Limit behaviour of the truncated pathwise Fourier-transformation of Lévy-driven CARMA processes for non-equidistant discrete time observations

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Supplementary Material

This online supplement contains all the proofs and some auxiliary results, as well as some further details of the simulations studies.

S1 Supplementary Material for Section 3

S1.1 Proof of Lemma 3.1

PROOF. Let ω be an arbitrary frequency. Observe that by Corollary 3.4 from Schlemm and Stelzer (2012) one has

$$\mathbf{b}^T(\mathbf{A}-i\omega I)^{-1}\mathbf{e} = -\frac{b(i\omega)}{a(i\omega)}.$$

Denote

$$F(t) = \mathbf{b}^T(\mathbf{A}-i\omega I)^{-1}e^{(\mathbf{A}-i\omega I)t}, \quad t \in [0,T],$$

$$G(t)=\int_0^t e^{-\mathbf{A}u}\mathbf{e}dL(u) \quad t\in[0,T].$$

Observe that $G(0)=0$ and since F is continuous and of finite variation, we get $[F,G]=0$, where $[\cdot,\cdot]$ denotes the usual quadratic covariation of semimartingales (see e.g. Protter (2004)). Applying the (multidimensional) integration by parts formula

$$\begin{aligned}\int_0^T dF(t)G(t) &= F(T)G(T)-F(0)G(0)-\int_0^T F(t)dG(t)-[F,G]\\ &= F(T)G(T)-\int_0^T F(t)dG(t)\end{aligned}$$

we obtain

$$\begin{aligned}&\int_0^T dF(t)G(t)=\int_0^T \mathbf{b}^T(\mathbf{A}-i\omega I)^{-1}(\mathbf{A}-i\omega I)e^{(\mathbf{A}-i\omega I)t}\int_0^t e^{-\mathbf{A}u}\mathbf{e}dL(u)dt\\ &=\int_0^T\int_0^t \mathbf{b}^T e^{\mathbf{A}(t-u)}\mathbf{e}dL(u)e^{-i\omega t}dt\\ &=\mathbf{b}^T(\mathbf{A}-i\omega I)^{-1}e^{(\mathbf{A}-i\omega I)T}\int_0^T e^{-\mathbf{A}t}\mathbf{e}dL(t)\\ &\quad -\int_0^T \mathbf{b}^T(\mathbf{A}-i\omega I)^{-1}e^{(\mathbf{A}-i\omega I)t}e^{-\mathbf{A}t}\mathbf{e}dL(t)\\ &=\mathbf{b}^T(\mathbf{A}-i\omega I)^{-1}e^{-i\omega T}\int_0^T e^{\mathbf{A}(T-t)}\mathbf{e}dL(t)+\frac{b(i\omega)}{a(i\omega)}\int_0^T e^{-i\omega t}dL(t).\end{aligned}$$

Thus

$$\begin{aligned}&\textstyle\int_0^T\int_0^t \mathbf{b}^T e^{\mathbf{A}(t-u)}\mathbf{e}dL(u)e^{-i\omega t}dt \qquad \text{(S1.1)}\\ &=\mathbf{b}^T(\mathbf{A}-i\omega I)^{-1}e^{-i\omega T}\textstyle\int_0^T e^{\mathbf{A}(T-t)}\mathbf{e}dL(t)+\frac{b(i\omega)}{a(i\omega)}\int_0^T e^{-i\omega t}dL(t).\end{aligned}$$

Using the form of the strictly stationary solution of (2.3) given in (2.4) we get

$$\int_0^T e^{\mathbf{A}(T-t)}\mathbf{e}dL(t) = \mathbf{X}(T) - e^{\mathbf{A}T}\mathbf{X}(0). \tag{S1.2}$$

Moreover, since $\int_0^T e^{(\mathbf{A}-i\omega I)t}dt = (i\omega I - \mathbf{A})^{-1}(I - e^{(\mathbf{A}-i\omega I)T})$, we have

$$\int_0^T \mathbf{b}^T e^{(\mathbf{A}-i\omega I)t}\mathbf{X}(0)dt = \mathbf{b}^T(i\omega I - \mathbf{A})^{-1}(I - e^{(\mathbf{A}-i\omega I)T})\mathbf{X}(0). \tag{S1.3}$$

We have

$$\begin{aligned}
\mathscr{F}_T(Y)(\omega) =& \frac{1}{\sqrt{T}}\int_0^T Y(t)e^{-i\omega t}dt \\
\overset{(2.7)}{=}& \frac{1}{\sqrt{T}}\int_0^T \left(\mathbf{b}^T e^{\mathbf{A}t}\mathbf{X}(0) + \int_0^t \mathbf{b}^T e^{\mathbf{A}(t-u)}\mathbf{e}dL(u)\right)e^{-i\omega t}dt \\
\overset{(S1.1),(S1.2),(S1.3)}{=}& \frac{1}{\sqrt{T}}\frac{b(i\omega)}{a(i\omega)}\int_0^T e^{-i\omega u}dL(u) \\
&+ \frac{1}{\sqrt{T}}\mathbf{b}^T(i\omega I - \mathbf{A})^{-1}\left(\mathbf{X}(0) - e^{-i\omega T}\mathbf{X}(T)\right).
\end{aligned}$$

To get the equivalent form note,

$$\begin{aligned}
&\sqrt{T}\mathscr{F}(Y)(\omega) = \mathbf{b}^T(i\omega I - \mathbf{A})^{-1}\left[\mathbf{e}\int_0^T e^{-i\omega u}dL(u) + \left(\mathbf{X}(0) - e^{-i\omega T}\mathbf{X}(T)\right)\right] \\
&\overset{(S1.2)}{=} \mathbf{b}^T(i\omega I - \mathbf{A})^{-1} \\
&\quad \times \left[\int_0^T \left(e^{-i\omega u} - e^{-i\omega T}e^{\mathbf{A}(T-u)}\right)\mathbf{e}dL(u) + \left(I - e^{(-i\omega I+\mathbf{A})T}\right)\mathbf{X}(0)\right],
\end{aligned}$$

which completes the proof of this Lemma. □

S1.2 Moments of the truncated Fourier transform

In addition to the main paper we calculate here moments of the truncated Fourier transform. First, recall the so-called *compensation formula*: If $(L_t)_{t\geq 0}$ is a Lévy process with finite first moments and f is a bounded deterministic function, then

$$\mathbb{E}\left[\int_0^T f(u)dL_u\right] = \mathbb{E}[L_1]\int_0^T f(s)ds. \tag{S1.4}$$

Secondly, observe that the solution of the system (2.2) and (2.3) is of the form (2.4), where $\mathbf{X}$ is the process with mean $m(t) = \mathbb{E}[\mathbf{X}(t)]$ and $P_X(t) = \mathbb{E}[\mathbf{X}(t)\mathbf{X}(t)^T]$ satisfying

$$\begin{aligned} m_X(t) &= e^{\mathbf{A}t}m_X(0) \\ P_X(t) &= e^{\mathbf{A}t}P_X(0)e^{\mathbf{A}^T t} + \sigma^2\int_0^t e^{\mathbf{A}(t-u)}\mathbf{e}\mathbf{e}^T e^{\mathbf{A}^\mathbf{T}(t-u)}du \end{aligned} \tag{S1.5}$$

In particular, for stationary processes these solutions are constant and the so-called Lyapunov equation

$$\mathbf{A}P_X + P_X\mathbf{A}^T + \sigma^2\mathbf{e}\mathbf{e}^T = 0 \tag{S1.6}$$

holds true. For Lévy-driven CARMA processes the form of the autocovariance function in terms of solutions of Lyapunov equations is formulated e.g. in Proposition 3.13 of Marquardt and Stelzer (2007).

We are first going to show that the truncated Fourier transform of a stationary CARMA process is a zero-mean random variable. Next, we find the covariance between the truncated Fourier transform at two different frequencies. As we have mentioned earlier, the spectral density function plays a central role.

Theorem S1.1. *Let* $\mathbf{X}$ *and* Y *be processes given by the state-space representation (2.2) and (2.3). Suppose that Assumptions 2.1, 2.2, 2.3. and 2.4 are satisfied.*

Then $\mathbb{E}(\mathscr{F}_T(Y)(\omega)) = 0$ *for all* $\omega \in \mathbb{R}$*. For* $\omega_1, \omega_2 \in \mathbb{R}$ *we have*

$$\mathbb{E}\left[\mathscr{F}_T(Y)(\omega_1)\mathscr{F}_T(Y)(\omega_2)\right] = \sigma^2 \frac{|b(i\omega_1)|^2}{|a(i\omega_1)|^2} + \frac{1}{T}K(T,\omega_1,-\omega_1), \quad \text{if} \quad \omega_1 = -\omega_2 \tag{S1.7}$$

and

$$\mathbb{E}\left[\mathscr{F}_T(Y)(\omega_1)\mathscr{F}_T(Y)(\omega_2)\right] = \frac{1}{T}K_1(T,\omega_1,\omega_2), \quad \text{if} \quad \omega_1 \neq -\omega_2, \tag{S1.8}$$

where K is a bounded function of T given by (S1.10) *below and*

$$K_1(T,\omega_1,\omega_2) = K(T,\omega_1,\omega_2) + \mathbf{b}^T(i\omega_1 I - \mathbf{A})^{-1}\sigma^2 \frac{1-\exp(-Ti(\omega_1+\omega_2))}{i(\omega_1+\omega_2)}\mathbf{e}\mathbf{e}^T(i\omega_2 I - \mathbf{A}^T)^{-1}\mathbf{b}.$$

PROOF. For the first part it is enough to observe that by the compensation formula $\mathbb{E}\left(\int_0^T e^{-i\omega u} dL(u)\right) = 0$ and $\mathbb{E}[\mathbf{X}(t)] = 0$. For the second part observe that using Lemma 3.1 and Formula (3.10) we have

$$\begin{aligned}\mathbb{E}[\mathscr{F}_T(Y)(\omega_1)\mathscr{F}_T(Y)(\omega_2)] &= \frac{1}{T}\mathbf{b}^T(i\omega_1 I - \mathbf{A})^{-1} \\ &\times \mathbb{E}\left[\left(\int_0^T \left(e^{-i\omega_1 u} - e^{-i\omega_1 T}e^{\mathbf{A}(T-u)}\right)\mathbf{e}dL(u) + \left(I - e^{(-i\omega_1 I + \mathbf{A})T}\right)\mathbf{X}(0)\right)\right. \\ &\times \left.\left(\int_0^T \mathbf{e}^T\left(e^{-i\omega_2 u} - e^{-i\omega_2 T}e^{\mathbf{A}^T(T-u)}\right)dL(u) + \mathbf{X}(0)^T\left(I - e^{(-i\omega_2 I + \mathbf{A}^T)T}\right)\right)\right] \\ &\times (i\omega_2 I - \mathbf{A}^T)^{-1}\mathbf{b} = \frac{1}{T}\mathbf{b}^T(i\omega_1 I - \mathbf{A})^{-1}\widetilde{I}(i\omega_2 I - \mathbf{A}^T)^{-1}\mathbf{b},\end{aligned}$$

where $\widetilde{I} = I_1 + I_2 + I_3 + I_4$ with

$$I_1 := \mathbb{E}\left[\int_0^T \left(e^{-i\omega_1 u} - e^{-i\omega_1 T}e^{\mathbf{A}(T-u)}\right)\mathbf{e}dL(u)\right.$$

$$\times \int_0^T \mathbf{e}^T \left(e^{-i\omega_2 u} - e^{-i\omega_2 T} e^{\mathbf{A}^T(T-u)}\right) dL(u)\Bigg]$$

$$I_2 := \mathbb{E}\left[\int_0^T \left(e^{-i\omega_1 u} - e^{-i\omega_1 T} e^{\mathbf{A}(T-u)}\right) \mathbf{e} dL(u) \mathbf{X}(0)^T \left(I - e^{(-i\omega_2 I + \mathbf{A}^T)T}\right)\right]$$

$$I_3 := \mathbb{E}\left[\left(I - e^{(-i\omega_1 I + \mathbf{A})T}\right) \mathbf{X}(0) \int_0^T \mathbf{e}^T \left(e^{-i\omega_2 u} - e^{-i\omega_2 T} e^{\mathbf{A}^T(T-u)}\right) dL(u)\right]$$

$$I_4 := \mathbb{E}\left[\left(I - e^{(-i\omega_1 I + \mathbf{A})T}\right) \mathbf{X}(0)\mathbf{X}(0)^T \left(I - e^{(-i\omega_2 I + \mathbf{A}^T)T}\right)\right].$$

