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# ON SOME CENTRAL AND NON-CENTRAL MULTIVARIATE CHI-SQUARE DISTRIBUTIONS

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*Abstract*. Let $R$ be a non-singular $m$-factorial correlation matrix, i.e. $R = D + AA'$ with a diagonal matrix $D > 0$ and a not necessarily definite matrix $AA'$ of the minimal possible rank $m$. From an expression for the general non-central multivariate $\chi^2$-distribution function with the accompanying correlation matrix $R$ some simpler cases are derived: The $p$-variate central $\chi^2$-distribution with $q$ degrees of freedom is given as a mixture with regard to a Wishart $W_m(q, I_m)$-distribution. For $m = 2$ several integral and series representations are derived including the limit case with exactly one zero on the diagonal of $D$. The two-factorial representation is applied to the four-variate $\chi^2$-distribution. Besides, it is used for Taylor approximations if $m > 2$. Furthermore, the non-central distribution function is given for $m = 1$ and applied to power calculations for some multivariate multiple comparisons with a control.

Key words and phrases: Multivariate chi-square distribution, multivariate gamma distribution, multivariate Rayleigh distribution, multivariate multiple comparisons with a control, power.

## 1. Introduction and Notation

The following notations are used throughout the paper: The spectral norm of any $p \times p$-matrix $A = (a_{ij})$ is denoted by $\|A\|$, $|A|$ is the determinant of $A$, $\dot{A}$ is defined by $A$-Diag$(a_{11}, \ldots, a_{pp})$, $A > 0$ means positive definiteness, etr$(A)$ stands for exp(trace$(A)$) and $(a^{ij}) = A^{-1}$. A $p \times p$-unit matrix is always denoted by $I_p$ or $I$. The Fourier transform (F.t.) or Laplace transform (L.t.) of a function $f$ is denoted by $\hat{f}$. The notation $\sum_{(n)}$ means a summation over all decompositions of a non-negative number $n = \sum n_i$ (or $\sum n_{ij}$) with nonnegative integers $n_i$, $i = 1, \ldots, p$ (or $n_{ij}, 1 \leq i < j \leq p$). Furthermore $n_{\cdot j} = \sum_{i=1}^{p} n_{ij}$ with $n_{ij} = n_{ji}$. Formulas from the handbook of mathematical functions by Abramowitz and Stegun (1965) are cited by "A.S." and their number.

Let $R = (r_{ij})$ denote a $p \times p$-correlation matrix and $Y$ a $p \times q$-matrix with $q$ independent $N(\mu_j, R)$-distributed columns. The joint distribution of the diagonal elements $X_i$ of the Wishart matrix $YY'$ is called the $p$-dimensional chi-square distribution in the sense of Krishnamoorthy and Parthasarathy (1951) with $q$ degrees of freedom, the accompanying correlation matrix $R$ and the non-centrality matrix $\Delta = MM'$, where $M$ is the expectation $E(Y)$ ($\chi_p^2(q, R, \Delta)$-distribution). It is comparatively easy to derive general representations of the corresponding distribution function (d.f.)

$$F_p(x_1, \ldots, x_p; q, R, \Delta) = \Pr\{X_1 \le x_1, \ldots, X_p \le x_p\} \tag{1.1}$$

or of the density (p.d.) $f_p$ from the characteristic function (ch.f.)

$$\hat{f}_p(t_1, \ldots, t_p; q, R, \Delta) = |I - 2iRT|^{-q/2} \operatorname{etr}(iT(I - 2iRT)^{-1}\Delta), \tag{1.2}$$

$$T = \operatorname{Diag}(t_1, \ldots, t_p), \qquad \text{(e.g. Jensen (1969), Sec. 2).}$$

In the central case $\Delta = 0$ is omitted in the above notations. For general formulas see Blumenson and Miller (1963), Miller (1964), Jensen (1970), Khatri, Krishnaiah and Sen (1977) and Royen (1991b, 1992). However, the general expressions are difficult to compute, and it is the aim of this paper to derive from the general formula in Theorem 3.1 some simpler cases, where the computation requires no more than series of uni- or bi-variate integrals and powers of linear or quadratic forms. The present paper was motivated mainly by the work at programs for certain multivariate multiple test procedures (cf. Sec. 6).

The following definition is fundamental:

**Definition 1.1.** Let $R$ be a non-singular $p \times p$-correlation matrix. $R$ is called *$m$-factorial* if $m$ is the smallest integer allowing a representation $R = D + AA'$ with a diagonal $D > 0$ and a not necessarily definite matrix $AA'$ of rank $m$.

With $D = W^{-2}$ note that

$$WRW = I_p + BB', \tag{1.3}$$

where the $p \times m$-matrix $B$ of rank $m$ has without loss of generality pairwise orthogonal columns $b_\mu$, some of them possibly imaginary. The char. roots $\beta_\mu \neq 1$ of $I_p + BB'$ coincide with the $m$ diagonal elements

$$\beta_\mu = 1 + b'_\mu b_\mu > 0 \tag{1.4}$$

of $I_m + B'B$.

