Supplementary Material to: Hypothesis Testing for Multiple Mean and Correlation Curves with Functional Data

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Appendix I. Description of the simulation.

1 Simulation

To investigate the finite sample properties of the proposed methods, and compare with the commonly used local linear smoothing (Loess) and spline methods, we present here several simulation studies, which are designed to mimic the practical situations with moderate sample sizes. We consider separately the tests for the equality of two mean curves and the tests for the correlation function between two stochastic processes. For each case, the simulation is based on 5000 replications, and the mean values of estimates over the replications are reported.

1.1 Testing the Equality of Mean Curves

1.1.1 Simulation for Test with Two-Sided Alternatives

For testing the hypotheses (9), we generate the observations $\{\mathbb{X}_{n_1}, \mathbb{Y}_{n_2}, (\mathbb{X}_{n_{xy}}, \mathbb{Y}_{n_{xy}})\}$ of $\{X(t), Y(t) : t \in \mathcal{T}\}$ using the data structure (2) with $n_1 = n_2 = 50$ and $n_{xy} = 20$ on $k(n) = 50$ equally spaced time points $\{t_j = j : j = 1, \ldots, 50\}$, so that $n_x = n_y = 30$. For subjects with only $X(t)$ or $Y(t)$ observed, we generate $X_i(t_j) + \epsilon_i(t_j) = \mu(t_j) + r_i \sin(8 + t_j/10)/30 + N(0, \sigma^2(t_j))$, $\mu(t) = \big[(t + E[r]) \sin(8 + t/10)\big]/30$, where the r_i's and r are iid random integers uniformly distributed on $\{1, \ldots, 50\}$ used to make the curves more wiggly looking; similarly, $Y_i(t_j) + \xi_i(t_j) = \eta(t_j) + C r_i \cos(8 + t_j/10)/100 + N(0, \sigma^2(t_j))$, $\eta(t) = \mu(t) + CE[r] \cos(8 + t_j/10)/100$. $\sigma^2(t) = 0.001 \times t$, r_i's and r are iid random integers uniformly distributed on $\{1, \ldots, 50\}$ used to make the curves more wiggly looking, and C is a constant characterizing the difference between $\mu(t)$ and $\eta(t)$. For subjects with $\big(X(t), Y(t)\big)$ observed, we generate $\big\{\big(X_i(t) + \epsilon_i(t_j), Y_i(t) + \xi_i(t_j)\big)^T \sim \mathbf{N}\big(\big(\mu(t), \eta(t)\big)^T, \Sigma(t)\big) : i = n_x + 1, \ldots, n_1\big\}$, where the covariance matrix $\Sigma(t)$ is composed by the variance $\sigma^2(t)$ and the correlation coefficient $\rho(t) = 0.01 \times (t/50)$.

1.1.2 Simulation for Test with One-Sided Alternatives

For testing the hypotheses (10), we generate the observations $\{\mathbb{X}_{n_1}, \mathbb{Y}^*_{n_2}, (\mathbb{X}_{n_{xy}}, \mathbb{Y}^*_{n_{xy}})\}$ of $\{X(t), Y(t) : t \in \mathcal{T}\}$ using the same method as Section 5.1.1, except that $Y_i(t_j) + \xi_i(t_j)$ is replaced with $Y_i^*(t_j) + \xi_i(t_j) \sim N\big(\eta^*(t_j), \sigma^2(t_j)\big)$, where $\eta^*(t) = \mu(t) + \big[C^* (t +$

$E[r])\big|\cos(8+t/3)\big|/100\big]$. Here, C^*, which plays a similar role as C, is a constant characterizing the difference between $\mu(t)$ and $\eta^*(t)$.