We have that $I_2 = I_3 = 0$ since $(L_t)_{t\geq 0}$ is independent of $\mathbf{X}(0)$. Observe that by the Itô isometry, the compensation formula and the fact that $\mathbb{E}[[L,L]_1] = \mathrm{Var}(L(1)) = \sigma^2$ we have

$$\begin{aligned}
I_1^1 :=& \mathbb{E}\left[\int_0^T e^{-i\omega_1 u} \mathbf{e} dL(u) \int_0^T \mathbf{e}^T e^{-i\omega_2 u} dL(u)\right] \\
=& \mathbb{E}\left[\int_0^T e^{-i(\omega_1+\omega_2)u} \mathbf{e}\mathbf{e}^T d[L,L]_u\right] \\
=& \mathbb{E}[[L,L]_1] \int_0^T e^{-i(\omega_1+\omega_2)u} \mathbf{e}\mathbf{e}^T du = \sigma^2 \int_0^T e^{-i(\omega_1+\omega_2)u} \mathbf{e}\mathbf{e}^T du.
\end{aligned}$$

Thus

$$I_1^1 = \begin{cases} \sigma^2 T \mathbf{e}\mathbf{e}^T, & \omega_1 = -\omega_2, \\ \sigma^2 \frac{1-\exp(-Ti(\omega_1+\omega_2))}{i(\omega_1+\omega_2)} \mathbf{e}\mathbf{e}^T, & \omega_1 \neq -\omega_2. \end{cases} \tag{S1.9}$$

Thus, if $\omega_1 = -\omega_2$, then

$$\begin{aligned}
&\frac{1}{T}\mathbf{b}^T (i\omega_1 I - \mathbf{A})^{-1} I_1^1 (i\omega_2 I - \mathbf{A}^T)^{-1} \mathbf{b} \\
&\quad = \frac{1}{T}\sigma^2 T \mathbf{b}^T (i\omega_1 I - \mathbf{A})^{-1} \mathbf{e}\mathbf{e}^T (i\omega_2 I - \mathbf{A}^T)^{-1} \mathbf{b} \\
&\quad = \sigma^2 \mathbf{b}^T (\mathbf{A} - i\omega_1 I)^{-1} \mathbf{e}\mathbf{e}^T (i\omega_1 I + \mathbf{A}^T)^{-1} \mathbf{b} \\
&\quad = \sigma^2 \left(-\frac{b(i\omega_1)}{a(i\omega_1)}\right)\left(-\frac{b(-i\omega_1)}{a(-i\omega_1)}\right) = \sigma^2 \frac{|b(i\omega_1)|^2}{|a(i\omega_1)|^2}.
\end{aligned}$$

Now

$$\begin{aligned}
I_1^2 :&= \mathbb{E}\left[\int_0^T e^{-i\omega_1 u}\mathbf{e}dL(u)\int_0^T \mathbf{e}^T e^{-i\omega_2 T} e^{\mathbf{A}^T(T-u)}dL(u)\right]\\
&= e^{-i\omega_2 T}\mathbb{E}\left[\int_0^T e^{-i\omega_1 u}\mathbf{e}\mathbf{e}^T e^{\mathbf{A}^T(T-u)}d[L,L]_u\right]\\
&= e^{-i\omega_2 T}\mathbb{E}[[L,L]_1]\int_0^T e^{-i\omega_1 u}\mathbf{e}\mathbf{e}^T e^{\mathbf{A}^T(T-u)}du\\
&= e^{-i\omega_2 T}\sigma^2\int_0^T e^{-i\omega_1 u}\mathbf{e}\mathbf{e}^T e^{\mathbf{A}^T(T-u)}du.
\end{aligned}$$

In the same way

$$\begin{aligned}
I_1^3 :&= \mathbb{E}\left[\int_0^T e^{-i\omega_1 T} e^{\mathbf{A}(T-u)}\mathbf{e}dL(u)\int_0^T \mathbf{e}^T e^{-i\omega_2 u}dL(u)\right]\\
&= e^{-i\omega_1 T}\sigma^2\int_0^T e^{\mathbf{A}(T-u)}\mathbf{e}\mathbf{e}^T e^{-i\omega_2 u}du.
\end{aligned}$$

Combining these two we arrive at

$$\begin{aligned}
I_1^2 + I_1^3 =& e^{-i(\omega_1+\omega_2)T}\sigma^2\left[\mathbf{e}\mathbf{e}^T(i\omega_1 I + \mathbf{A}^T)^{-1}\left(e^{(i\omega_1 I+\mathbf{A}^T)T} - I\right)\right.\\
&\left. + (i\omega_2 I + \mathbf{A})^{-1}\left(e^{(i\omega_2 I+\mathbf{A})T} - I\right)\mathbf{e}\mathbf{e}^T\right].
\end{aligned}$$

Now

$$\begin{aligned}
I_1^4 :&= \mathbb{E}\left[\int_0^T e^{-i\omega_1 T} e^{\mathbf{A}(T-u)}\mathbf{e}dL(u)\int_0^T \mathbf{e}^T e^{-i\omega_2 T} e^{\mathbf{A}^T(T-u)}dL(u)\right]\\
&= e^{-i(\omega_1+\omega_2)T}\sigma^2\int_0^T e^{\mathbf{A}(T-u)}\mathbf{e}\mathbf{e}^T e^{\mathbf{A}^T(T-u)}du.
\end{aligned}$$

Now

$$I_4 = \mathbb{E}[\mathbf{X}(0)\mathbf{X}(0)^T] - e^{-i\omega_1 T}e^{\mathbf{A}T}\mathbb{E}[\mathbf{X}(0)\mathbf{X}(0)^T] - e^{-i\omega_2 T}\mathbb{E}[\mathbf{X}(0)\mathbf{X}(0)^T]e^{\mathbf{A}^T T}$$

$$+ e^{-i(\omega_1+\omega_2)T} e^{\mathbf{A}T} \mathbb{E}[\mathbf{X}(0)\mathbf{X}(0)^T] e^{\mathbf{A}^T T}.$$

By stationarity we have

$$\mathbb{E}[\mathbf{X}(0)\mathbf{X}(0)^T] =: P_X = P_X(0) = P_X(T),$$

where P_X satisfies (S1.6). Combining this with (S1.5) we obtain

$$\begin{aligned} I_1^4 + I_4 =& P_X - e^{-i\omega_1 T} e^{\mathbf{A}T} P_X - e^{-i\omega_2 T} P_X e^{\mathbf{A}^T T} + e^{-i(\omega_1+\omega_2)T} e^{\mathbf{A}T} P_X e^{\mathbf{A}^T T} \\ &+ e^{-i(\omega_1+\omega_2)T} (P_X - e^{\mathbf{A}T} P_X e^{\mathbf{A}^T T}) \\ =& e^{-i\omega_1 T} P_X \left(I - e^{\mathbf{A}T} \right) + e^{-i\omega_2 T} \left(I - e^{\mathbf{A}^T} \right) P_X \\ &+ P_X \left(1 - e^{-i\omega_1 T} - e^{-i\omega_2 T} + e^{-i(\omega_1+\omega_2)T} \right). \end{aligned}$$

Since $\mathbf{A}$ is a stable matrix, $e^{\mathbf{A}T}$ is bounded. Thus

$$\begin{aligned} K(T,\omega_1,\omega_2) =& \mathbf{b}^T (i\omega_1 I - \mathbf{A})^{-1} \Bigg[e^{-i(\omega_1+\omega_2)T} \sigma^2 \Bigg[\mathbf{e}\mathbf{e}^T (i\omega_1 I + \mathbf{A}^T)^{-1} \\ &\times \left(e^{(i\omega_1 I + \mathbf{A}^T)T} - I \right) + (i\omega_2 I + \mathbf{A})^{-1} \left(e^{(i\omega_2 I + \mathbf{A})T} - I \right) \mathbf{e}\mathbf{e}^T \Bigg] \\ &+ e^{-i\omega_1 T} P_X \left(I - e^{\mathbf{A}^T T} \right) + e^{-i\omega_2 T} \left(I - e^{\mathbf{A}T} \right) P_X \qquad \text{(S1.10)} \\ &+ P_X \left(1 - e^{-i\omega_1 T} - e^{-i\omega_2 T} + e^{-i(\omega_1+\omega_2)T} \right) \Bigg] (i\omega_2 I - \mathbf{A}^T)^{-1} \mathbf{b} \end{aligned}$$

is bounded in T for fixed $\omega_1, \omega_2 \in \mathbb{R}$. □

Now we give the form of the covariance matrix. Put

$$\Sigma(\omega_1,\omega_2) := [\Sigma_{ij}]_{1\le i,j\le 4} = \mathbb{E}\left[\begin{bmatrix}\Re\mathscr{F}_T(Y)(\omega_1)\\ \Im\mathscr{F}_T(Y)(\omega_1)\\ \Re\mathscr{F}_T(Y)(\omega_2)\\ \Im\mathscr{F}_T(Y)(\omega_2)\end{bmatrix}\begin{bmatrix}\Re\mathscr{F}_T(Y)(\omega_1)\\ \Im\mathscr{F}_T(Y)(\omega_1)\\ \Re\mathscr{F}_T(Y)(\omega_2)\\ \Im\mathscr{F}_T(Y)(\omega_2)\end{bmatrix}^T\right].$$

Theorem S1.2. *Let* $\mathbf{X}$ *and* Y *be processes given by the state-space representation (2.2) and (2.3). Suppose that Assumptions 2.1, 2.2, 2.3 and 2.4 are satisfied. For* $\omega_1 \neq \omega_2$ *and* $\omega_1 \neq -\omega_2$ *there exists a bounded matrix* $K_2 \in \mathbb{C}^{4\times 4}$ *such that*

$$\Sigma(\omega_1,\omega_2) = \frac{1}{2}\sigma^2 \operatorname{diag}\left(\frac{|b(i\omega_1)|^2}{|a(i\omega_1)|^2}, \frac{|b(i\omega_1)|^2}{|a(i\omega_1)|^2}, \frac{|b(i\omega_2)|^2}{|a(i\omega_2)|^2}, \frac{|b(i\omega_2)|^2}{|a(i\omega_2)|^2}\right) + \frac{1}{T}K_2.$$

Proof. For $k,l = 1,2$ let us denote

$$\Sigma_1(\omega_1,\omega_2) := \mathbb{E}\left[\Re\mathscr{F}_T(Y)(\omega_1)\Re\mathscr{F}_T(Y)(\omega_2)\right],$$

$$\Sigma_2(\omega_1,\omega_2) := \mathbb{E}\left[\Im\mathscr{F}_T(Y)(\omega_1)\Im\mathscr{F}_T(Y)(\omega_2)\right],$$

$$\Sigma_3(\omega_1,\omega_2) := \mathbb{E}\left[\Re\mathscr{F}_T(Y)(\omega_1)\Im\mathscr{F}_T(Y)(\omega_2)\right].$$

All entries $\Sigma_{i,j}$ of the matrix Σ are of one of the above forms. Indeed, Σ_{11}, Σ_{33} are of the form Σ_1 for $k = l$ and $k,l \in \{1,2\}$. Similarly, Σ_{22}, Σ_{44} are of the form Σ_2 for $k = l$ and $k,l \in \{1,2\}$. Moreover, Σ_{13}, Σ_{31} are of the form Σ_1 for $k \neq l$ and $k,l \in \{1,2\}$ and Σ_{24}, Σ_{42} are of the form Σ_2 for $k \neq l$ and $k,l \in \{1,2\}$. All other elements are of the form Σ_3.