For an $m$-factorial $R$ the d.f. $F_p(\cdot; q, R)$ is given in Corollary 1a of Theorem 3.1 as a mixture with regard to a Wishart $W_m(q, I_m)$-distribution. The one-factorial case was treated in Royen (1991a, b). For the two-factorial case several representations of $F_p$ are given in Corollary 1b of Theorem 3.1. Together with the limit case with only $p-1$ positive elements in $D$ (cf. Theorem 4.2) many four-variate $\chi^2$-distributions are covered by these correlation matrices (cf. Sec. 5).

If there exists a "good" approximation of $R$ by an $m$-factorial $R_0$ (practically $m = 1$ or $m = 2$) then a Taylor polynomial approximation with $R - R_0$ (or some transformed deviations) can be derived for the d.f., which generalizes the corresponding approach for the multivariate normal distribution in Royen (1987). This method is described concisely at the end of Section 3.

For the non-central case with a one-factorial $R$ and any non-centrality matrix $\Delta$ a comparatively simple expansion is found in Corollary 2a of Theorem 3.1, which is simplified further if $\mathrm{rank}(\Delta) = 1$.

## 2. Some Preliminaries

Let $g_r(x)$ be the gamma density $x^{r-1}\exp(-x)/\Gamma(r)$ and $G_r(x)$ the corresponding d.f.. We need the functions

$$g_{r+n}^{(n)}(x) = \frac{d^n}{dx^n} g_{r+n}(x) = L_n^{(r-1)}(x) g_r(x) \Big/ \binom{r+n-1}{n} \tag{2.1}$$

and

$$h_{r,n}(x) = (-1)^n L_n^{(r-1)}(2x) g_r(x) \Big/ \binom{r+n-1}{n} \tag{2.2}$$

with the generalized Laguerre polynomials $L_n^{(r-1)}$ (A.S.22.11.6).

An always absolutely convergent expansion for central multivariate $\chi^2$-probabilities of (bounded or unbounded) rectangular regions was given in Royen (1991 b) using the functions

$$H_{r,n}(x) = \int_0^x h_{r,n}(\xi) d\xi. \tag{2.3}$$

For $n \to \infty$ its order is

$$H_{r,n}(x) = O(n^{-r/2-1/4}) \tag{2.4}$$

with an $O$-constant depending on $r$ and $x$. Also,

$$G_{r+n}^{(n)}(x) = \int_0^x g_{r+n}^{(n)}(\xi) d\xi = O(n^{-r/2-1/4}) \tag{2.5}$$

with an $O$-constant only depending on $r$.

With any scale factors $v_j = w_j^2/2$ we define for any real numbers $t_j$, $j = 1, \ldots, p$:

$$\begin{aligned} z_j &= (1 - it_j/v_j)^{-1}, \quad u_j = 1 - z_j = (-it_j/v_j) z_j, \\ w_j &= z_j - u_j = \exp(i\phi_j), \quad \phi_j = 2\arctan(t_j/v_j), \end{aligned} \tag{2.6}$$

and the diagonal matrices $T, W, V = W^2/2, Z, U, \Omega$ with the corresponding elements $t_j, w_j, v_j, z_j, u_j$ and $\omega_j$. With $v = 1$, $x > 0$ and any real or complex $y$ we obtain, by Fourier transform, the relations

| $\hat f_{r,n}(t,y)$ | $f_{r,n}(x,y)$ | |
|---|---|---|
| $z^{r+n}\exp(-yu)$ | $g_{r+n}(x,y)$ | (2.7a) |
| $z^r u^n \exp(-yu)$ | $g^{(n)}_{r+n}(x,y)$ | (2.7b) |
| $z^r \omega^n \exp(-yu)$ | $h_{r,n}(x,y)$ | (2.7c) |

and

$$F_{r,n}(x,y)=\int_0^x f_{r,n}(\xi,y)d\xi=$$

$$G_{r+n}(x,y)=e^{-y}\sum_{m=0}^{\infty}G_{r+m+n}(x)y^m/m!, \tag{2.8a}$$

$$G^{(n)}_{r+n}(x,y)=e^{-y}\sum_{m=0}^{\infty}G^{(n)}_{r+m+n}(x)y^m/m!=\sum_{m=0}^{\infty}G^{(m+n)}_{r+m+n}(x)(-y)^m/m!, \tag{2.8b}$$

$$H_{r,n}(x,y)=e^{-y}\sum_{m=0}^{\infty}H_{r+m,n}(x)y^m/m!=\ e^{-y/2}\sum_{m=0}^{\infty}H_{r,m+n}(x)(y/2)^m/m!. \tag{2.8c}$$