1.2 Testing Correlation Functions

1.2.1 Simulation for Test with Two-Sided Alternatives

For simplicity, each of our simulated samples contains $n_1 = n_2 = n_{xy} = 50$ subjects observed on $k(n) = 50$ time points $\big\{t_j = j: \; j = 1, \ldots, 50\big\}$, so that, in view of (2), the sample contains only the paired observations $\Big\{\big(X_i(t_j)+\epsilon_i(t_j), Y_i(t_j)+\xi_i(t_j)\big)^T: i = 1, \ldots, 50; \, j = 1, \ldots, 50\Big\}$. For testing the hypotheses in (15) using the test statistic S_n, we generate in each sample $X_i(t_j)+\epsilon_i(t_j) \sim N\big(\mu(t_j), \sigma_1^2(t_j)\big)$, where $\mu(t) = t\,\sin(8+t/10)/30$ and $\sigma_1^2(t) = 0.01\times t$, and, conditional on $X_i(t_j) = x_i(t_j)$, $Y_i(t_j)+\xi_i(t_j)$ has the conditional normal distribution,

$$Y_i(t_j)+\xi_i(t_j)\big|x_i(t_j)+\epsilon_i(t_j) \sim N\Big(\mu(t_j)+\rho(t_j)\big[\sigma_2(t_j)/\sigma_1(t_j)\big]\big[x_i(t_j)+\epsilon_i(t_j)-\mu(t_j)\big], \big(1-\rho^2\big)\sigma_2^2(t_j)\Big), \tag{29}$$

where $\sigma_2^2(t) = 0.01 \times t$ and $\rho(t) = \rho\sin(8 - t/10)$ for some $\rho \geq 0$. Here, for any $t \in \{1, \ldots, 50\}$, $\rho(t)$ is the true correlation coefficient between $X_i(t)$ and $Y_i(t)$, and ρ determines the difference of the correlation curve $R(t)$ from zero.

Appendix II. Proofs.

Proof of Theorem 1 (i). Since, by (3) and (4), $X_{i,k(n)}(\cdot)$ and $Y_{i,k(n)}(\cdot)$ are two stochastic processes on $\mathcal{T}$, we denote by

$$\mu_{k(n)}(t) = E[X_{i,k(n)}(t)] \quad \text{and} \quad \eta_{k(n)}(t) = E[Y_{i,k(n)}(t)] \qquad (A.1)$$

the expectations of the random variables $X_{i,k(n)}(t)$ and $Y_{i,k(n)}(t)$, respectively, for each fixed $t \in \mathcal{T}$. Then, we have

$$\begin{aligned} &\sqrt{\tfrac{n_1 n_2}{n}} \Big\{ \big[\widehat{\mu}_{n_1}(t) - \mu(t)\big] - \big[\widehat{\eta}_{n_2}(t) - \eta(t)\big] \Big\} \\ &= \sqrt{\tfrac{n_1 n_2}{n}} \Big\{ \big[\widehat{\mu}_{n_1}(t) - \mu_{k(n)}(t)\big] - \big[\widehat{\eta}_{n_2}(t) - \eta_{k(n)}(t)\big] \Big\} \qquad (A.2) \\ &\quad + \sqrt{\tfrac{n_1 n_2}{n}} \Big\{ \big[\mu_{k(n)}(t) - \mu(t)\big] - \big[\eta_{k(n)}(t) - \eta(t)\big] \Big\}. \end{aligned}$$

Note that by (2), (4) and the assumption $E[\epsilon_i(t)] = 0$ for all t,

$$\begin{aligned} \mu_{k(n)}(t) &= \frac{(t_{j+1} - t)E[X_i(t_j) + \epsilon_i(t_{j+1})] + (t - t_j)E[X_i(t_{j+1}) + \epsilon_i(t_{j+1})]}{t_{j+1} - t_j} \\ &= \frac{(t_{j+1} - t)\mu(t_j) + (t - t_j)\mu(t_{j+1})}{t_{j+1} - t_j}, \qquad t \in [t_j, t_{j+1}]. \end{aligned}$$

Thus, $\mu_{k(n)}(\cdot)$ is the linear interpolation of $\mu(\cdot)$ on the $[t_j, t_{j+1})]$'s for $j = 0, \ldots, k(n) + 1$ with $t_0 = \inf\{t \in \mathcal{T}\}$ and $t_{k(n)+1} = \sup\{t \in \mathcal{T}\}$. Similarly, $\eta_{k(n)}(\cdot)$ is the linear interpolation of $\eta(\cdot)$ on $\mathcal{T}$.

