqrt{2\pi}}\cdot {}_2F_1(1,1;\frac{n+1}{2};1-v)\right.$$
$$\left. + \frac{u^2}{n}\cdot {}_2F_1(1,\frac{3}{2};\frac{n+2}{2};1-v)\right], \quad (16)$$

where

$$R(n) = \left(\Gamma\left(\frac{n-1}{2}\right)\right)^{-1}\Gamma\left(\frac{n}{2}\right).$$

When $v = 1$ the expressions in (15) and (16) are simplified since then all the hypergeometric functions in (15) and (16) equal 1. When $u = 0$, $v = 1$ the results in (15) and (16) are in accordance with the results for $\hat{C}_{pm}$ by Chan, Cheng, and Spiring (1988a) and by PKJ. When $u = 1$, $v = 1$ the results in (15) and (16) are in accordance with the results for $\hat{C}_{pmk}$ in PKJ, apart from some misprints in their formulas. (See also Kotz and Johnson (1993).)

To obtain the expected value, variance and mean square error of the estimator $C^*_{p,n-1}(u,v)$ in (8) the results in this section can be used together with the relationship in (9).

6. Comparisons of Indices

To explore the behavior of the estimators for different values of u and v the expected values, variances, and mean square errors were calculated, using Mathematica, for different values of the parameters

$$u, v, n, a = \frac{|\mu - T|}{\sigma}, \quad \text{and} \quad b = \frac{d}{\sigma}. \tag{17}$$

In accordance with PKJ we did the calculations using $a = 0(0.5)2.0$, $b = 2(1)6$, $n = 10(10)50$. Furthermore, $u = 0(1)5$ and $v = 0(1)5$ were used. Only integer values of u and v were utilized since it seems not to be of any interest to consider more complicated indices. Only parts of the results from the calculations are presented here.

In the class of indices studied we are looking for an index that is sensitive to departures from the target value, T, especially in the case when σ is small. In such a case it is more difficult to detect that the process is not capable in the sense that it is not on target. From the study of $C_p(u,v)$ in Section 3 we saw that large values of u and v will make the index $C_p(u,v)$ more sensitive to departures from the target value.

When calculating the expected values of the estimators for the parameters studied we find that all estimators are biased to a larger or smaller extent. The larger the n-value the smaller the bias. When the process is not on target, that is, when $a > 0$, the bias is positive when using both the estimator $C^*_{p,n}(u,v)$ and the estimator $C^*_{p,n-1}(u,v)$. When the process is on target, that is, when $a = 0$, then the bias can be positive or negative. In Table 1 the relative bias of $C^*_{p,n}(u,v)$, when $a = 0$, is given for $n = 30$, $b = 3, 5$, $u = 0(1)5$, and $v = 0(1)5$.

We see from Table 1 that, when the process is on target, the relative bias is negative and quite large in absolute value when u and v are large. The same pattern as in Table 1 can be seen for the estimator $C^*_{p,n-1}(u,v)$ and for other values of b and n. When $n = 10$ the pattern is most distinct. (See Vännman (1993).)

Table 1. The relative bias of $C^*_{p,n}(u,v)$, when the process is on target and $n = 30$.

$b=3$	v					
u	0	1	2	3	4	5
0	0.044	0.026	0.010	−0.004	−0.017	−0.029
1	−0.006	−0.023	−0.038	−0.051	−0.062	−0.073
2	−0.057	−0.072	−0.085	−0.097	−0.108	−0.117
3	−0.108	−0.121	−0.133	−0.143	−0.153	−0.162
4	−0.158	−0.170	−0.180	−0.189	−0.198	−0.206
5	−0.209	−0.219	−0.228	−0.236	−0.243	−0.250

$b=5$	v					
u	0	1	2	3	4	5
0	0.044	0.026	0.010	−0.004	−0.017	−0.029
1	0.014	−0.003	−0.019	−0.032	−0.044	−0.056
2	−0.016	−0.033	−0.047	−0.060	−0.071	−0.082
3	−0.047	−0.062	−0.076	−0.088	−0.099	−0.109
4	−0.077	−0.092	−0.104	−0.115	−0.126	−0.135
5	−0.108	−0.121	−0.133	−0.143	−0.153	−0.162

When the process is on target, and at the same time σ is so small that the process can be considered capable, it is an undesirable property to have an index whose estimator largely underestimates the true index. One way of reasoning for obtaining a suitable index is to look for an index whose estimator has a small bias when $a = 0$ and at the same time has small variability. To combine these two criteria we can search for values of u and v that will give small values of the mean square error of the estimator when $a = 0$, where a is given in (17). One suggestion for obtaining meaningful indices is using the following criteria.