$F_{r,n}(x)$ is written instead of $F_{r,n}(x,0)$. Since the functions $G_{r+n}(x,y)$, $(y\in\mathbb{R})$ are the most important elements in the formulas of the following sections we list some further representations, as follows:

$$G_r(x,y)=\sum_{m=0}^{\infty}G^{(m)}_{r+m}(x)(-y)^m/m!=e^{-y/2}\sum_{m=0}^{\infty}H_{r,m}(x)(y/2)^m/m! \tag{2.9a}$$

$$=e^{-y}\int_0^x {}_0F_1(r;\xi y)g_r(\xi)d\xi=e^{-y}\sum_{m=0}^{\infty}{}_0F_1(r+1+m;xy)g_{r+1+m}(x) \tag{2.9b}$$

$$=(x/\pi)^{1/2}\int_{-1}^{1}G_{r-1/2}((1-c^2)x)\exp(-(\sqrt{y}-c\sqrt{x})^2)dc\quad(r\geq 1/2;G_0(x)\equiv 1) \tag{2.9c}$$

$$=\begin{cases}\Phi(\sqrt{2y}+\sqrt{2x})-\Phi(\sqrt{2y}-\sqrt{2x})-e^{-x-y}\sum\limits_{k=1}^{r-1/2}(\sqrt{x/y})^{r-k}I_{r-k}(2\sqrt{xy}),\\ \text{if}\quad r=1/2,3/2,\dots, & (2.9\text{d})\\ \\ e^{-x-y}(\frac{x}{y})^{r/2}\frac{1}{\pi}\int_0^{\pi}\frac{y\cos(r\phi)-\sqrt{xy}\cos((r-1)\phi)}{y-2\sqrt{xy}\cos(\phi)+x}\exp(2\sqrt{xy}\cos(\phi))d\phi+\delta(x-y),\\ \text{if}\quad r=0,1,2,\dots;y>0;\delta(x-y)=\begin{cases}1, & y<x,\\ 1/2, & y=x,\\ 0, & y>x.\end{cases} & (2.9\text{e})\end{cases}$$

It should be noted that the spherical Bessel functions are elementary. The formulas (2.9a) are verified by characteristic functions and the remaining ones by A.S.9.6.47, A.S.9.6.18/19, A.S.9.6.33 and A.S.6.5.29.

Now several expressions are summarized for the ch.f. (1.2) with an $m$-factorial $R$. For this we define with $B$ from (1.3) and $\beta_\mu$ from (1.4) the following quantities:

$$\widetilde{B} = \widetilde{B}_k = \begin{cases} B, & \text{if } k = 1, \quad (2.10a) \\ B\mathrm{Diag}(\beta_1^{-1/2}, \ldots, \beta_m^{-1/2}), & \text{if } k = 2, \quad (2.10b) \\ B\mathrm{Diag}((1+\beta_1)^{-1/2}, \ldots, (1+\beta_m)^{-1/2}), & \text{if } k = 3, \quad (2.10c) \end{cases}$$

$$C = \widetilde{B}\widetilde{B}' = \begin{cases} WRW - I, & \text{if } k = 1, \\ I - (WRW)^{-1}, & \text{if } k = 2, \\ I - 2(I + WRW)^{-1}, & \text{if } k = 3, \end{cases} \tag{2.11}$$

$$D = \begin{cases} \frac{1}{2} W\Delta W, & \text{if } k = 1, \\ \frac{1}{2}(I - C)W\Delta W(I - C), & \text{if } k = 2, 3, \end{cases} \tag{2.12}$$

$$\alpha = \begin{cases} 1, & \text{if } k = 1, 2, \\ 1/4, & \text{if } k = 3, \end{cases} \qquad \alpha^* = \begin{cases} 1, & \text{if } k = 1, 2, \\ 1/2, & \text{if } k = 3, \end{cases} \tag{2.13}$$

$$c = \begin{cases} 1, & \text{if } k = 1, \\ |I - C|, & \text{if } k = 2, 3, \end{cases} \qquad c^* = \begin{cases} 1, & \text{if } k = 1, \\ c^r \mathrm{etr}(-\alpha B'DB), & \text{if } k = 2, 3, \end{cases} \tag{2.14}$$

and with $U, Z, \Omega$ from (2.6):

$$Y = \begin{cases} -U, & \text{if } k = 1, \\ Z, & \text{if } k = 2, \\ \Omega, & \text{if } k = 3. \end{cases} \tag{2.15}$$

**Lemma 2.1.** *For an $m$-factorial correlation matrix $R$ the ch.f. in* (1.2) *with $q = 2r$ is given by*

$$\hat{f}(t_1, \ldots, t_p; q, R, \Delta) =$$
$$c^* \left( \prod_{j=1}^{p} z_j^r \right) \mathrm{etr}(-UD)|I_p - CY|^{-r} \mathrm{etr}(\alpha^*(I_m - \widetilde{B}_k' Y \widetilde{B}_k)^{-1} \widetilde{B}_k' Y D Y \widetilde{B}_k),$$
$$k = 1, 2, 3. \tag{2.16}$$

*In particular for $m = 1$, $B = b$ the last factor in* (2.16) *is simplified to*

$$\exp(\alpha(1 + b'Ub)^{-1} b'YDYb) \tag{2.17}$$

*and*

$$c|I_p - CY|^{-1} = |I_p + bb'U|^{-1} = (1 + b'Ub)^{-1}. \tag{2.18}$$

*If* $\mathrm{rank}(C) = 2$ *then*

$$|I_p - CY| = 1 - \sum_j c_{jj} y_j + \sum_{i<j} (c_{ii}c_{jj} - c_{ij}^2) y_i y_j. \tag{2.19}$$