By the assumptions that $\mu(\cdot)$ and $\eta(\cdot)$ are Lipschitz continuous on $\mathcal{T}$ which is bounded, it follows that $\mu_{k(n)}(t)$ and $\eta_{k(n)}(t)$ are uniformly continuous on $\mathcal{T}$. Thus, we have

$$\inf_{s \in [t_j, t_{j+1})} \mu(s) \le \mu_{k(n)}(t) \le \sup_{s \in [t_j, t_{j+1})} \mu(s)$$

and

$$\inf_{s \in [t_j, t_{j+1})} \eta(s) \le \eta_{k(n)}(t) \le \sup_{s \in [t_j, t_{j+1})} \eta(s)$$

for $t \in [t_j, t_{j+1})$, $j = 0, \dots, k(n)+1$. Let $\delta_{k(n)} = \max\{t_{j+1} - t_j : j = 0, 1, ..., k(n)\}$. The assumption of first order Lipschitz continuity implies there are $0 < c_1, c_2 < \infty$, such that

$$\sup_{s,t\in\mathcal{T}, |t-s|\le\delta_{k(n)}} |\mu(t) - \mu(s)| \le c_1 \delta_{k(n)} \text{ and } \sup_{s,t\in\mathcal{T}, |t-s|\le\delta_{k(n)}} |\eta(t) - \eta(s)| \le c_2 \delta_{k(n)}.$$

Thus by the condition $\sqrt{n}\delta_{k(n)} \to 0$, we get

$$\sqrt{\frac{n_1 n_2}{n}} \sup_{t\in\mathcal{T}} \Big| \big[\mu_{k(n)}(t) - \mu(t)\big] - \big[\eta_{k(n)}(t) - \eta(t)\big] \Big| \le (c_1 + c_2)\sqrt{n}\delta_{k(n)} \to 0. \qquad (A.3)$$

Now, it suffices to show that in $\ell^\infty(\mathcal{T})$,

$$\sqrt{\frac{n_1 n_2}{n}} \Big\{ \big[\widehat{\mu}_{n_1}(\cdot) - \mu_{k(n)}(\cdot)\big] - \big[\widehat{\eta}_{n_2}(\cdot) - \eta_{k(n)}(\cdot)\big] \Big\} \overset{D}{\Rightarrow} W(\cdot). \qquad (A.4)$$

To prove (A.4), it suffice to show, in $\ell^\infty(\mathcal{T})$,

$$\sqrt{n_1}\big[\widehat{\mu}_{n_1}(\cdot) - \mu_{k(n)}(\cdot)\big] \overset{D}{\Rightarrow} W_1(\cdot) \quad \text{and} \quad \sqrt{n_2}\big[\widehat{\eta}_{n_2}(\cdot) - \eta_{k(n)}(\cdot)\big] \overset{D}{\Rightarrow} W_1(\cdot). \qquad (5)$$

for some Gaussian processes $W_1(\cdot)$ and $W_2(\cdot)$.

We will use Theorem 2.11.23 in van der Vaart and Wellner (1996, P.221) to prove (A.5). We only show the first in (A.5), that for the second is the same. Denote $\tilde{X}_i(\cdot) = X_i(\cdot) + \epsilon_i(\cdot)$, then

$$\hat{\mu}_{n_1}(t) \frac{1}{n_1} \sum_{i=1}^{n_1} \frac{(t - t_j)\tilde{X}_i(t_j) + (t_{j+1} - t)\tilde{X}_i(t_{j+1})}{t_{j+1} - t_j} := \frac{1}{n_1} \sum_{i=1}^{n_1} g_{n,t}(\tilde{X}_i), \qquad t \in [t_j, t_{j+1}],$$

where $g_{n,t}$ is the linear interpolation functional, with knots $\{t_1, ..., t_{k(n)}\}$, evaluated at $t \in \mathcal{T}$, and $\mu_{k(n)}(t) = E g_{n,t}(\tilde{X}_i) := P g_{n,t}(\tilde{X}_i)$. Denote P_{n_1} the empirical measure of $\tilde{X}_1, ..., \tilde{X}_{n_1}$, then the first in (A.5) is written as

$$n_1^{-1/2}(P_{n_1} - P) g_{n,\cdot}(\tilde{X}_i) \overset{D}{\Rightarrow} W_1(\cdot), \qquad \text{in } \ell^\infty(\mathcal{T}). \qquad (A.6)$$

To show the above, we only need to check the conditions of Theorem 2.11.23.