Observe that for each ω we have

$$\Re\mathscr{F}_T(Y)(\omega) = \frac{\mathscr{F}_T(Y)(\omega) + \mathscr{F}_T(Y)(-\omega)}{2},$$

$$\Im\mathscr{F}_T(Y)(\omega) = \frac{\mathscr{F}_T(Y)(\omega) - \mathscr{F}_T(Y)(-\omega)}{2i}.$$

Using Theorem S1.1 we obtain

$$\Sigma_1(\omega_1,\omega_2) := \begin{cases} \sigma^2 \frac{|b(0)|^2}{|a(0)|^2} + \frac{1}{T}K(0), & \omega_1 = \omega_2 = 0; \\ \frac{1}{2}\sigma^2 \frac{|b(i\omega_1)|^2}{|a(i\omega_1)|^2} + \frac{1}{T}K_{1,1}(\omega_1), & \omega_1 = \omega_2; \\ \frac{1}{2}\sigma^2 \frac{|b(i\omega_1)|^2}{|a(i\omega_1)|^2} + \frac{1}{T}K_{1,2}(\omega_1), & \omega_1 = -\omega_2; \\ \frac{1}{T}K_{1,3}(\omega_1,\omega_2), & \omega_1 \neq \omega_2 \text{ and } \omega_1 \neq -\omega_2, \end{cases}$$

$$\Sigma_2(\omega_1,\omega_2) := \begin{cases} 0, & \omega_1 = 0 \text{ or } \omega_2 = 0; \\ \frac{1}{2}\sigma^2 \frac{|b(i\omega_1)|^2}{|a(i\omega_1)|^2} + \frac{1}{T}K_{2,1}(\omega_1), & \omega_1 = \omega_2; \\ -\frac{1}{2}\sigma^2 \frac{|b(i\omega_1)|^2}{|a(i\omega_1)|^2} - \frac{1}{T}K_{2,2}(\omega_1), & \omega_1 = -\omega_2; \\ \frac{1}{T}K_{2,3}(\omega_1,\omega_2), & \omega_1 \neq \omega_2 \text{ and } \omega_1 \neq -\omega_2, \end{cases}$$

$$\Sigma_3(\omega_1,\omega_2) := \begin{cases} 0, & \omega_2 = 0; \\ \frac{1}{T}K_{3,1}(\omega_1), & \omega_1 = \omega_2 \text{ or } \omega_1 = -\omega_2; \\ \frac{1}{T}K_{3,2}(\omega_1,\omega_2), & \omega_1 \neq \omega_2 \text{ and } \omega_1 \neq -\omega_2. \end{cases}$$

Here K is given by (S1.10) and $K_{i,j}$ are bounded in T for $i,j = 1,2,3$. □

S1.3 Proof of Lemma 3.2

Proof. Observe that

$$\begin{aligned}|\tilde{Z}(T)| &= \left| \frac{1}{\sqrt{T}} \mathbf{b}^T (i\omega I - A)^{-1} \left[\mathbf{X}(0) - e^{-i\omega T} \mathbf{X}(T) \right] \right| \\ &\leq \frac{1}{\sqrt{T}} \left| \mathbf{b}^T (i\omega I - A)^{-1} \mathbf{X}(0) \right| + \frac{1}{\sqrt{T}} \left| \mathbf{b}^T (i\omega I - A)^{-1} \mathbf{X}(T) \right|.\end{aligned}$$

Obviously,

$$\lim_{T\to\infty} \frac{1}{\sqrt{T}} \left| \mathbf{b}^T (i\omega I - A)^{-1} \mathbf{X}(0) \right| = 0 \quad a.s. \text{ as } T \to \infty.$$

Because of stationarity, $\mathbf{X}(T)$ is bounded in probability and $\frac{1}{\sqrt{T}}$ converges to zero thus

$$\frac{1}{\sqrt{T}} \left| \mathbf{b}^T (i\omega I - A)^{-1} \mathbf{X}(T) \right| \to 0 \text{ in probability.}$$

Therefore

$$\mathbb{P} - \lim_{T\to\infty} |\tilde{Z}(T)| = 0.$$

This completes the proof. □

S1.4 Proof of Theorem 3.3

Proof. Observe that $\int_0^T dL(t) = L(T)$, thus $Z(T) = \frac{1}{\sqrt{T}} \frac{b(0)}{a(0)} L(T)$. By the standard Central Limit Theorem $d - \lim_{T\to\infty} \frac{1}{\sqrt{T}} L(T) = \mathcal{N}(0, \sigma^2)$. Therefore $d - \lim_{T\to\infty} \frac{1}{\sqrt{T}} \frac{b(0)}{a(0)} L(T) = \mathcal{N}\left(0, \left(\frac{b(0)}{a(0)}\right)^2 \sigma^2\right)$.

Observe that for all $n \in \mathbb{N}$ the random variable $\frac{a(0)Z(n)}{b(0)\sigma} \sim \mathcal{N}(0,1)$. Then by the continuous mapping theorem we have $d - \lim_{T\to\infty} \frac{1}{\sigma^2} \left| \frac{a(0)Z(T)}{b(0)} \right|^2 \sim \chi^2(1)$. □

S1.5 Proof of Theorem 3.4

Proof. We first show that $\frac{1}{\sqrt{N}}\int_0^{\frac{2\pi N}{\omega}} e^{-i\omega t}dL(t)$ is asymptotically normal. For $N \in \mathbb{N}$ and $j \in \{0,\dots,N-1\}$ put

$$X_j := \begin{bmatrix} X_j^1 \\ X_j^2 \end{bmatrix} := \begin{bmatrix} \int_{2\pi j/\omega}^{2\pi(j+1)/\omega} \cos(\omega t)dL(t) \\ \int_{2\pi j/\omega}^{2\pi(j+1)/\omega} \sin(\omega t)dL(t) \end{bmatrix}.$$

Observe that X_j are independent and identically distributed random vectors with mean zero and the covariance matrix $\widetilde{\Sigma_1} := \frac{\sigma^2\pi}{\omega} I_{2\times 2}$. Therefore,

$$\int_0^{\frac{2\pi N}{\omega}} e^{-i\omega t}dL(t) = \sum_{j=0}^{N-1} X_j.$$

Applying the classical CLT we obtain

$$\sqrt{N}\left(\frac{1}{N}\sum_{j=0}^{N-1} X_j\right) = \frac{1}{\sqrt{N}}\int_0^{\frac{2\pi N}{\omega}} e^{-i\omega t}dL(t) \to \mathscr{N} \sim \mathscr{N}(0,\widetilde{\Sigma_1}) \quad \text{as} \quad N \to \infty.$$

So $\frac{\sqrt{\omega}}{\sqrt{2\pi N}}\int_0^{\frac{2\pi N}{\omega}} e^{-i\omega t}dL(t) \to \mathscr{N}(0,\Sigma_1)$, where $\Sigma_1 = \frac{\omega}{2\pi}\widetilde{\Sigma_1} = \frac{\sigma^2}{2} I_{2\times 2}$. Put

$$A := \begin{bmatrix} \Re\left(\frac{b(i\omega)}{a(i\omega)}\right) & \Im\left(\frac{b(i\omega)}{a(i\omega)}\right) \\ \Im\left(\frac{b(i\omega)}{a(i\omega)}\right) & -\Re\left(\frac{b(i\omega)}{a(i\omega)}\right) \end{bmatrix}.$$

Observe that

$$A \begin{bmatrix} \frac{\sqrt{\omega}}{\sqrt{2\pi N}} \int_0^{2\pi N/\omega} \cos(\omega t)dL(t) \\ \frac{\sqrt{\omega}}{\sqrt{2\pi N}} \int_0^{2\pi N/\omega} \sin(\omega t)dL(t) \end{bmatrix} = \begin{bmatrix} \Re Z(2\pi N) \\ \Im Z(2\pi N) \end{bmatrix}.$$

Thus $Z = AX$ is normally distributed with mean zero and the covariance matrix $\Sigma = A\Sigma_1 A^T = \frac{\sigma^2}{2}\left|\frac{b(i\omega)}{a(i\omega)}\right|^2 I_{2\times 2}$. □

S1.6 Proof of Theorem 3.5

PROOF We use the notation of the proof of Theorem 3.4. Thus $|Z|^2$ is proportional to chi-square random variables with two degrees of freedom, i.e. $|Z|^2 = \frac{\sigma^2}{2}\left|\frac{b(i\omega)}{a(i\omega)}\right|^2 X$, where $X \sim \chi^2(2)$.

Thus $|Z|^2 \sim \Gamma\left(1, \frac{\sigma^2}{2}\left|\frac{b(i\omega)}{a(i\omega)}\right|^2\right)$ so $|Z|^2 \sim \operatorname{Exp}\left(\sigma^2\left|\frac{b(i\omega)}{a(i\omega)}\right|^2\right)$. □

S1.7 Proof of Theorem 3.6

PROOF For fixed $n \in \mathbb{N}$ and $k = 1, \ldots, n$, put

$$X_k^{(2i-1)}(\omega_i) := \int_{2(k-1)\pi}^{2k\pi} \cos(\omega_i t)dL(t),$$

$$X_k^{(2i)}(\omega_i) := \int_{2(k-1)\pi}^{2k\pi} \sin(\omega_i t)dL(t), \quad i = 1, \ldots, d.$$

Let $\left(s_n^{(2i-1)}\right)^2 = \sum_{k=1}^n \operatorname{Var}\left[X_k^{(2i-1)}(\omega_i)\right]$ and $\left(s_n^{(2i)}\right)^2 = \sum_{k=1}^n \operatorname{Var}\left[X_k^{(2i)}(\omega_i)\right]$. Put

$$Z_n^{(2i-1)}(\omega_i) := \frac{\sum_{k=1}^n X_k^{(2i-1)}(\omega_i)}{s_n^{(2i-1)}}, \quad Z_n^{(2i)}(\omega_i) := \frac{\sum_{k=1}^n X_k^{(2i)}(\omega_i)}{s_n^{(2i)}}.$$

Then we will show that by the Cramer-Wold-device the random vector $\mathbf{Z} \in \mathbb{R}^{2d}$ with $\mathbf{Z} = \left[Z_n^{(2i-1)}(\omega_i), Z_n^{(2i)}(\omega_i)\right]_{i=1,\ldots,d}^T$ converges to $\mathcal{N}(0, I_{2d\times 2d})$ in distribution.