**Proof of Lemma 2.1.** The formulas for the central case are from Royen (1991b). With $T = \mathrm{Diag}(t_1, \ldots, t_p)$ we find

$$
\begin{aligned}
\mathrm{tr}(iT(I-2iRT)^{-1}\Delta) &= \frac{1}{2}\mathrm{tr}(2iTW^{-2}(I-WRW2iTW^{-2})^{-1}W\Delta W) \\
&= \mathrm{tr}((I-Z^{-1})(I-(I+C_1)(I-Z^{-1}))^{-1}D_1) \\
&= \mathrm{tr}((Z-I)(Z-(I+C_1)(Z-I))^{-1}D_1) \\
&= -\,\mathrm{tr}(U(I+C_1U)^{-1}D_1) \qquad (2.20)\\
&= -\,\mathrm{tr}(UD_1) + \mathrm{tr}(U(I+C_1U)^{-1}C_1UD_1). \qquad (2.21)
\end{aligned}
$$

With $C_1 = BB'$ and $(I_p + BB'U)^{-1}B = B(I_m + B'UB)^{-1}$ it follows that the second term in (2.21) is equal to $\mathrm{tr}((I_m + B'UB)^{-1}B'UD_1UB)$.

To show the second formula with $Y = Z$ we start from (2.20) with $U = I - Z$ and $C_2 = I - (WRW)^{-1}$. It is

$$
\begin{aligned}
&-\,\mathrm{tr}((I-Z)(I+(WRW-I)(I-Z))^{-1}D_1) \\
&= -\,\mathrm{tr}((I-Z)(WRW-(WRW-I)Z)^{-1}D_1) \\
&= -\,\mathrm{tr}((I-C_2Z-(WRW)^{-1}Z))(I-C_2Z)^{-1}(WRW)^{-1}D_1) \\
&= -\,\mathrm{tr}(BB'D_2) - \mathrm{tr}(D_2) + \mathrm{tr}(Z(I-C_2Z)^{-1}D_2) \\
&= -\,\mathrm{tr}(B'D_2B) - \mathrm{tr}(UD_2) + \mathrm{tr}(Z(I-C_2Z)^{-1}C_2ZD_2).
\end{aligned}
$$

With $C_2 = \widetilde{B}_2\widetilde{B}_2'$ and $\widetilde{B}_2(I_m - \widetilde{B}_2'Z\widetilde{B}_2)^{-1} = (I_p - \widetilde{B}_2\widetilde{B}_2'Z)^{-1}\widetilde{B}_2$ the last trace is equal to $\mathrm{tr}((I_m - \widetilde{B}_2'Z\widetilde{B}_2)^{-1}\widetilde{B}_2'ZD_2Z\widetilde{B}_2)$.

In a similar way the formula with $Y = \Omega$ is shown, starting again from (2.20) with $2U = I - \Omega$.

In the proof of Theorem 3.1 we have to justify a change of the order of integration over $\mathbb{R}_+^p$ and $\{S_{m\times m} > 0\}$ using the following lemma:

**Lemma 2.2.** *Let $S$ be $W_m(2r, I_m)$-distributed ($r \geq 1$), $b^1, \ldots, b^p$ the rows of the matrix $B$ in* (1.3) *and $\beta_1, \ldots, \beta_m$ the numbers from* (1.4). *Then, for any non-negative numbers $x_j$, $d_{jj}$ and any $n_j \in \mathbb{N}_0$, $j = 1, \ldots, p$, the expectation*

$$E\left(\prod_{j=1}^{p} |g_{r+n_j}(x_j; d_{jj} + \frac{1}{2}b^j S b^{j'})|\right) \qquad (2.22)$$

*is bounded by $\prod_{\beta_\mu<1}\beta_\mu^{-r}$.*

**Proof.** Set $S/2 = Y^{1/2}CY^{1/2}$, where $Y = \mathrm{Diag}(Y_1, \ldots, Y_m)$ has independent gamma-distributed elements and $C$ with $c_{\mu\mu} \equiv 1$ is distributed independently of $Y$. From the F.t. (2.7c) we obtain

$$|h_{r+n,k}(x)| \leq B\Big(\frac{r-1+n}{2}, \frac{1}{2}\Big)/(2\pi) < 1 \qquad (r+n \geq 3/2),$$

and from (2.2), A.S.22.14.13 the bound $|h_{1,k}(x)| \leq 1$. Now it follows with the second series in (2.9a), $b'_\mu b_\nu = \sum_{j=1}^{p} b_{j\mu} b_{j\nu} = 0$, $1 \leq \mu < \nu \leq m$, and $\delta = \frac{1}{2}\sum d_{jj}$ that (2.22) is bounded by

$$\exp(-\delta)\int_{\mathbb{R}_+^m} E\Big(\sum_{k=0}^{\infty}\sum_{(k)}\prod_{j=1}^{p}|\frac{1}{2}d_{jj}+\frac{1}{2}b^j Y^{1/2}CY^{1/2}b^{j'}|^k/k_j!\Big) \times \prod_{\mu=1}^{m}\exp(-\frac{1}{2}b'_\mu b_\mu y_\mu)g_r(y_\mu)dy_\mu \tag{2.23}$$

with the expectation $E$ referring to a possibly singular distribution of $C$. After the substitution $y_\mu = (1 + b'_\mu b_\mu/2)y_\mu$ the inner sum is written as