For any $s, t \in \mathcal{T}$, let $\rho(s,t) = |t-s|$, then $(\mathcal{T}, \rho)$ is a totally bounded semi-metric space. Let $\tilde{X}$ be an iid copy of the $\tilde{X}_i$'s, and define $\tilde{Y}_i$ and $\tilde{Y}$ similarly. Let $\mathcal{G}_n = \{g_{n,t}(\tilde{X}) : t \in \mathcal{T}\}$. Then $G_n = G = \sup_{t\in\mathcal{T}}[\tilde{X}^2(t) + \tilde{Y}^2(t)]^{1/2}$ is an envelope for $\mathcal{G}_n$. By the given condition $PG^2 < \infty$, so $PG_n^2 = PG = O(1)$, and $P[G_n^2 I(G_n > \delta\sqrt{n})] = P[G^2 I(G > \delta\sqrt{n})] \to 0$ for every $\delta > 0$. Also, for every $\delta_n \to 0$, by the given conditon $\delta_n \to 0$ $\sup_{|t-s|\le\delta_n} E\Big(\big[X(t)+\epsilon(t)-X(s)-\epsilon(s)\big]^2 + \big[Y(t)+\eta(t)-Y(s)-\eta(s)\big]^2\Big) \to 0$,

$$\sup_{\rho(s,t)<\delta_n} P\big(g_{n,s}(\tilde{X}) - g_{n,t}(\tilde{X})\big)^2 \le \sup_{\rho(s,t)<\delta_n} P\big(\tilde{X}(s) - \tilde{X}(t)\big)^2 \to 0.$$

Thus (2.11.21) in van der Vaart and Wellner (1996, P.220) is satisfied.

Let $N_{[\,]}\big(\epsilon, \mathcal{G}_n, L_2(P)\big)$ be the number of ϵ-brackets needed to cover $\mathcal{G}_n$ under the $L_2(P)$ metric. Since for each n, there is one member $g_{n,t} \in \mathcal{G}_n$, let $l_{n,t} = u_{n,t} = g_{n,t}$, then $l_{n,t}(\tilde{X}) \le g_{n,t}(\tilde{X}) \le u_{n,t}(\tilde{X})$ $(t \in \mathcal{T})$, and for all $\epsilon > 0$, $P\big(u_{n,t}(\tilde{X}) - l_{n,t}(\tilde{X})\big)^2 = 0 < \epsilon\|G\|_{L_2(P)}$, i.e., $(l_{n,t}, u_{n,t})$ is a ϵ-bracket of $\mathcal{G}_n$ under the $L_2(P)$ norm. Here we have $N_{[\,]}\big(\epsilon\|G\|_{L_2(P)}, \mathcal{G}_n, L_2(P)\big) = 1$, thus

$$\int_0^{\delta_n} \sqrt{\log N_{[\,]}\big(\epsilon\|G\|_{L_2(P)}, \mathcal{G}_n, L_2(P)\big)}\, d\epsilon \to 0, \qquad \text{for every} \quad \delta_n \to 0.$$

Now, by Theorem 2.11.23 in van der Vaart and Wellner (1996, P.221), (A.6) is true.