We first apply the Lindeberg-Feller Central Limit Theorem (see Billingsley (1995), for instance) to each coordinate of the vector $\mathbf{Z}$. Observe that for all

$i = 1, \dots, d$ by the Itô isometry we obtain

$$\begin{aligned}\operatorname{Var}\left(X_k^{(2i-1)}(\omega_i)\right) &= \operatorname{Var}\left(\int_{2(k-1)\pi}^{2k\pi} \cos(\omega_i t) dL(t)\right) = \sigma^2 \int_{2(k-1)\pi}^{2k\pi} \cos^2(\omega_i t) dt \\ &= \sigma^2 \frac{4\pi\omega_i + \sin(4\pi\omega_i k) - \sin(4\pi\omega_i (k-1))}{4\omega_i}.\end{aligned}$$

Thus

$$\left(s_n^{(2i-1)}\right)^2 = \sum_{k=1}^{n} \operatorname{Var}\left[X_k^{(2i-1)}(\omega_i)\right] = \sigma^2 \frac{4n\pi\omega_i + \sin(4\pi\omega_i n)}{4\omega_i}.$$

In the same way,

$$\left(s_n^{(2i)}\right)^2 = \sum_{k=1}^{n} \operatorname{Var}\left[X_k^{(2i)}(\omega_i)\right] = \sigma^2 \frac{4n\pi\omega_i - \sin(4\pi\omega_i n)}{4\omega_i}.$$

Observe that

$$\lim_{n\to\infty} \frac{1}{n} \left(s_n^{(2i-1)}\right)^2 = \lim_{n\to\infty} \frac{1}{n} \left(s_n^{(2i)}\right)^2 = \sigma^2 \pi.$$

If the Lindeberg condition is satisfied, the $2i$-th, respectively $2i-1$-th coordinate of $\mathbf{Z}$ for $i = 1, \dots, d$, i.e.

$$\begin{aligned}Z_n^{(2i-1)}(\omega_i) &= \frac{2\sqrt{\omega_i}}{\sigma\sqrt{4\pi n\omega_i + \sin(4\pi n\omega_i)}} \int_0^{2\pi n} \cos(\omega_i t) dL(t) \\ Z_n^{(2i)}(\omega_i) &= \frac{2\sqrt{\omega_i}}{\sigma\sqrt{4\pi n\omega_i - \sin(4\pi n\omega_i)}} \int_0^{2\pi n} \sin(\omega_i t) dL(t)\end{aligned}$$

converges to $\mathcal{N}(0,1)$. Taking

$$Y_n^{(2i-1)}(\omega_i) = \frac{\sigma\sqrt{4\pi n\omega_i + \sin(4\pi n\omega_i)}}{2\sqrt{2\pi n\omega_i}},$$

$$Y_n^{(2i)}(\omega_i) = \frac{\sigma\sqrt{4\pi n\omega_i - \sin(4\pi n\omega_i)}}{2\sqrt{2\pi n\omega_i}}$$

and noting that

$$\lim_{n\to\infty} Y_n^{(2i-1)}(\omega_i) = \frac{\sigma}{\sqrt{2}}, \quad \lim_{n\to\infty} Y_n^{(2i)}(\omega_i) = \frac{\sigma}{\sqrt{2}}$$

is constant at all frequencies, by Slutsky arguments for $i = 1,\dots,d$ we get

$$\frac{1}{\sqrt{2\pi n}} \int_0^{2\pi n} \cos(\omega_i t) dL(t) = Z_n^{(2i-1)}(\omega_i) Y_n^{(2i-1)}(\omega_i) \to \mathcal{N}\left(0, \frac{\sigma^2}{2}\right), \quad \text{(S1.11)}$$

$$\frac{1}{\sqrt{2\pi n}} \int_0^{2\pi n} \sin(\omega_i t) dL(t) = Z_n^{(2i)}(\omega_i) Y_n^{(2i)}(\omega_i) \to \mathcal{N}\left(0, \frac{\sigma^2}{2}\right). \quad \text{(S1.12)}$$

Now we are going to prove the Lindeberg condition for odd coordinates of **Z** (for the even ones an analogous reasoning holds), i.e. for all $\varepsilon > 0$ it holds

$$\lim_{n\to\infty} \frac{1}{\left(s_n^{(2i-1)}\right)^2} \sum_{k=1}^n \mathbb{E}\left[\left(X_k^{(2i-1)}(\omega_i)\right)^2 \mathbb{1}_{\left\{|X_k^{(2i-1)}(\omega_i)| > \varepsilon s_n^{(2i-1)}\right\}}\right] = 0.$$

Observe that if the random variables $\{X_k^{(2i-1)}(\omega_i)\}$ are uniformly square integrable, then they satisfy the Lindeberg condition. Indeed,

$$\begin{aligned}
&\left(s_n^{(2i-1)}\right)^{-2} \sum_{k=1}^n \mathbb{E}\left[\left(X_k^{(2i-1)}(\omega_i)\right)^2 \mathbb{1}_{\left\{\left|X_k^{(2i-1)}(\omega_i)\right| > \varepsilon s_n^{(2i-1)}\right\}}\right] \\
&= \left(s_n^{(2i-1)}\right)^{-2} \sum_{k=1}^n \mathbb{E}\left[\left(X_k^{(2i-1)}(\omega_i)\right)^2 \mathbb{1}_{\left\{\left|X_k^{(2i-1)}(\omega_i)\right|^2 > \left(\varepsilon s_n^{(2i-1)}\right)^2\right\}}\right] \\
&\le \left(s_n^{(2i-1)}\right)^{-2} n \sup_{k=1,\dots,n} \mathbb{E}\left[\left(X_k^{(2i-1)}(\omega_i)\right)^2 \mathbb{1}_{\left\{\left|X_k^{(2i-1)}(\omega_i)\right|^2 > \left(\varepsilon s_n^{(2i-1)}\right)^2\right\}}\right] \\
&= \left(\sigma^2 \frac{4n\pi\omega_i + \sin(4\pi\omega_i n)}{4\omega_i}\right)^{-1} n \\
&\quad \times \sup_{k=1,\dots,n} \mathbb{E}\left[\left(X_k^{(2i-1)}(\omega_i)\right)^2 \mathbb{1}_{\left\{\left|X_k^{(2i-1)}(\omega_i)\right|^2 > \left(\varepsilon s_n^{(2i-1)}\right)^2\right\}}\right]
\end{aligned}$$

Since $\lim_{n\to\infty}\left(\sigma^2\frac{4n\pi\omega_i+\sin(4\pi\omega_i n)}{4\omega_i}\right)^{-1}n \to \frac{1}{\pi\sigma^2}$, uniform square integrability implies in this case the Lindeberg condition. It remains to show the uniform square integrability of $\left\{X_k^{(2i-1)}(\omega_i)\right\}_{k\in\mathbb{N}}$.

Assume first, that our driving process $(L(t))_{t\geq 0}$ is of bounded variation. Then

$$M_k = \left|\int_{2(k-1)\pi}^{2k\pi}\cos(\omega_i t)dL_t\right| \leq \int_{2(k-1)\pi}^{2k\pi}|\cos(\omega_i t)|d|L_t| \leq \int_{2(k-1)\pi}^{2k\pi}d|L_t|,$$

where $|\cdot|$ denotes the total variation of the process. It is clear that $\int_{2(k-1)\pi}^{2k\pi}d|L_t| \overset{d}{=} \int_0^{2\pi}d|L_t|$. We have

$$\begin{aligned}\mathbb{E}\left[|M_k|^2\mathbb{1}_{\{|M_k|>K\}}\right] &\leq \mathbb{E}\left[\left|\int_{2(k-1)\pi}^{2k\pi}d|L_t|\right|^2\mathbb{1}_{\left\{\left|\int_{2(k-1)\pi}^{2k\pi}d|L_t|\right|>K\right\}}\right]\\ &= \mathbb{E}\left[\left|\int_0^{2\pi}d|L_t|\right|^2\mathbb{1}_{\left\{\left|\int_0^{2\pi}d|L_t|\right|>K\right\}}\right].\end{aligned}$$

By the square integrability of $\int_0^{2\pi}d|L_t|$, which is implied by the square integrability of $(L(t))_{t\geq 0}$ we obtain the uniform integrability of $(M_k)_{k\in\mathbb{N}}$.

Now we assume that $(L(t))$ is a square integrable martingale with finite moments of all orders. Observe that X_k is square integrable for all $k\in\mathbb{N}$. By the Burkholder-Davis-Gundy Inequality (see e.g. Protter (2004)) for each $p\geq 1$ there exists a positive constant C_p such that

$$\begin{aligned}&\mathbb{E}\left[\left(\int_{2(k-1)\pi}^{2k\pi}\cos(\omega_i t)dL(t)\right)^p\right]\\ &\quad\leq C_p\mathbb{E}\left[\left[\int_{2(k-1)\pi}^{2k\pi}\cos(\omega_i t)dL(t),\int_{2(k-1)\pi}^{2k\pi}\cos(\omega_i t)dL(t)\right]^{p/2}\right].\end{aligned}$$

Since

$$\left[\int_{2(k-1)\pi}^{2k\pi}\cos(\omega_i t)dL(t),\int_{2(k-1)\pi}^{2k\pi}\cos(\omega_i t)dL(t)\right]=\int_{2(k-1)\pi}^{2k\pi}\cos^2(\omega_i t)d[L,L]_t,$$

using the above inequality for $p=4$ we obtain

$$\begin{aligned}\mathbb{E}\left[\left(\int_{2(k-1)\pi}^{2k\pi}\cos(\omega_i t)dL(t)\right)^4\right]&\le C_4\mathbb{E}\left[\int_{2(k-1)\pi}^{2k\pi}\cos^2(\omega_i t)d[L,L]_t\right]\\&\le\sigma^2\int_{2(k-1)\pi}^{2k\pi}\cos^2(\omega_i t)dt<C\end{aligned}$$

for some constant C. Since $\{X_k^{(2i-1)}(\omega_i)\}$ are square integrable and bounded in $L^4(\Omega,\mathscr{F},\mathbb{P})$, they are uniformly square integrable.

As any Lévy process is by the Lévy-Itô decomposition the sum of a finite variation Lévy process and an independent square integrable martingale with moments of all orders, we obtain the claimed uniform squre integrability for all driving Lévy processes.

Likewise one shows that $\theta^T Z$ converges in distribution to $\mathscr{N}\left(0,\frac{\sigma^2}{2}\theta^T\theta\right)$ for all $\theta\in\mathbb{R}^{2d}$. So the Cramer-Wold device concludes.

Hence, $\left[Z_n^{(2i-1)}(\omega_i)Y_n^{(2i-1)}(\omega_i),Z_n^{(2i)}(\omega_i)Y_n^{(2i)}(\omega_i)\right]_{i=1,\ldots,d}^T$ converges in distribution to $\mathscr{N}\left(0,\frac{\sigma^2}{2}I_{2d\times 2d}\right)$ and thus using Lemma 3.2 and equations (S1.11), (S1.12) $\left[\mathscr{F}_T(\omega_j)\right]_{j=1,\ldots,d}^T$ converges to $\mathscr{N}\left(0,\frac{\sigma^2}{2}\mathbf{B}\right)$, where $\mathbf{B}$ is defined above. Repeating the reasoning from the proof of Theorem 3.5 we obtain that the vector $\left[|\mathscr{F}_T(\omega_j)|^2\right]_{j=1,\ldots,d}^T$ converges to a vector of independent, exponentially distributed random variables with $\mathrm{Exp}\left(\sigma^2\left|\frac{b(i\omega_j)}{a(i\omega_j)}\right|^2\right)$ for $j=1,\ldots,d$. $\square$

S1.8 Proof of Theorem 3.7

We begin by establishing an error bound of the trapezoidal method for non-equidistant data. For a very accessible presentation of quadrature rules we refer to Talvila and Wiersma (2012). Recall the basic properties of the trapezoidal rule:

Lemma S1.3. *Let $f\colon [a,b] \to \mathbb{R}$ be a twice continuously differentiable function. Write*

$$\int_a^b f(x)dx = \frac{b-a}{2}[f(a)+f(b)] + E^T(f). \tag{S1.13}$$

Then

$$|E^T(f)| \leq \frac{(b-a)^3}{12} \sup_{x\in[a,b]} |f''(x)|. \tag{S1.14}$$

For the composite trapezoidal rule for an equidistant grid $a < a+(b-a)\frac{1}{n} < \dots, a+(b-a)\frac{i}{n} < \dots < b$ we have

$$\int_a^b f(x)dx = \frac{b-a}{2n}\left[f(a)+2\sum_{i=1}^{n-1} f\left(a+(b-a)\frac{i}{n}\right)+f(b)\right] + E_n^T(f). \tag{S1.15}$$

Then

$$|E_n^T(f)| \leq \frac{(b-a)^3}{12n^2} \sup_{x\in[a,b]} |f''(x)|. \tag{S1.16}$$

A proof can be found e.g. in Talvila and Wiersma (2012).