$$\frac{1}{k!}\left(\sum_{j=1}^{p}|\frac{1}{2}d_{jj}+\widetilde{b}^j Y^{1/2}CY^{1/2}\widetilde{b}^{j'}|\right)^k \tag{2.24}$$

with

$$\widetilde{b}_{j\mu} = b_{j\mu}(2 + b'_\mu b_\mu)^{-1/2}, \quad |\widetilde{b}'_\mu \widetilde{b}_\mu| = |b'_\mu b_\mu|/(2 + b'_\mu b_\mu) < 1 \tag{2.25}$$

since $\beta_\mu = 1 + b'_\mu b_\mu > 0$. For a pure imaginary or real $B$ it is

$$\sum_{j=1}^{p}|\frac{1}{2}d_{jj}+\widetilde{b}^j Y^{1/2}CY^{1/2}\widetilde{b}^{j'}| \leq \delta + \sum_{\mu=1}^{m}|\widetilde{b}'_\mu \widetilde{b}_\mu|y_\mu \tag{2.26}$$

since $Y^{1/2}CY^{1/2} \geq 0$ and $\widetilde{b}'_\mu \widetilde{b}_\nu = 0$, $(\mu \neq \nu)$. If $B$ contains real and imaginary columns the quadratic form $Q_j = \widetilde{b}^j Y^{1/2}CY^{1/2}\widetilde{b}^{j'}$ can be written as $Q_j = Q_{j11} - Q_{j22} + 2iQ_{j12}$ with real forms. Then $Q_{j11}Q_{j22} - Q_{j12}^2 \geq 0$ follows from $Y^{1/2}CY^{1/2} \geq 0$ and therefore $|Q_j| \leq Q_{j11} + Q_{j22}$, which leads again to the bound (2.26).

Now it follows from (2.24), (2.25) and (2.26) that (2.23) is bounded by

$$\prod_{\mu=1}^{m}(1+\frac{1}{2}b'_\mu b_\mu)^{-r}\int_0^\infty \exp(|\widetilde{b}'_\mu \widetilde{b}_\mu|y_\mu)g_r(y_\mu)dy_\mu = \prod_{\beta_\mu<1}\beta_\mu^{-r}.$$

## 3. The Main Results

Even for the central general $\chi_p^2(q,R)$-density "simple" formulas are only available in a symbolic form (cf. also Blumenson and Miller (1963)). With the operator

$$\partial^{-1} = \mathrm{Diag}\Big((\frac{\partial}{\partial x_1})^{-1},\ldots,(\frac{\partial}{\partial x_p})^{-1}\Big),$$

any scale factors $v_j = w_j^2/2$ and $C = I_p - (WRW)^{-1}$, the series derived from the ch.f. (2.16) with $\Delta = 0$ and $k = 2$ can be written as

$$f_p(x_1,\ldots,x_p;2r,R) = |WRW|^{-r}\left(\prod_{j=1}^{p}\frac{v_j^r\exp(-v_jx_j)}{\Gamma(r)}\right)|I_p - CV\partial^{-1}|^{-r}\prod_{j=1}^{p}x_j^{r-1}.$$

Using the formula for Laplace transforms of zonal polynomials (Johnson and Kotz (1972), Chap. 38, Sec. 4) the operator is easily shown to coincide with the generalized hypergeometric function ${}_1F_0(r;CV\partial^{-1})$, $(2r \geq p)$. Zonal polynomials however do not seem to be useful for a direct representation of the $\chi_p^2(2r,R)$-distribution. With the central functions $F_{r,n}(x)$ in (2.8), $c$ from (2.14) and all principal minor arrays $C_J$ of $C = C_k$ in (2.11) with row and column indices $j \in J \subseteq \{1,\ldots,p\}$, $(J \neq \emptyset)$, we have the following expansions for the d.f.:

$$F_p(x_1,\ldots,x_p;2r,R) = c^r\sum_{n=0}^{\infty}\sum_{(n)}c(n_1,\ldots,n_p;r)\prod_{j=1}^{p}F_{r,n_j}(v_jx_j), \tag{3.1}$$

$$c(n_1,\ldots,n_p;r) = \frac{s^n}{\Gamma(r)}\sum\Gamma\Big(r+\sum n_J\Big)\prod_J(-|C_J|)^{n_J}/n_J!,\ \ s = \begin{cases}1, & \text{if } k=1,\\ -1, & \text{if } k=2,3,\end{cases}$$

where the last sum extends over the $n_J$ with $\sum_{j\in J}n_J = n_j$, $j = 1,\ldots,p$. The series with Laguerre polynomials ($f_{r,n} = g_{r+n}^{(n)}$, cf. (2.1), (2.5)) are absolutely convergent if $\|C\| < 1$. This can always be achieved by suitable factors $v_j$. The remaining series are always absolutely convergent. For more details see Royen (1991b) and (1992).

For the actual computation of probabilities simpler formulas are desirable. For the integration over a density separated arguments $x_j$ are preferable as given in the following, not necessarily real, mixture representation of $F_p$.