Next we identify the weak limit $W(\cdot)$. For each fixed interpolation $g_{n,t} \in \mathcal{G}_n$, $g_{n,t}(\tilde{X})$ is a random function in t, so $W(\cdot)$ is a process on $\mathcal{T}$. For each positive integer k and fixed $(t_1, ..., t_k)$, by central limit theorm for double array, $(W(t_1), ..., W(t_k))^T$ is the weak limit of the vector

$$\sqrt{\frac{n_1 n_2}{n}}\Big\{\big[\widehat{\mu}_{n_1}(\cdot)-\mu_{k(n)}(t_j)\big]-\big[\widehat{\eta}_{n_2}(\cdot)-\eta_{k(n)}(t_j)\big] : j=1,...,k\Big\}.$$

So $(W(t_1),...,W(t_k))^T$ is a mean zero normal random vector, and by the uniform weak convergence showed above, $W(\cdot)$ is a Gaussian process on $\mathcal{T}$. Clearly $E[W(\cdot)]=0$. The covariance function $R(s,t)=E[W(s)W(t)]$ is given by

$$\begin{aligned} R(s,t) &= \lim_{n\to\infty}\Big(\frac{n_1 n_2}{n}\Big)Cov\Big\{\big[\widehat{\mu}_{n_1}(s)-\mu_{k(n)}(s)\big]-\big[\widehat{\eta}_{n_2}(s)-\eta_{k(n)}(s)\big], \\ &\qquad \big[\widehat{\mu}_{n_1}(t)-\mu_{k(n)}(t)\big]-\big[\widehat{\eta}_{n_2}(t)-\eta_{k(n)}(t)\big]\Big\}. \end{aligned}$$

Since

$$\begin{aligned} & Cov\Big\{\big[\widehat{\mu}_{n_1}(s)-\mu_{k(n)}(s)\big]-\big[\widehat{\eta}_{n_2}(s)-\eta_{k(n)}(s)\big], \big[\widehat{\mu}_{n_1}(t)-\mu_{k(n)}(t)\big]-\big[\widehat{\eta}_{n_2}(t)-\eta_{k(n)}(t)\big]\Big\} \\ &= Cov\Big\{\Big[\frac{1}{n_1}\sum_{i=1}^{n_x}\big[X_{i,k(n)}(s)-\mu_{k(n)}(s)\big]+\frac{1}{n_1}\sum_{i=n_x+1}^{n_1}\big[X_{i,k(n)}(s)-\mu_{k(n)}(s)\big] \\ &\quad -\frac{1}{n_2}\sum_{i=n_x+1}^{n_1}\big[Y_{i,k(n)}(s)-\eta_{k(n)}(s)\big]-\frac{1}{n_2}\sum_{i=n_1}^{n_2}\big[Y_{i,k(n)}(s)-\eta_{k(n)}(s)\big]\Big], \\ &\quad \Big[\frac{1}{n_1}\sum_{i=1}^{n_x}\big[X_{i,k(n)}(t)-\mu_{k(n)}(t)\big]+\frac{1}{n_1}\sum_{i=n_x+1}^{n_1}\big[X_{i,k(n)}(t)-\mu_{k(n)}(t)\big] \\ &\quad -\frac{1}{n_2}\sum_{i=n_x+1}^{n_1}\big[Y_{i,k(n)}(t)-\eta_{k(n)}(t)\big]-\frac{1}{n_2}\sum_{i=n_1}^{n_2}\big[Y_{i,k(n)}(t)-\eta_{k(n)}(t)\big]\Big]\Big\} \\ &= \frac{1}{n_1}Cov\big[X_{1,k(n)}(s),X_{1,k(n)}(t)\big]-\frac{n_{xy}}{n_1 n_2}Cov\big[X_{1,k(n)}(s),Y_{1,k(n)}(t)\big] \\ &\quad -\frac{n_{xy}}{n_1 n_2}Cov\big[Y_{1,k(n)}(s),X_{1,k(n)}(t)\big]+\frac{1}{n_2}Cov\big[Y_{1,k(n)}(s),Y_{1,k(n)}(t)\big], \end{aligned}$$

we have

$$\begin{aligned} R(s,t) &= \lim_{n\to\infty}\Big(\frac{n_1 n_2}{n}\Big)\Big\{n_1^{-1}\,Cov\big[X_{1,k(n)}(s),X_{1,k(n)}(t)\big] - \frac{(n_1-n_x)}{n_1 n_2}\,Cov\big[X_{1,k(n)}(s),Y_{1,k(n)}(t)\big] \\ &\quad -\frac{(n_1-n_x)}{n_1 n_2}\,Cov\big[Y_{1,k(n)}(s),X_{1,k(n)}(t)\big] + n_2^{-1}\,Cov\big[Y_{1,k(n)}(s),Y_{1,k(n)}(t)\big]\Big\} \\ &= \gamma_2 R_{11}(s,t) - \gamma_{12}[R_{12}(s,t)+R_{21}(s,t)] + \gamma_1 R_{22}(s,t). \qquad \square \end{aligned}$$