Now we are going to formulate a version of the trapezoidal rule for non-equidistant points. We assume that we have some control on the maximal distance between observations.

Lemma S1.4. *Let $a = x_0 < x_1 < \cdots < x_{N-1} < x_N = b$ be an arbitrary partition of the interval $[a,b]$ and assume that $f\colon [a,b] \to \mathbb{R}$ is a twice continuously differentiable function. Put $h_{\max} = \max_{j=0,\ldots,N-2}(x_{j+1} - x_j)$. Then*

$$\int_a^b f(x)dx = \sum_{i=0}^{N-1} \frac{x_{j+1} - x_j}{2}[f(x_j) + f(x_{j+1})] + E^T(f),$$

where $|E^T(f)| \leq N\|f''\|_\infty \frac{h_{\max}^3}{12}$.

PROOF

Let us write

$$[a,b] = \bigcup_{j=0}^{N-1} [x_j, x_{j+1}], \quad I_j := [x_j, x_{j+1}]$$

and apply Lemma S1.3 for each interval I_j. Therefore

$$\int_{x_j}^{x_{j+1}} f(x)dx = \frac{x_{j+1} - x_j}{2}[f(x_j) + f(x_{j+1})] + E_j^T(f),$$

with

$$|E_j^T(f)| \leq \frac{|x_{j+1} - x_j|^3}{12} \sup_{x\in[x_j,x_{j+1}]} |f''(x)|.$$

For each $i = 0, 1, \ldots, N-1$ we have

$$\sup_{x\in[x_j,x_{j+1}]} |f''(x)| \leq \sup_{x\in[a,b]} |f''(x)| =: \|f''\|_\infty.$$

Therefore

$$\int_a^b f(x)dx = \sum_{i=0}^{N-1} \frac{x_{j+1} - x_j}{2}[f(x_j) + f(x_{j+1})] + E^T(f),$$

where

$$|E^T(f)| = \left|\sum_{i=0}^{N-1} E_j^T(f)\right| \leq \|f''\|_\infty \sum_{i=0}^{N-1} \frac{(x_{j+1}-x_j)^3}{12}$$
$$\leq \|f''\|_\infty \sum_{i=0}^{N-1} \frac{h_{\max}^3}{12} = N\|f''\|_\infty \frac{h_{\max}^3}{12}.$$

This completes the proof. □

We use some results and ideas from Brockwell and Schlemm (2013). The aim is to find an approximation similar to Proposition 5.4 of Brockwell and Schlemm (2013) of the integral appearing in the truncated Fourier transform in the case that the observations of the process Y are given on a non-equidistant grid. Let

$$T_{[0,T]}^N f = \sum_{j=0}^{N-1} \frac{x_{j+1}-x_j}{2}[f(x_j)+f(x_{j+1})]$$

be the trapezoidal rule discussed in Lemma S1.4. Recall first the Fubini type theorem for stochastic integrals from Brockwell and Schlemm (2013).

Lemma S1.5. *(Brockwell and Schlemm, 2013, Theorem 2.4) Let $[a,b] \subset \mathbb{R}$ be a bounded interval and $(L(t))_{t\geq 0}$ be a Lévy process with finite second moments. Assume that $F\colon [a,b]\times\mathbb{R}\to\mathbb{R}^d$ is a bounded function $\mathscr{B}([a,b])\otimes\mathscr{B}([-s,t])$-measurable for all $s,t\in(0,\infty)$ and the family $\{u\mapsto F(s,u)\}_{u\in[a,b]}$ is uniformly absolutely integrable and uniformly converges to zero as $|u|\to 0$. Then*

$$\int_a^b \int_{\mathbb{R}} F(s,u)dL(u)ds = \int_{\mathbb{R}} \int_a^b F(s,u)dsdL(u) \quad a.s. \tag{S1.17}$$

In the paper Brockwell and Schlemm (2013) the assumption about measurability in the statement of the theorem is not explicitely stated. However, an inspection of their proof combined with results from Veraar (2012) shows that the precise statement has to be in the above form.

Secondly, note that for non-equidistant data the corresponding error estimation (Brockwell and Schlemm, 2013, Proposition A.6) has the following form:

Proposition S1.6. *Let $[a,b] \subset \mathbb{R}$ be a compact interval and use the notation of Lemma S1.4.*

1. *If $f\colon [a,b] \to \mathbb{R}$ is twice continuously differentiable, then*

$$\left| \int_a^b f(s)ds - T^N_{[a,b]}f \right| \leq N \|f''\|_\infty \frac{h^3_{\max}}{12}.$$

2. *If $F\colon [a,b] \to \mathbb{R}^d$ is twice continuously differentiable, then*

$$\left\| \int_a^b F(s)ds - T^N_{[a,b]}F \right\| \leq \sqrt{d} N \|F''\|_\infty \frac{h^3_{\max}}{12},$$

where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^d$.

Here $\|F''\|_\infty := \max_{i=1,\dots,d} \sup_{t_i \in [a,b]} \|F''(t_i)\|$.

Put

$$E^{T,N}_{fg} := T^N_{[0,T]} f(\cdot)g(\cdot) - \int_0^T g(s)f(s)ds.$$

PROOF OF THEOREM 4.7. Assume that we have observed the process Y on the grid $0 = x_0^{(T)} < x_1^{(T)} < \cdots < x_{N(T)-1}^{(T)} = T$. We have

$$T_{[0,T]}^{N} FY = \sum_{j=0}^{N(T)-1} \alpha_j^{(N(T))} Y(x_j^{(T)}), \tag{S1.18}$$

where $\alpha_j^{(N(T))}$ $(j = 0, \ldots, N(T)-1)$ are the coefficients given by (3.12) and (3.13).

Observe that $f(a) = \int f(s)\delta_a(s)ds$. Moreover, for all $j = 0, \ldots, N(T)-1$ we know that $x_j^{(T)} \in [0,T]$, therefore for all $u \in [0,T]$ and for all $j = 0, \ldots, N(T)-1$ we have

$$\mathbb{1}_{[u,T]}\left(x_j^{(T)}\right) = \mathbb{1}_{\left[0,x_j^{(T)}\right]}(u).$$

Thus by (S1.18) we have

$$\begin{aligned}
T_{[0,T]}^{N} FY &= \sum_{j=0}^{N(T)-1} \alpha_j^{(N(T))} Y\left(x_j^{(T)}\right) = \sum_{j=0}^{N(T)-1} \alpha_j^{(N(T))} \int_{-\infty}^{x_j^{(T)}} \mathbf{b}^T e^{\mathbf{A}\left(x_j^{(T)}-u\right)} \mathbf{e} dL^*(u) \\
&= \sum_{j=0}^{N(T)-1} \alpha_j^{(N(T))} \left(\int_{-\infty}^{0} \mathbf{b}^T e^{\mathbf{A}\left(x_j^{(T)}-u\right)} \mathbf{e} dL^*(u) \right. \\
&\quad \left. + \int_{0}^{x_j^{(T)}} \mathbf{b}^T e^{\mathbf{A}\left(x_j^{(T)}-u\right)} \mathbf{e} dL^*(u) \right) \\
&= \int_{-\infty}^{0} \sum_{j=0}^{N(T)-1} \alpha_j^{(N(T))} \mathbf{b}^T e^{\mathbf{A}\left(x_j^{(T)}-u\right)} \mathbf{e} dL^*(u) \\
&\quad + \int_{0}^{T} \sum_{j=0}^{N(T)-1} \alpha_j^{(N(T))} \mathbf{b}^T e^{\mathbf{A}\left(x_j^{(T)}-u\right)} \mathbf{e} \mathbb{1}_{\left[0,x_j^{(T)}\right]}(u) dL^*(u) \\
&= \int_{-\infty}^{0} \sum_{j=0}^{N(T)-1} \alpha_j^{(N(T))} \mathbf{b}^T e^{\mathbf{A}\left(x_j^{(T)}-u\right)} \mathbf{e} dL^*(u) \\
&\quad + \int_{0}^{T} \sum_{j=0}^{N(T)-1} \alpha_j^{(N(T))} \mathbf{b}^T e^{\mathbf{A}\left(x_j^{(T)}-u\right)} \mathbf{e} \mathbb{1}_{[u,T]}\left(x_j^{(T)}\right) dL^*(u)
\end{aligned}$$

$$
\begin{aligned}
&= \int_{-\infty}^{0} \int_{0}^{T} \sum_{j=0}^{N(T)-1} \alpha_j^{(N(T))} \delta_{x_j^{(T)}}(s) \mathbf{b}^T e^{\mathbf{A}(s-u)} \mathbf{e} ds dL^*(u) \\
&\quad + \int_{0}^{T} \int_{u}^{T} \sum_{j=0}^{N(T)-1} \alpha_j^{(N(T))} \delta_{x_j^{(T)}}(s) \mathbf{b}^T e^{\mathbf{A}(s-u)} \mathbf{e} ds dL^*(u) \\
&= \int_{-\infty}^{T} \int_{\max\{0,u\}}^{T} \sum_{j=0}^{N(T)-1} \alpha_j^{(N(T))} \delta_{x_j^{(T)}}(s) \mathbf{b}^T e^{\mathbf{A}(s-u)} \mathbf{e} ds dL^*(u).
\end{aligned}
$$

Thus using the representation (2.6) and the Fubini-type Theorem S1.5 we have

$$
\begin{aligned}
\int_0^T F(s) Y(s) ds &= \int_0^T F(s) \int_{-\infty}^{s} \mathbf{b}^T e^{\mathbf{A}(s-u)} \mathbf{e} dL^*(u) ds \\
&= \int_{-\infty}^{T} \int_{\max\{0,u\}}^{T} F(s) \mathbf{b}^T e^{\mathbf{A}(s-u)} \mathbf{e} ds dL^*(u) \\
&= \int_{-\infty}^{0} \int_{0}^{T} F(s) \mathbf{b}^T e^{\mathbf{A}(s-u)} \mathbf{e} ds dL^*(u) \\
&\quad + \int_{0}^{T} \int_{u}^{T} F(s) \mathbf{b}^T e^{\mathbf{A}(s-u)} \mathbf{e} ds dL^*(u).
\end{aligned}
$$

Thus

$$
\begin{aligned}
E_{FY}^{T,N} &= T_{[0,T]}^{N} FY - \int_0^T F(s) Y(s) ds \\
&= \int_0^T \left(\sum_{j=0}^{N(T)-1} \alpha_j^{(N(T))} \delta_{x_j^{(T)}}(s) - F(s) \right) Y(s) ds \\
&= \int_{-\infty}^{0} \int_{0}^{T} \left(\sum_{j=0}^{N(T)-1} \alpha_j^{(N(T))} \delta_{x_j^{(T)}}(s) - F(s) \right) \mathbf{b}^T e^{\mathbf{A}(s-u)} \mathbf{e} ds dL^*(u) \\
&\quad + \int_{0}^{T} \int_{u}^{T} \left(\sum_{j=0}^{N(T)-1} \alpha_j^{(N(T))} \delta_{x_j^{(T)}}(s) - F(s) \right) \mathbf{b}^T e^{\mathbf{A}(s-u)} \mathbf{e} ds dL^*(u).
\end{aligned}
$$