1. Only indices with small bias and small mean square error, when the process is on target, will be considered.

2. Among the possible u- and v-values obtained, indices will be chosen with respect to their sensitivity to departures from the target value in the sense that the expected value of the estimator of the index ought to be sensitive to departures from the target value, especially in the case of small σ. Also the mean square errors, when the process is not on target, will be taken into consideration.

As an example consider the case when $n = 30$ and the estimator $C^*_{p,n}(u,v)$. In Table 2 the values, when $a = 0$, of $\mathrm{MSE}(C^*_{p,n}(u,v))$, multiplied by 100, are given for $n = 30$, $b = 3, 5$, $u = 0(1)5$, and $v = 0(1)5$. We see from Table 2 that when $n = 30$ the smallest values of the mean square error are obained for $u = 0$ and $v = 2$ or 3 but the mean square error does not vary too much close to these values. We also see from Table 2 that large values of u and v give rise to large mean square errors and hence indices with large values of u and v are not suitable to use. The same pattern is found for the other values of b and n not given in Table 2. Choosing the indices with, for instance, the seven smallest mean square errors for $b = 3$ from Table 2 will give us $u = 0$ together with $v = 1, 2, 3, 4, 5$ and $u = 1$ together with $v = 0, 1$. Correspondingly for $b = 5$ we find $u = 0$ together with $v = 1, 2, 3, 4$ and $u = 1$ together with $v = 0, 1, 2$. Selecting the indices common to both groups, which seems to be a reasonable strategy, gives us $u = 0$ together with $v = 1, 2, 3, 4$ and $u = 1$ together with $v = 0, 1$ to consider further.

Table 2. $100 \times \mathrm{MSE}(C^*_{p,n}(u,v))$, when the process is on target and $n = 30$.

$b = 3$	v					
u	0	1	2	3	4	5
0	2.235	1.963	1.889	1.927	2.035	2.187
1	1.999	2.080	2.266	2.507	2.779	3.070
2	2.585	2.937	3.322	3.718	4.116	4.511
3	3.995	4.535	5.058	5.561	6.045	6.510
4	6.227	6.874	7.474	8.036	8.566	9.068
5	9.283	9.954	10.570	11.143	11.680	12.185

$b = 5$	v					
u	0	1	2	3	4	5
0	6.209	5.453	5.248	5.353	5.652	6.077
1	5.540	5.400	5.649	6.109	6.696	7.361
2	5.695	6.089	6.730	7.496	8.332	9.204
3	6.672	7.518	8.490	9.515	10.560	11.606
4	8.473	9.688	10.931	12.166	13.380	14.566
5	11.096	12.598	14.051	15.448	16.792	18.084

Next we compare the indices, among those obtained above, with respect to their sensitivity to departures from the target value. PKJ showed that $\hat{C}_{pmk}$ is more sensitive to departures from the target value than are $\hat{C}_{pm}$ and $\hat{C}_{pk}$. Hence the cases $u = 0$, $v = 1$, corresponding to C_{pm}, and $u = 1$, $v = 0$, corresponding to C_{pk}, are not of interest to study further.

We then consider, for different values of a and b, the expected values and the values of the mean square error of $C^*_{p,n}(u,v)$ when $u=0$ and $v=2,3,4$ as well as $u=1$ and $v=1$. In Tables 3 and 4 these values are given for $n=30$.

Table 3. The expected value of $C^*_{p,n}(u,v)$, when $n=30$.

	$u=0, v=2$					$u=1, v=1$				
	a					a				
b	0	0.5	1.0	1.5	2.0	0	0.5	1.0	1.5	2.0
2	0.673	0.557	0.393	0.288	0.224	0.635	0.462	0.244	0.097	0.002
3	1.010	0.835	0.589	0.432	0.336	0.977	0.768	0.485	0.284	0.152
4	1.346	1.113	0.785	0.576	0.448	1.319	1.074	0.725	0.471	0.303
5	1.683	1.391	0.982	0.720	0.560	1.661	1.379	0.965	0.659	0.453
6	2.020	1.670	1.178	0.864	0.672	2.003	1.685	1.206	0.846	0.604

	$u=0, v=3$					$u=0, v=4$				
	a					a				
b	0	0.5	1.0	1.5	2.0	0	0.5	1.0	1.5	2.0
2	0.664	0.517	0.341	0.243	0.186	0.655	0.485	0.305	0.214	0.163
3	0.996	0.775	0.511	0.364	0.280	0.983	0.728	0.458	0.321	0.245
4	1.327	1.033	0.681	0.485	0.373	1.310	0.971	0.611	0.427	0.326
5	1.659	1.291	0.852	0.607	0.466	1.638	1.213	0.763	0.534	0.408
6	1.991	1.550	1.022	0.728	0.559	1.965	1.456	0.916	0.641	0.489