Proof of Theorem 2. (i). By Theorem 1, we have that, under H_0 of (9),

$$L_n \xrightarrow{D} \frac{1}{|\mathcal{T}|}\int_{\mathcal{T}} W^2(t)dt. \tag{A.9}$$

Since $R(\cdot,\cdot)$ is *almost everywhere* continuous and $\mathcal{T}$ is bounded, $R^2(\cdot,\cdot)$ is integrable, that is, $\int_{\mathcal{T}}\int_{\mathcal{T}} R^2(s,t)\,dsdt < \infty$. By Mercer's Theorem (cf. Theorem 5.2.1 of Shorack and Wellner (1986), page 208), we have that

$$R(s,t) = \sum_{j=1}^{\infty} \lambda_j h_j(s) h_j(t), \tag{A.10}$$

where $\lambda_j \geq 0$, $j = 1, 2, \ldots$, are the eigenvalues of $R(\cdot,\cdot)$, and $h_j(\cdot)$, $j = 1, 2, \ldots$, are the corresponding orthonormal eigenfunctions.

Let $\{Z_1, \ldots, Z_m, \ldots\}$ be the set of independent identically distributed random variables with $Z_m \sim N(0,1)$. Then $\mathbb{Z}(t) = \sum_{j=1}^{\infty}\sqrt{\lambda_j} Z_j h_j(t)$ is a Gaussian process on $\mathcal{T}$ with mean zero and covariance function $R(s,t)$. Thus, the two stochastic processes, $W(t)$ and $\mathbb{Z}(t)$, have the same distribution on $\mathcal{T}$,

$$W(t) \stackrel{d}{=} \mathbb{Z}(t) = \sum_{j=1}^{\infty}\sqrt{\lambda_j} Z_j h_j(t) \tag{A.11}$$

and, by (A.9) and (A.10),

$$\frac{1}{|\mathcal{T}|}\int_{\mathcal{T}} W^2(t)dt \stackrel{d}{=} \frac{1}{|\mathcal{T}|}\int_{\mathcal{T}} \Big[\sum_{j=1}^{\infty}\sqrt{\lambda_j}Z_j h_j(t)\Big]^2 dt = \frac{1}{|\mathcal{T}|}\sum_{j=1}^{\infty}\lambda_j Z_j^2. \qquad (A.12)$$

The result of Theorem 2(i) follows from (A.9), (A.11) and (A.12).

(ii). By Theorem 1, we have that, under H_0 of (10),

$$D_n \stackrel{D}{\rightarrow} \frac{1}{|\mathcal{T}|}\int_{t\in\mathcal{T}} W(t)\,dt = U, \qquad (A.13)$$

where U has normal distribution with mean zero. To compute the variance of U, we consider the partition $\big\{[s_j, s_{j+1}) : j = 1, \ldots, m\big\}$ of $\mathcal{T}$ with $\delta = \max\big\{s_{j+1} - s_j : j = 1, \ldots, m\big\}$. Then, it follows from (A.13) that

$$U = \lim_{\delta\to 0}\sum_{j=1}^{m} W(s_j)(s_{j+1} - s_j). \qquad (A.14)$$

Since $E\big[W(s_j)\big] = 0$ for each fixed j, we have that, by (A.14) and the continuity condition of $R(\cdot,\cdot)$,

$$\begin{aligned} Var(U) &= \lim_{\delta\to 0}\sum_{i=1}^{m}\sum_{j=1}^{m} E\big[W(s_i)\,W(s_j)\big]\,(s_{i+1} - s_i)(s_{j+1} - s_j) \\ &= \lim_{\delta\to 0}\sum_{i=1}^{m}\sum_{j=1}^{m} R(s_i, s_j)\,(s_{i+1} - s_i)(s_{j+1} - s_j) = \int_{s\in\mathcal{T}}\int_{t\in\mathcal{T}} R(s,t)\,ds\,dt. \end{aligned}$$