Let us denote

$$\Gamma^{(N)}(u) := \int_0^T \left(\sum_{j=0}^{N(T)-1} \alpha_j^{(N(T))} \delta_{x_j^{(T)}}(s) - F(s) \right) \mathbf{b}^T e^{\mathbf{A}(s-u)} \mathbf{e} ds, \quad u \le 0, \tag{S1.19}$$

$$G^{(N)}(u) := \int_u^T \left(\sum_{j=0}^{N(T)-1} \alpha_j^{(N(T))} \delta_{x_j^{(T)}}(s) - F(s) \right) \mathbf{b}^T e^{\mathbf{A}(s-u)} \mathbf{e} ds, \quad u \in [0,T]. \tag{S1.20}$$

By Assumption 2.3 we know that there exist positive constants α, β such that

$$\|\exp(\mathbf{A}t)\| \le \beta \exp(-\alpha t). \tag{S1.21}$$

Note that by Lemma S1.4 and Proposition S1.6 we have

$$\begin{aligned} &\left\| \int_{u_0}^T \left(\sum_{j=0}^{N(T)-1} \alpha_j^{(N(T))} \delta_{x_j^{(T)}}(s) - F(s) \right) \mathbf{b}^T e^{\mathbf{A}(s-u)} \mathbf{e} ds \right\|_{\mathbb{R}^d} \\ &= \left\| \sum_{j=0}^{N(T)-1} \alpha_j^{(N(T))} \delta_{x_j^{(T)}}(s) \mathbf{b}^T e^{\mathbf{A}(x_j-u)} \mathbf{e} - \int_{u_0}^T F(s) \mathbf{b}^T e^{\mathbf{A}(s-u)} \mathbf{e} ds \right\|_{\mathbb{R}^d} \\ &\le \sqrt{d} N(T) \|\widetilde{F}''(u)\|_\infty \frac{h_{\max}^3(T)}{12} \end{aligned}$$

with $\widetilde{F}(s) = F(s)\mathbf{b}^T e^{\mathbf{A}(s-u)}\mathbf{e}$ and $u_0 \in [0,T]$. If $u_0 = 0$, then there exist $\widetilde{\alpha} > 0$ and $D > 0$ such that $\|\widetilde{F}''(u)\|_\infty \le D \exp(\widetilde{\alpha} u)$ for $u \le 0$. Therefore there exists a constant $D_1 > 0$ such that

$$\|\Gamma^{(N)}(u)\|_{\mathbb{R}^d} \le \sqrt{d} N(T) \|\widetilde{F}|''_{[0,T]}\|_\infty \frac{h_{\max}^3(T)}{12} \le D_1 N(T) h_{\max}^3(T) \exp(\widetilde{\alpha} u), u \le 0.$$

If now $u_0 = u$ is any element of $[0,T]$, then there exists $D > 0$ such that $\|\widetilde{F}''|_{[u,T]}\|_\infty \le D$ for $u \in [0,T]$. Therefore there exists a constant $D_2 > 0$ such that for $u \in [0,T]$ we have

$$\|G^{(N)}(u)\|_{\mathbb{R}^d} \leq \sqrt{d}N(T)\|\widetilde{F}|''_{[u,T]}\|_\infty \frac{h^3_{\max}(T)}{12} \leq D_2 N(T) h^3_{\max}(T).$$

By the Itô isometry

$$\begin{aligned}
\left\| \int_{-\infty}^0 \Gamma^{(N)}(u) dL^*(u) \right\|_{L^2}^2 &= \mathbb{E}\left[\left(\int_{-\infty}^0 \Gamma^{(N)}(u) dL^*(u) \right)^T \left(\int_{-\infty}^0 \Gamma^{(N)}(u) dL^*(u) \right) \right] \\
&= \mathbb{E}\left[\left(\int_{-\infty}^0 \Gamma^{(N)}(u) dL^*(u) \right)^T \left(\int_{-\infty}^0 \Gamma^{(N)}(u) dL^*(u) \right) \right] \\
&= \sigma^2 \left(\int_{-\infty}^0 [\Gamma^{(N)}(u)]^T \Gamma^{(N)}(u) du \right) \\
&\leq \sigma^2 \int_{-\infty}^0 \|\Gamma^{(N)}(u)\|^2_{\mathbb{R}^d} du \\
&\leq \sigma^2 \int_{-\infty}^0 \left(D_1 N(T) h^3_{\max}(T) \exp(\widetilde{\alpha} u) \right)^2 du \\
&= \sigma^2 N(T)^2 h^6_{\max}(T) D_1^2 \int_{-\infty}^0 \exp(2\widetilde{\alpha} u) du \\
&= D_\Gamma N(T)^2 h^6_{\max}(T),
\end{aligned}$$

where $D_\Gamma > 0$ is a constant. In a similar way we obtain

$$\begin{aligned}
\left\| \int_0^T G^{(N)}(u) dL(u) \right\|_{L^2}^2 &= \mathbb{E}\left[\left(\int_0^T G^{(N)}(u) dL(u) \right)^T \left(\int_0^T G^{(N)}(u) dL(u) \right) \right] \\
&= \mathbb{E}\left[\left(\int_0^T G^{(N)}(u) dL(u) \right)^T \left(\int_0^T G^{(N)}(u) dL(u) \right) \right] \\
&= \sigma^2 \left(\int_0^T [G^{(N)}(u)]^T G^{(N)}(u) du \right) \\
&\leq \sigma^2 \int_0^T \|G^{(N)}(u)\|^2_{\mathbb{R}^d} du \\
&\leq \sigma^2 \int_0^T \left(D_2 N(T) h^3_{\max}(T) \right)^2 du
\end{aligned}$$

$$= D_G N(T)^2 T h_{\max}^6(T)$$

for some constant $D_G > 0$. Therefore

$$\|E_{FY}^{T,N}\|_{L^2}^2 \leq 2\left[\left\|\int_{-\infty}^0 \Gamma^{(N)}(u)dL^*(u)\right\|_{L^2}^2 + \left\|\int_0^T G^{(N)}(u)dL(u)\right\|_{L^2}^2\right],$$

thus

$$\|E_{FY}^{T,N}\|_{L^2}^2 \leq C_1\left((C_2+T)N(T)^2 h_{\max}^6(T)\right)^2,$$

where C_1, C_2 are positive constants. If $\lim_{T\to\infty} TN(T)^2 h_{\max}^6(T) = 0$, then we have $\lim_{T\to\infty}\|E_{FY}^{T,N}\|_{L^2}^2 = 0$. This completes the proof. □

S1.9 Proof of Theorem 3.8

PROOF We identify $\mathbb{C}$ with $\mathbb{R}^2$ in the canonical way. Applying Theorem 3.7 for $d = 2$, $F(t) = [\cos(\omega t), -\sin(\omega t)]^T$ we get

$$\mathbb{E}\left[\left\|\sum_{j=0}^{N(T)-1} \alpha_j^{(N(T))} Y\left(x_j^{(T)}\right) - \int_0^T Y(t)\begin{bmatrix}\cos(\omega t)\\ -\sin(\omega t)\end{bmatrix} dt\right\|^2\right]$$
$$\leq C_1(C_2+T)N(T)^2 h_{\max}^6(T).$$

Dividing both sides by $T > 0$ we obtain

$$\mathbb{E}\left[\left\|\sum_{j=0}^{N(T)-1} \frac{\alpha_j^{(N(T))}}{\sqrt{T}} Y\left(x_j^{(T)}\right) - \frac{1}{\sqrt{T}}\int_0^T Y(t)\begin{bmatrix}\cos(\omega t)\\ -\sin(\omega t)\end{bmatrix} dt\right\|^2\right]$$

$$\leq \frac{C_1 C_2}{T} + C_1 N(T)^2 h_{\max}^6(T).$$

Passing to the limit with $T \to \infty$ and recalling $\lim_{T\to\infty} N(T) h_{\max}^3(T) = 0$ we get the assertion. □

S1.10 Proof of Theorem 3.9

PROOF. Put

$$Z(T) := \frac{1}{\sqrt{T}} \frac{b(i\omega)}{a(i\omega)} \int_0^T e^{-i\omega t} dL(t)$$

and consider the following two-dimensional random vectors:

$$\mathbf{Z}_n := \begin{bmatrix} \Re(Z(T)) \\ \Im(Z(T)) \end{bmatrix}, \quad \mathbf{U}_n := \begin{bmatrix} \Re(\mathscr{F}_T Y(\omega)) \\ \Im(\mathscr{F}_T Y(\omega)) \end{bmatrix}, \quad \mathbf{V}_n := \begin{bmatrix} \Re(\mathscr{T}_T Y(\omega)) \\ \Im(\mathscr{T}_T Y(\omega)) \end{bmatrix}.$$

Observe that it is enough to consider the above limits for $T = n$. By Lemma 3.2 we know that

$$\mathbb{P} - \lim_{n\to\infty} \|\mathbf{U}_n - \mathbf{Z}_n\| = \mathbf{0}.$$

From Theorem 3.4 we get

$$d - \lim_{n\to\infty} \mathbf{Z}_n = \mathscr{N}(0, \Sigma).$$

Therefore

$$d - \lim_{n\to\infty} \mathbf{U}_n = \mathscr{N}(0, \Sigma).$$

By Theorem 3.8 we have

$$\mathbb{P} - \lim_{n\to\infty} (\mathbf{V}_n - \mathbf{U}_n) = 0.$$

Therefore,

$$d - \lim_{n\to\infty} \mathbf{V}_n = \mathcal{N}(0,\Sigma).$$

In the same way we obtain

$$d - \lim_{n\to\infty} |\mathbf{Z}|^2 = \mathrm{Exp}\left(\sigma^2 \left|\frac{b(i\omega)}{a(i\omega)}\right|^2\right).$$