From Table 3 we see that $u=0$, $v=2$ will give us the index, among those considered, that is least sensitive to departures from the target value. Hence we exclude that case. We also exclude the case when $u=1$, $v=1$ for not being sensitive enough to departures from the target value when σ is small, that is, when b, defined in (17), is large. Hence we end up with the two indices corresponding to $u=0$, $v=3$ and $u=0$, $v=4$, which are fairly equivalent with respect to mean square error and sensitivity to departures from the target value. The estimator $C^*_{p,n}(0,4)$ is somewhat more sensitive to departures from the target value but will give rise to a small but negative bias, while $C^*_{p,n}(0,3)$ is almost unbiased, when the process is on target. Comparing the mean square errors of the two estimators when the process is not on target, we see from Table 4 that they are fairly equivalent. For $n=50$, by calculating the expected values of $C^*_{p,n}(0,3)$ and $C^*_{p,n}(0,4)$ we find that, for all values of a and b, the estimators have small bias. For numerical details see Vännman (1993), where also an example for $n=10$ is studied and the same conclusion regarding suitable indices is reached.

Table 4. $100\times\text{MSE}(C^*_{p,n}(u,v))$, when $n = 30$.

	$u=0, v=2$					$u=1, v=1$				
	a					a				
b	0	0.5	1.0	1.5	2.0	0	0.5	1.0	1.5	2.0
2	0.840	0.736	0.285	0.094	0.036	1.074	1.089	0.492	0.189	0.078
3	1.889	1.655	0.641	0.211	0.081	2.080	2.028	0.942	0.370	0.156
4	3.359	2.942	1.139	0.376	0.144	3.522	3.302	1.552	0.618	0.262
5	5.248	4.560	1.780	0.587	0.225	5.400	4.910	2.322	0.931	0.398
6	7.558	6.662	2.563	0.844	0.324	7.715	6.853	3.252	1.310	0.561

	$u=0, v=3$					$u=0, v=4$				
	a					a				
b	0	0.5	1.0	1.5	2.0	0	0.5	1.0	1.5	2.0
2	0.857	0.828	0.258	0.074	0.026	0.904	0.906	0.234	0.061	0.021
3	1.927	1.863	0.581	0.166	0.059	2.035	2.040	0.526	0.136	0.047
4	3.426	3.312	1.032	0.295	0.106	3.617	3.627	0.935	0.242	0.084
5	5.353	5.175	1.613	0.460	0.165	5.565	5.668	1.461	0.379	0.131
6	7.709	7.452	2.323	0.663	0.238	8.138	8.161	2.104	0.545	0.188

It is of importance to notice that when n is small the mean square error for all estimators studied is quite large compared to the corresponding capability index. This can be seen from Table 5, which gives the square root of the mean square error for $C^*_{p,n}(0,4)$ for some parameter values. Hence we can conclude that we need quite large sample sizes to reduce the variability due to estimation.

Table 5. The square root of the mean square error for $C^*_{p,n}(0,4)$ and the corresponding index, for $a = 0, 1$, $b = 3, 5$ and $n = 10, 30, 50$.

			$\sqrt{\text{MSE}(C^*_{p,n}(0,4))}$		
a	b	$C_p(0,4)$	$n=10$	$n=30$	$n=50$
0	3	1.000	0.272	0.143	0.107
0	5	1.667	0.454	0.238	0.179
1	3	0.447	0.151	0.073	0.054
1	5	0.745	0.251	0.121	0.090

7. Concluding Remarks

By considering the class of indices described by $C_p(u,v)$ in (6) and the properties of estimators of the corresponding indices we can choose values of u and v, that is, a suitable index, so that the index and its estimator satisfy criteria of interest, not necessarily the same as those suggested here.

It is of interest to note that studying the properties of $C_p(u,v)$ alone, and not at the same time taking the properties of its estimator into account, might mislead us when searching for a suitable index. Hence its is of importance always to consider the properties of the estimator, such as the behavior of the expected value and the mean square error, when deciding which of the capability indices to use.

A more comprehensive and detailed version of this paper is given in Vännman (1993), including the results for the estimator $C^*_{p,n-1}(u,v)$.

The distribution of $C^*_{p,n}(u,v)$ has been derived in Vännman and Kotz (1994), where, also, suitable criteria for choosing an index from the family are suggested, based on a decision rule that can be used to determine whether the process is capable or not.

Appendix

We derive the rth moment of $C^*_{p,n}(u,v)$ and introduce the notation

$$\xi = \frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i-\bar{X})^2, \quad \eta' = \frac{n(\bar{X}-T)^2}{\sigma^2}, \quad \text{and} \quad D = \frac{d\sqrt{n}}{\sigma}. \tag{18}$$

In this notation the estimator $C^*_{p,n}(u,v)$ in (7) becomes:

$$C^*_{p,n}(u,v) = \frac{D-u\sqrt{\eta'}}{3\sqrt{\xi+v\eta'}}. \tag{19}$$

Under the assumption of normality we have that ξ and η' in (18) are independent random variables, and that ξ is distributed according to a central χ^2-distribution with $n-1$ degrees of freedom. Furthermore we have that η' is distributed according to a non-central χ^2-distribution with 1 degree of freedom and non-centrality parameter λ given in (11).