The result of Theorem 2(ii) follows from (A.13) and $Var(U)$. □

Proof of Theorem 3

The proof is similar to derivation of (A.5) by substituting $\mu(\cdot)$ and $\mu_n(\cdot)$ with $\boldsymbol{\mu}(\cdot)$ and $\widehat{\boldsymbol{\mu}}_n(\cdot)$, respectively. The difference here is that we have order two polynomial interpolations for the terms $\big(X^2_{i,k(n)}(\cdot),\, Y^2_{i,k(n)}(\cdot),\, X_{i,k(n)}(\cdot)Y_{i,k(n)}(\cdot)\big)$ in addition to the linear interpolations for $X_{i,k(n)}(\cdot)$ and $Y_{i,k(n)}(\cdot)$ with G^2 playing the role of G in the derivation of (A.5). The rest of the derivation is proceeded the same way. Then, the delta method leads to the claimed result. To identify the matrix covariance function $\Omega(s,t)$, we note that

$$\boldsymbol{\mu}_n(t) = \frac{1}{n}\sum_{i=1}^{n}\big(X_{i,k(n)}(t), Y_{i,k(n)}(t), X^2_{i,k(n)}(t), Y^2_{i,k(n)}(t), X_{i,k(n)}(t)\, Y_{i,k(n)}(t)\big)' := \frac{1}{n}\sum_{i=1}^{n}\mathbf{Z}_i(t).$$

Then, $\Omega(s,t) = Cov\big(\mathbf{Z}(s),\, \mathbf{Z}(t)\big)$ gives the expression for $\Omega(s,t)$. $\square$

Proof of Theorem 4.

The proof here is focused on the derivation of (28), as the proof of (27) can be proceeded using the same approach here and the results of Theorem 3. We first note that $W_n(t) = H\big[\boldsymbol{\mu}_n(t)\big]$ and, under H_0 of (15) and (16), $H\big[\boldsymbol{\mu}(t)\big] = 0$. It follows that

$$\sqrt{n}W_n(t) = \sqrt{n}\Big\{H\big[\boldsymbol{\mu}_n(t)\big] - H\big[\boldsymbol{\mu}(t)\big]\Big\} = \big[1 + o_p(1)\big]\sqrt{n}\,\dot{H}\big[\boldsymbol{\mu}(t)\big]\,\big[\boldsymbol{\mu}_n(t) - \boldsymbol{\mu}(t)\big].$$

Using result of Theorem 3, the delta method and the similar derivations as the proof of Theorem 1, we have

$$\sqrt{n}W_n(\cdot) \stackrel{P}{\to} \widetilde{W}(\cdot) \;\text{ in }\; \ell^\infty(\mathcal{T}) \;\text{ uniformly on } \mathcal{P},$$

where $\widetilde{W}(\cdot)$ is the mean zero Gaussian process on $\mathcal{T}$ with covariance function

$$\begin{aligned} Q(s,t) &= Cov\Big\{\dot{H}\big[\boldsymbol{\mu}(s)\big]\,\mathbf{X}(s),\, \dot{H}\big[\boldsymbol{\mu}(t)\big]\,\mathbf{Z}(t)\big]\Big\} \\ &= \dot{H}\big[\boldsymbol{\mu}(s)\big]\,Cov\big[\mathbf{Z}(s),\,\mathbf{Z}(t)\big]\,\dot{H}'\big[\boldsymbol{\mu}(t)\big] = \dot{H}\big[\boldsymbol{\mu}(s)\big]\,\Omega(s,t)\,\dot{H}'\big[\boldsymbol{\mu}(t)\big]. \end{aligned}$$

The rest of the proof is the same as in that of Theorem 2. □