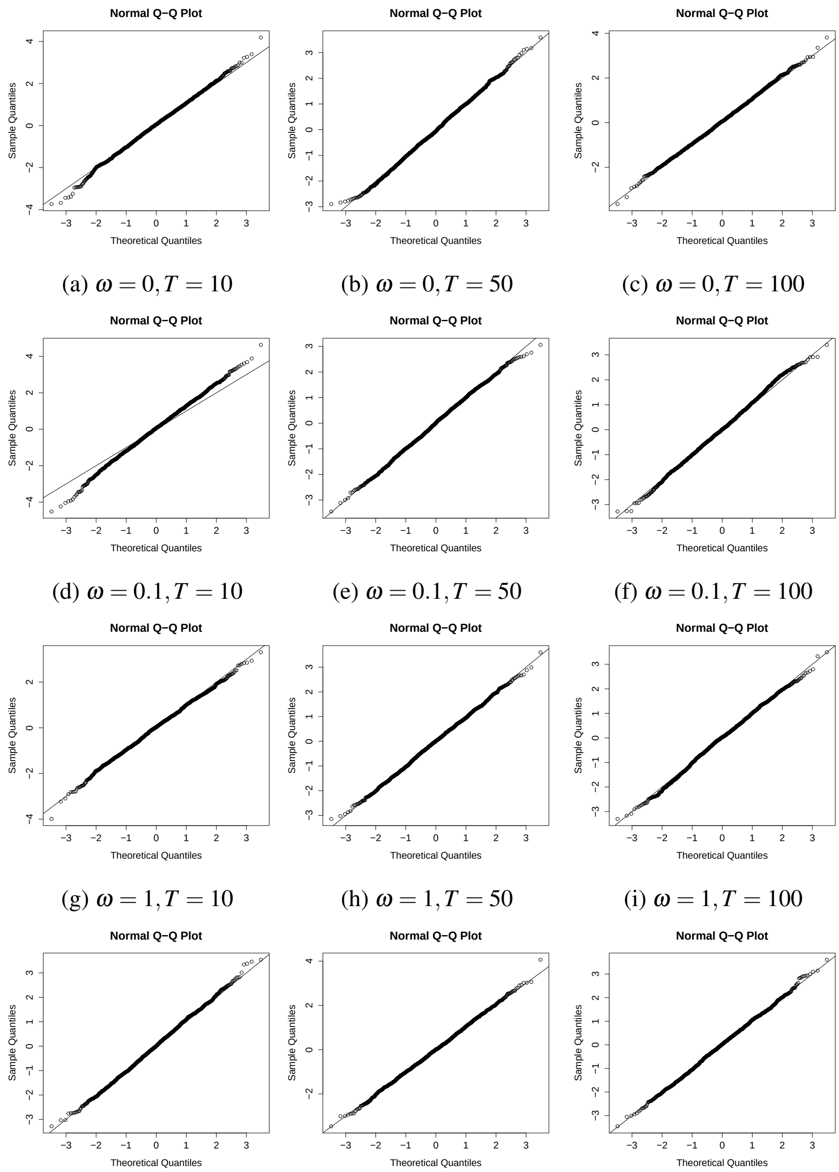
In order to obtain the assertion for $\omega = 0$ we repeat the above reasonings applying Theorem 3.3 instead of Theorem 3.4 □

S2 Supplementary Material for Section 4

S2.1 Plots for the CARMA(2,1) simulation study

This section presents the plots for the simulation study for CARMA(2,1) processes, i.e. Example 4.2. QQ-plots showing the results for 2000 simulated paths for the four different frequencies and three different combinations of time horizon and maximum grid width can be found in Figures S.1, S.3 and S.5 for the driving Lévy process being a standard Brownian motion, a Variance Gamma and a two-sided Poisson process, respectively. Likewise, Figures S.2, S.4 and S.6 show corresponding histograms.

So on top of the QQ plots we now also provide histograms in Figures S.2, S.4 and S.6, respectively, together with plots of the limiting normal density. To us it seems very hard to see the convergence to normality with increasing T in the histograms, which reflects the fact that it is essentially the tails which need to converge and they are much clearer visible in the QQ plots than in histograms. It is also not easy to see in them that for $\omega = 0.1, T = 10$ the variance of the simulated values is different from the asymptotic theoretical one. The only thing one notices is that for $\omega = 0.1, T = 10$ the histogram routine of R tends to use very different bins than in all the other cases. Note that all histograms were obtained using the default parameters of the `hist` function in R, so the binning was done by the standard automatic selection to give "nice" histograms.



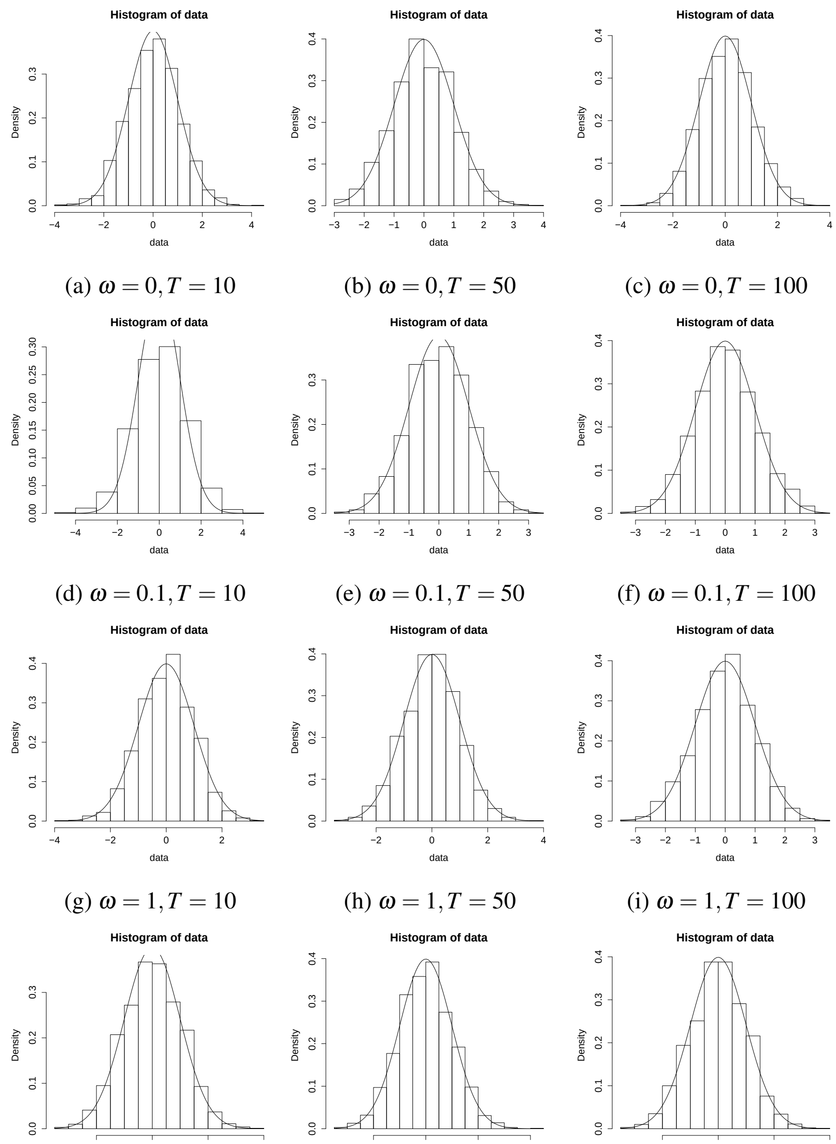
(a) $\omega = 0, T = 10$ (b) $\omega = 0, T = 50$ (c) $\omega = 0, T = 100$

(d) $\omega = 0.1, T = 10$ (e) $\omega = 0.1, T = 50$ (f) $\omega = 0.1, T = 100$

(g) $\omega = 1, T = 10$ (h) $\omega = 1, T = 50$ (i) $\omega = 1, T = 100$

(j) $\omega = 10, T = 10$ (k) $\omega = 10, T = 50$ (l) $\omega = 10, T = 100$

Figure S.1: Normal QQ plots for the real part of the truncated Fourier transform of the simulated CARMA(2,1) processes driven by standard Brownian Motion for the frequencies $0, 0.1, 1, 10$ (rows) and time horizons/maximum non-equidistant grid sizes $10/0.1, 50/0.05, 100/0.01$ (columns). The theoretical quantiles are coming from the (limiting) law described in Theorem 3.9.



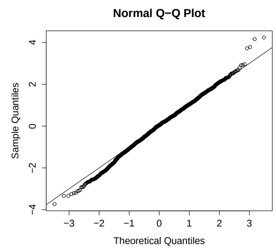
(a) $\omega = 0, T = 10$ (b) $\omega = 0, T = 50$ (c) $\omega = 0, T = 100$

(d) $\omega = 0.1, T = 10$ (e) $\omega = 0.1, T = 50$ (f) $\omega = 0.1, T = 100$

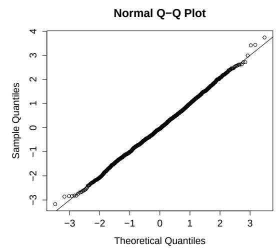
(g) $\omega = 1, T = 10$ (h) $\omega = 1, T = 50$ (i) $\omega = 1, T = 100$

(j) $\omega = 10, T = 10$ (k) $\omega = 10, T = 50$ (l) $\omega = 10, T = 100$

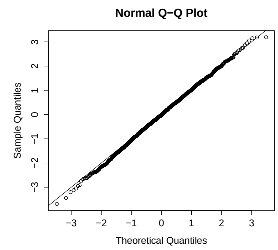
Figure S.2: Histograms and limiting density for the real part of the truncated Fourier transform of the simulated CARMA(2,1) processes driven by standard Brownian Motion for the frequencies $0, 0.1, 1, 10$ (rows) and time horizons/maximum non-equidistant grid sizes $10/0.1, 50/0.05, 100/0.01$ (columns)



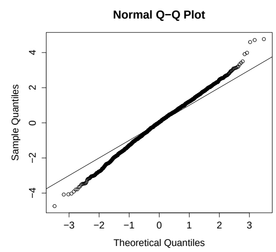
(a) $\omega = 0, T = 10$



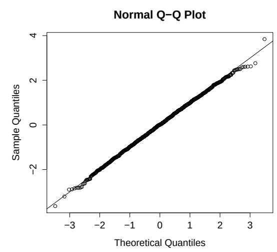
(b) $\omega = 0, T = 50$



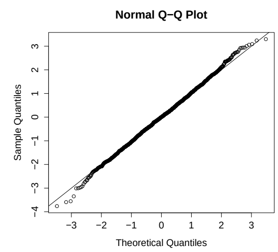
(c) $\omega = 0, T = 100$



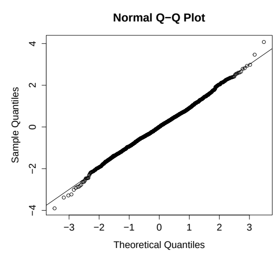
(d) $\omega = 0.1, T = 10$



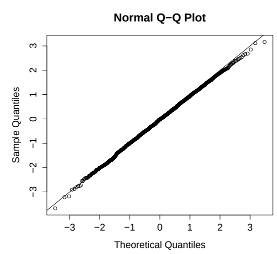
(e) $\omega = 0.1, T = 50$



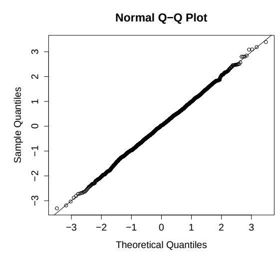
(f) $\omega = 0.1, T = 100$



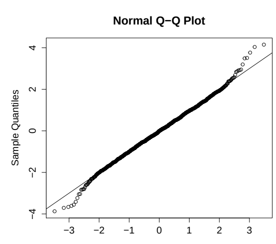
(g) $\omega = 1, T = 10$



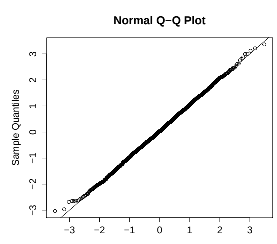
(h) $\omega = 1, T = 50$



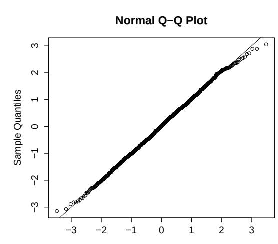
(i) $\omega = 1, T = 100$



(j) $\omega = 10, T = 10$

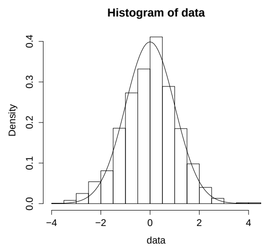


(k) $\omega = 10, T = 50$

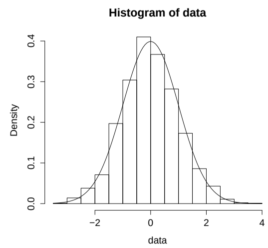


(l) $\omega = 10, T = 100$

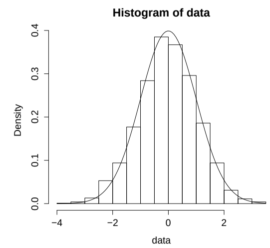
Figure S.3: Normal QQ plots for the real part of the truncated Fourier transform of the simulated CARMA(2,1) processes driven by a Variance Gamma process for the frequencies $0, 0.1, 1, 10$ (rows) and time horizons/maximum non-equidistant grid sizes $10/0.1, 50/0.05, 100/0.01$ (columns). The theoretical quantiles are coming from the (limiting) law described in Theorem 3.9.