With the notations (2.10)$-$(2.15) we define the polynomials $P(Y)$ by

$$c\frac{P(Y;\widetilde{B},D)}{|I_p - \widetilde{B}\widetilde{B}'Y|} = \alpha^*\mathrm{tr}((I_p - CY)^{-1}CYDY) = \alpha^*\mathrm{tr}((\widetilde{B}(I_m - \widetilde{B}'Y\widetilde{B})^{-1}\widetilde{B}'YDY)$$

and

$$Q_n = \left\{\begin{array}{ll}P^n/n!, & \text{if } Y = -U\\ \mathrm{etr}(-\alpha B'DB)P^n/n!, & \text{if } Y = Z, Y = \Omega\end{array}\right\} = \sum q_n(N_1,\ldots,N_p)\prod_{j=1}^{p}y_j^{Nj}.$$

With the functions $F_{r,n}$ from (2.8) and the diagonal elements $d_{jj}$ of $D$ in (2.12) the following theorem holds:

**Theorem 3.1.** *The d.f. of the* $\chi_p^2(q,R,\Delta)$*-distribution with an* $m$*-factorial* $R = W^{-1}(I_p + BB')W^{-1}$ *is given by*

$$\begin{aligned} &F_p(x_1,\ldots,x_p;q,R,\Delta) \\ &= \sum_{n=0}^{\infty} E\Big(\sum q_n(N_1,\ldots,N_p)\prod_{j=1}^{p} F_{r,N_j}(v_jx_j, d_{jj} + \frac{1}{2}b^jS_{2(r+n)}b^{j'})\Big) \qquad (3.2) \\ &(r = q/2, v_j = w_j^2/2) \end{aligned}$$

*with the expectations referring to the* $W_m(2(r+n), I_m)$*-distributed Wishart matrices* $S_{2(r+n)}$.

**Corollary 1a.**

$$F_p(x_1,\ldots,x_p;q,R) = E\left(\prod_{j=1}^{p} G_r(v_jx_j, \frac{1}{2}b^jS_qb^{j'})\right). \qquad (3.3)$$

*If* $m = 2$ *we write*

$$\frac{1}{2}b^jS_qb^{j'} = b_{j1}^2Y_1 + b_{j2}^2Y_2 + 2b_{j1}b_{j2}(Y_1Y_2)^{1/2}\cos\Phi, \qquad (3.4)$$

*where the independent random variables* $Y_i$ *and* $\Phi$ *have the densities* $g_r(y_i)$ *and*

$$f_r(\phi) = (\sin^2\phi)^{r-1}/B(1/2, r-1/2), \quad (0 \le \phi \le \pi,\ r \ge 1). \qquad (3.5)$$

**Corollary 1b.** *If* $R$ *is two-factorial then the central d.f.* $F$ *is given by any of the following representations* (3.6) − (3.8)*:*

$$\begin{aligned} &\sum_{n=0}^{\infty}\binom{r-1+n}{n}\int_0^{\infty}\int_0^{\infty}\Big((2n)!\sum_{(2n)}\prod_{j=1}^{p} G_{r+n_j}(v_jx_j, b_{j1}^2y_1 + b_{j2}^2y_2)(b_{j1}b_{j2})^{n_j}/n_j!\Big) \\ &\qquad\qquad \times g_{r+n}(y_1)g_{r+n}(y_2)dy_1dy_2. \qquad (3.6) \end{aligned}$$

With $C = B(\mathrm{Diag}(\beta_1,\beta_2))^{-1}B'$, $\mathrm{tr}(C) < 1$ (always satisfied if at least one $\beta_\mu = 1 + b'_\mu b_\mu < 1$) and $\gamma_{ij} = -(c_{ii}c_{jj} - c_{ij}^2)$, $(i \neq j)$:

$$\begin{aligned} &(\beta_1\beta_2)^{-r}(\Gamma(r))^{-1}\sum_{n=0}^{\infty}\Gamma(r+n)/(1-\mathrm{tr}(C))^{r+n} \qquad (3.7) \\ &\times\int_0^{\infty}\Big(\sum_{(2n)}\Big(\sum_{n_{\cdot j}=n_j}\prod_{i<j}\gamma_{ij}^{n_{ij}}/n_{ij}!\Big)\prod_{j=1}^{p} G_{r+n_j}\Big(v_jx_j, c_{jj}y/(1-\mathrm{tr}(C))\Big)\Big)g_{r+n}(y)dy. \end{aligned}$$

$$c^r\sum_{n=0}^{\infty}\sum_{(n)} c(n_1,\ldots,n_p;r)\prod_{j=1}^{p} F_{r,n_j}(v_jx_j) \qquad (3.8)$$

with

$$c = c_k = \begin{cases} 1, & \text{if } F_{r,n} = G_{r+n}^{(n)},\ k=1, \\ (\beta_1\beta_2)^{-1}, & \text{if } F_{r,n} = G_{r+n},\ k=2, \\ \Big((1+\beta_1)(1+\beta_2)/4\Big)^{-1}, & \text{if } F_{r,n} = H_{r,n},\ k=3, \end{cases}$$

and

$$c(n_1,\dots,n_p;r) = (\Gamma(r))^{-1} \sum_{n_{\cdot j}=n_j} \Gamma(r+\frac{1}{2}(n+\sum_{j=1}^{p} n_{jj})) \prod_{1\le i\le j\le p} \gamma_{ij}^{n_{ij}}/n_{ij}!,$$

$$\gamma_{jj} = \begin{cases} -c_{jj}, & \text{if } k=1, \\ c_{jj}, & \text{if } k=2,3, \end{cases} \quad \gamma_{ij} \text{ as in (3.7) } (i\ne j), \quad C=(c_{ij}) \text{ from (2.11)}.$$