Using the binomial theorem we can write the rth moment of $C^*_{p,n}(u,v)$ in (19) as

$$E((C^*_{p,n}(u,v))^r) = 3^{-r}\sum_{i=0}^{r}(-u)^i\binom{r}{i}D^{r-i}E(\eta'^{i/2}(\xi+v\eta')^{-r/2}). \tag{20}$$

In (20) the expression $(-u)^i$ should be interpreted as 1 when $i=0$, also for the case $u=0$. Now we rewrite the expected value in the right hand side of (20), using the fact that a non-central χ^2-distribution with 1 degree of freedom and non-centrality parameter λ can be written as a mixture of central χ^2-distributions with $1+2j$ degrees of freedom and Poisson weights

$$\frac{e^{-\lambda/2}}{j!}\left(\frac{\lambda}{2}\right)^j.$$

See, for instance, Johnson and Kotz (1970b, p. 132). Let η_j be distributed according to a central χ^2-distribution with $1+2j$ degrees of freedom. Then we get

$$E(\eta'^{i/2}(\xi+v\eta')^{-r/2}) = \sum_{j=0}^{\infty}\frac{e^{-\lambda/2}}{j!}\left(\frac{\lambda}{2}\right)^j E(\eta_j^{i/2}(\xi+v\eta_j)^{-r/2}). \tag{21}$$

Let $\tau_j = \xi+\eta_j$ and $\zeta_j = \eta_j/\tau_j$. Under the assumption of normality we have that ζ_j and τ_j are independent random variables (see, for instance, Johnson and Kotz (1970a)) and that ζ_j is distributed according to a Beta$((1+2j)/2$, $(n-1)/2)$-distribution. Furthermore we have that τ_j is distributed according to a central χ^2-distribution with $n+2j$ degrees of freedom. Using ζ_j and τ_j and their independence we can rewrite the expected value in the right hand side of (21) to get

$$E(\eta_j^{i/2}(\xi+v\eta_j)^{-r/2}) = E(\tau_j^{-(r-i)/2})E(\zeta_j^{i/2}(1+(v-1)\zeta_j)^{-r/2}). \tag{22}$$

Since ζ_j is Beta-distributed we can write

$$\begin{aligned}
&E(\zeta_j^{i/2}(1+(v-1)\zeta_j)^{-r/2})\\
&= \frac{\Gamma\left(\frac{n}{2}+j\right)}{\Gamma\left(\frac{1}{2}+j\right)\Gamma\left(\frac{n-1}{2}\right)}\int_0^1 x^{j+(i-1)/2}(1-x)^{(n-3)/2}(1+(v-1)x)^{-r/2}dx\\
&= \frac{\Gamma\left(\frac{n}{2}+j\right)\Gamma\left(\frac{i+1}{2}+j\right)}{\Gamma\left(\frac{1}{2}+j\right)\Gamma\left(\frac{n+i}{2}+j\right)}\cdot {}_2F_1\Big(\frac{r}{2},\frac{i+1}{2}+j;\frac{n+i}{2}+j;1-v\Big),
\end{aligned} \tag{23}$$

where ${}_2F_1$ is the hypergeometric function (see, for instance, Abramowitz and Stegun (1965)).

Combining the results from (20)-(23) with the expected value of τ_j (see, for instance, Johnson and Kotz (1970a)) we get

$$\begin{aligned}
E((C_{p,n}^*(u,v))^r) = 3^{-r}\sum_{i=0}^{r}(-u)^i\binom{r}{i}\left(\frac{D}{\sqrt{2}}\right)^{r-i}e^{-\lambda/2}\\
\cdot\sum_{j=0}^{\infty}\frac{\left(\frac{\lambda}{2}\right)^j}{j!}\frac{\Gamma\left(\frac{n-r+i}{2}+j\right)\Gamma\left(\frac{i+1}{2}+j\right)}{\Gamma\left(\frac{1}{2}+j\right)\Gamma\left(\frac{n+i}{2}+j\right)}\cdot {}_2F_1\Big(\frac{r}{2},\frac{i+1}{2}+j;\frac{n+i}{2}+j;1-v\Big),
\end{aligned} \tag{24}$$

where D is given in (18) and λ is given in (11). In (24) the expression $(-u)^i$ should be interpreted as 1 when $i=0$, also for the case $u=0$. Letting $r=1$ and 2 in (24) we obtain the expected values given in (10) and (12), respectively.

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