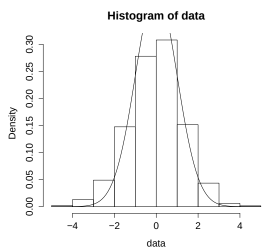
(a) $\omega = 0, T = 10$



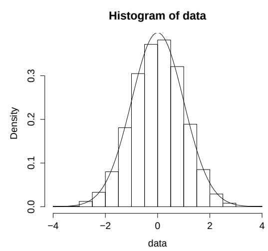
(b) $\omega = 0, T = 50$



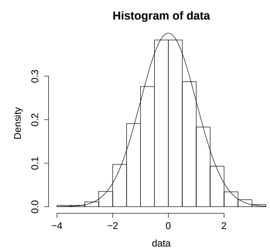
(c) $\omega = 0, T = 100$



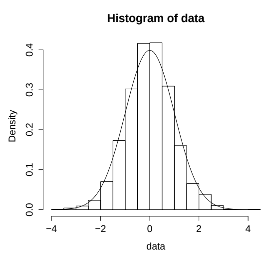
(d) $\omega = 0.1, T = 10$



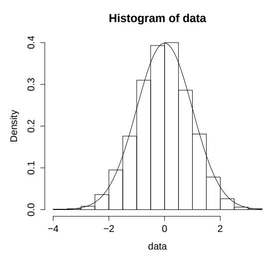
(e) $\omega = 0.1, T = 50$



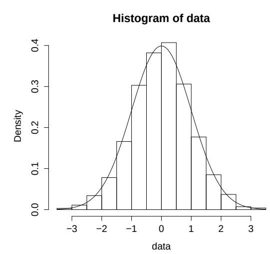
(f) $\omega = 0.1, T = 100$



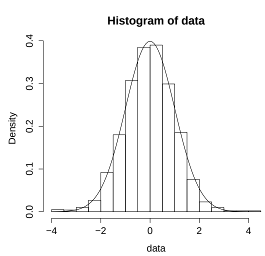
(g) $\omega = 1, T = 10$



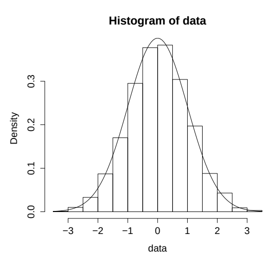
(h) $\omega = 1, T = 50$



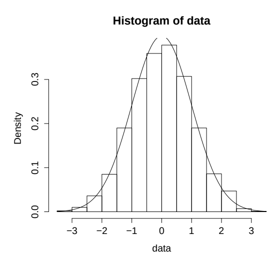
(i) $\omega = 1, T = 100$



(j) $\omega = 10, T = 10$

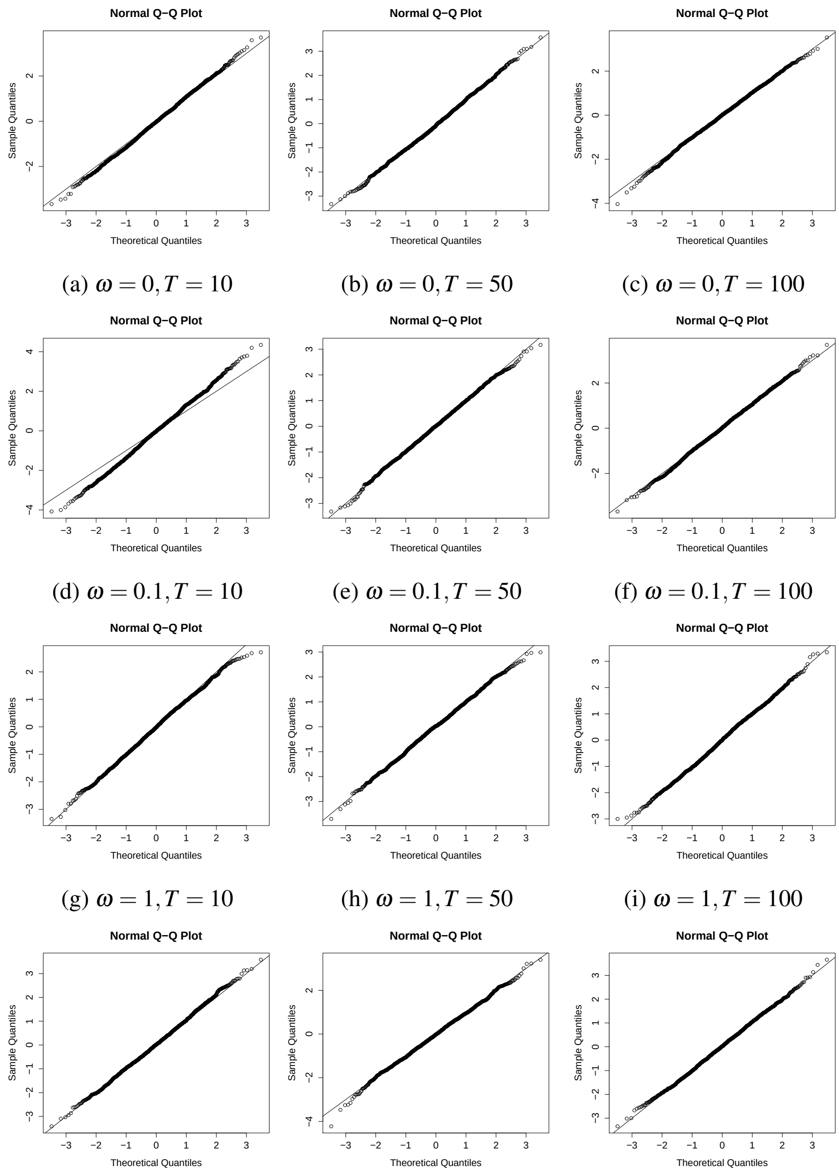


(k) $\omega = 10, T = 50$



(l) $\omega = 10, T = 100$

Figure S.4: Histograms and limiting density for the real part of the truncated Fourier transform of the simulated CARMA(2,1) processes driven by a Variance Gamma process for the frequencies $0, 0.1, 1, 10$ (rows) and time horizons/maximum non-equidistant grid sizes $10/0.1, 50/0.05, 100/0.01$ (columns)



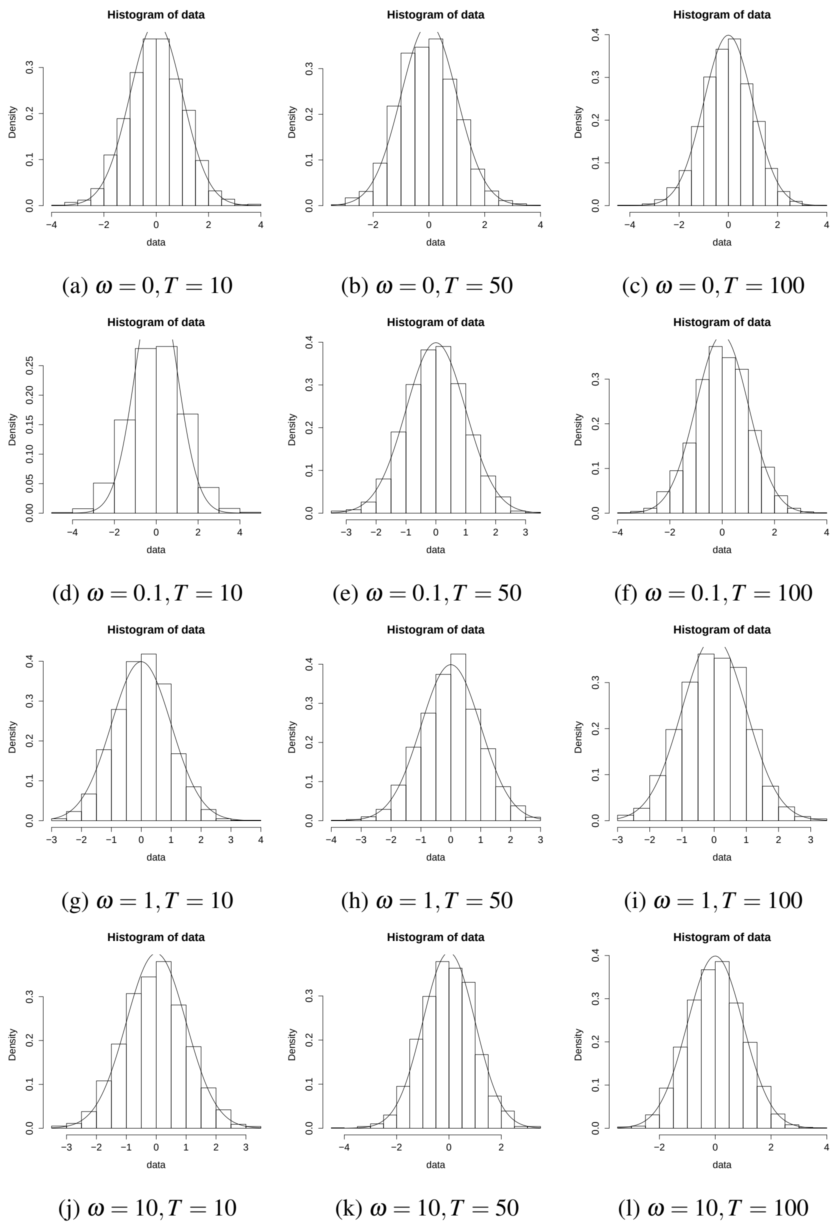
(a) $\omega = 0, T = 10$ (b) $\omega = 0, T = 50$ (c) $\omega = 0, T = 100$

(d) $\omega = 0.1, T = 10$ (e) $\omega = 0.1, T = 50$ (f) $\omega = 0.1, T = 100$

(g) $\omega = 1, T = 10$ (h) $\omega = 1, T = 50$ (i) $\omega = 1, T = 100$

(j) $\omega = 10, T = 10$ (k) $\omega = 10, T = 50$ (l) $\omega = 10, T = 100$

Figure S.5: Normal QQ plots for the real part of the truncated Fourier transform of the simulated CARMA(2,1) processes driven by a two sided Poisson process for the frequencies $0, 0.1, 1, 10$ (rows) and time horizons/maximum non-equidistant grid sizes $10/0.1, 50/0.05, 100/0.01$ (columns). The theoretical quantiles are coming from the (limiting) law described in Theorem 3.9.



(a) $\omega = 0, T = 10$ (b) $\omega = 0, T = 50$ (c) $\omega = 0, T = 100$

(d) $\omega = 0.1, T = 10$ (e) $\omega = 0.1, T = 50$ (f) $\omega = 0.1, T = 100$

(g) $\omega = 1, T = 10$ (h) $\omega = 1, T = 50$ (i) $\omega = 1, T = 100$

(j) $\omega = 10, T = 10$ (k) $\omega = 10, T = 50$ (l) $\omega = 10, T = 100$

Figure S.6: Histograms and limiting density for the real part of the truncated Fourier transform of the simulated CARMA(2,1) processes driven by a two-sided Poisson process for the frequencies $0, 0.1, 1, 10$ (rows) and time horizons/maximum non-equidistant grid sizes $10/0.1, 50/0.05, 100/0.01$ (columns)

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