For the absolute convergence of (3.8) $\|B'B\| < 1$ is supposed only if $F_{r,n} = G_{r+n}^{(n)}$. Also, the inequalities $\|C_3\| < \|C_k\|$, $(k=1,2)$ and $\|C_3\| < 1$ hold with any scale matrix $W$. Since $\max(\varepsilon^{-n}\prod_{j=1}^{p} G_{r+n_j}(x_j) | \sum n_j = n) \to 0$ for every $\varepsilon > 0$ the condition $\|C_2\| < 1$ is not necessary for the convergence of (3.8) with $F_{r,n} = G_{r+n}$ (Royen (1991b, 1992)).

**Corollary 2a.** *If $R = W^{-1}(I_p + bb')W^{-1}$ is one-factorial then*

$$\begin{aligned} &F_p(x_1,\dots,x_p;q,R,\Delta) \\ &= d^* \sum_{n=0}^{\infty} \int_0^{\infty} \Big(\sum_{(2n)} d(n_1,\dots,n_p) \prod_{j=1}^{p} F_{r,n_j}(v_jx_j, d_{jj}+b_j^2 y) b_j^{n_j}\Big) g_{r+n}(y)dy \end{aligned} \tag{3.9}$$

*with*

$$d^* = d_k^* = \begin{cases} 1, & \text{if } k=1, \\ \exp(-\alpha b'Db), & \text{if } k=2,3 \ (\alpha \text{ from (2.13)},\ D=(d_{ij}) \text{ from (2.12)}), \end{cases} \tag{3.10}$$

$$\alpha b'Db = \begin{cases} \frac{1}{2}b'W\Delta Wb, & \text{if } k=1, \\ \frac{1}{2}\beta^{-2}b'W\Delta Wb, & \text{if } k=2, \quad (\beta = 1+b'b) \\ \frac{1}{2}(1+\beta)^{-2}b'W\Delta Wb, & \text{if } k=3, \end{cases} \tag{3.11}$$

*and*

$$d(n_1,\dots,n_p) = \alpha^n \sum_{\substack{n_{\cdot j}+n_{jj}=n_j \\ j=1,\dots,p}} \prod_{1\le i\le j\le p} d_{ij}'^{\,n_{ij}}/n_{ij}!,\ d_{ij}' = \begin{cases} d_{jj}, & \text{if } i=j, \\ 2d_{ij}, & \text{if } i\ne j. \end{cases} \tag{3.12}$$

For identical correlations see (6.1). In particular with $\Delta = (\delta_i\delta_j)$, $D = (d_id_j)$ and $f_r$ from (3.5) we obtain

**Corollary 2b.** *If* $R$ *is one-factorial and* $\operatorname{rank}(\Delta) = 1$ *then*

$$\begin{aligned} &F_p(x_1, \ldots, x_p; q, R, \Delta) \\ &= \int_0^\infty \int_0^\pi \prod_{j=1}^p G_r(v_j x_j, v_j \delta_j^2 + b_j^2 y + 2b_j \delta_j \sqrt{v_j y} \cos(\phi)) f_r(\phi) d\phi g_r(y) dy \qquad (3.13) \\ &= d^* \sum_{n=0}^\infty \frac{\alpha^n}{n!} \int_0^\infty \Big((2n)! \sum_{(2n)} \prod_{j=1}^p F_{r,n_j}(v_j x_j, d_j^2 + b_j^2 y)(b_j d_j)^{n_j}/n_j!\Big) g_{r+n}(y) dy. \qquad (3.14) \end{aligned}$$

**Proof of Theorem 3.1.** The L.t. of

$$E\left(\prod_{j=1}^p v_j g_{r+n_j}(v_j x_j, d_{jj} + \frac{1}{2} b^j S b^{j'})\right) \qquad (3.15)$$

is given by

$$\left(\prod_{j=1}^p z_j^{r+n} \exp(-d_{jj} u_j)\right) E\left(\operatorname{etr}(-\frac{1}{2} S B' U B)\right) \qquad (3.16)$$

($W_m(\nu, I_m)$-distributed $S$, $\nu = 2(r+n)$, $z_j = (1 + t_j/v_j)^{-1}, u_j = 1 - z_j$).

The change of the order of integration is justified by Lemma 2.2 for all $t_j > 0$. Since $\operatorname{Re}(I_m + B'UB) > 0$ (cf. Anderson (1984), Sec. 7.3, (11)) the expectation in (3.16) is equal to

$$|I_m + B'UB|^{-\nu/2} = |I_p + BB'U|^{-\nu/2}.$$

Since $u^n$, $\omega^n$ are finite linear combinations of powers of $z$, the identity of (3.15), (3.16) also holds if the pairs $(g_{r+n}; z^{r+n})$ are replaced by $(g_{r+n}^{(n)}; z^r u^n)$ or $(h_{r,n}; z^r \omega^n)$.

For $q = 2r > 1$ it follows from the central case in (2.16) with $k = 3$ that the F.t. $\hat{f}_p(\cdot; q, R)$ belongs to the Hilbert space $\mathcal{L}^2(\mathbb{R}^p)$ since $\|\widetilde{B}_3 \widetilde{B}_3'\| < 1$. This also holds for $\hat{f}_p(\cdot; q, R, \Delta)$ since the exponent $\operatorname{tr}(-iT(I - 2iRT)^{-1}\Delta)$ in (1.2) is bounded according to Lemma 2.1. After having expanded the last factor of (2.16) into the exponential series it follows with the corresponding partial sums $s_n$ that $\|s_n - \hat{f}_p\|_2 \to 0$ and

$$\left|\int_{\mathcal{A}} (s_n - f_p) dx\right| < \Big(\int_{\mathcal{A}} dx\Big)^{1/2} \|s_n - f_p\|_2 \to 0$$

for any bounded region $\mathcal{A} \subseteq \mathbb{R}_+^p$. In particular with rectangular regions $\mathcal{A} = \mathrm{X}_{j=1}^p (0, x_j)$ this entails (3.2).

**Proof of Corollary 1b.** From (3.3), (3.4), (3.5) we obtain with the identities

$$G_r(x,\lambda_1+\lambda_2)=\exp(-\lambda_2)\sum_n G_{r+n}(x,\lambda_1)\lambda_2^n/n!,\ (\lambda_2=2b_{j1}b_{j2}(y_1y_2)^{1/2}\cos(\phi)),$$

$$\int_0^\pi(\cos^2\phi)^n f_r(\phi)d\phi=\frac{\Gamma(r)\Gamma(n+1/2)}{\Gamma(1/2)\Gamma(r+n)}$$

and

$$\begin{aligned}&2^{2n}\frac{\Gamma(r)\Gamma(n+1/2)}{\Gamma(1/2)\Gamma(r+n)}(\Gamma(r))^{-2}(y_1y_2)^{r-1+n}\\ =&\binom{r-1+n}{n}(2n)!(\Gamma(r+n))^{-2}(y_1y_2)^{r-1+n}\end{aligned}$$

the series (3.6). Because of $|G_{r+n}(x,\delta)|\le\exp(\max(0,-\delta))G_{r+n}(x),(\delta\in\mathbb{R}$, cf. A.S.9.1.62) and $\max(\varepsilon^{-n}\prod_{j=1}^p G_{r+n_j}(x_j)|\sum n_j=n)\to 0$ for every $\varepsilon>0$, the series (3.6) is majorized by

$$\sum_n\binom{r-1+n}{n}(\varepsilon\sum_j|b_{j1}b_{j2}|)^{2n}\prod_{b'_\mu b_\mu<0}(1+b'_\mu b_\mu)^{-r-n}.$$

The series (3.7) follows by inversion from the L.t.

$$(\prod_{j=1}^p z_j^r)|I-CY|^{-r}=(\prod_{j=1}^p z_j^r)\sum_n\binom{r-1+n}{n}Q^n(1-L)^{-r-n}$$

$$(L=\sum_j c_{jj}y_j,\quad Q=\sum_{i<j}\gamma_{ij}y_iy_j,\quad Y=Z)$$

with

$$(\prod_{j=1}^p z_j^{r+n_j})(1-L)^{-r-n}=\int_0^\infty(\prod_{j=1}^p z_j^{r+n_j})\exp(yL)g_{r+n}(y)dy\quad(\mathrm{Re}(L)<1)$$

and it is majorized by

$$\sum_n\binom{r-1+n}{n}(\varepsilon^2\sum_{i<j}|\gamma_{ij}|)^n(1-\sum_{c_{jj}>0}c_{jj})^{-r-n}.$$

Finally (3.8) follows from the general expansion (3.1) with (2.19).

**Proof of Corollary 2a and 2b.** The identity (3.9) follows from Lemma 2.1 with $m=1$ and (2.18). In particular with $d_{ij}=d_id_j$ and $b'YDYb=$

$(\sum_{j=1}^{p} b_j d_j y_j)^2$ we obtain (3.14). The identity of (3.13) with (3.14) is verified with $F_{r,n} = G_{r+n}^{(n)}$ by integration over $\phi$ using the relation

$$G_r(x, \lambda_1 + \lambda_2) = \sum_{n=0}^{\infty} G_{r+n}^{(n)}(x, \lambda_1)(-\lambda_2)^n/n!, \quad (\lambda_2 = 2b_j\delta_j\sqrt{v_j y}\cos(\phi))$$

or directly by the L.t. of the corresponding density.

If $R$ is not $m$-factorial ($m \leq 2$) then an approximation of $R$ by an $m$-factorial $R_0$ may be useful. Let $H = (h_{ij})$ be the difference $C - C_0$ of the corresponding matrices $C$ and $C_0$ with $\operatorname{rank}(C_0) = m$ (e.g. $C_0 = B_0 B_0' = W_0 R_0 W_0 - I$, $C = W_0 R W_0 - I$, cf. (2.11)). The Taylor polynomial $T_2(H; C_0)$ of 2nd degree of the Taylor expansion with center $C_0$ of the central ch.f. (2.16) is given by

$$c^r \left(\prod_{j=1}^{p} z_j^r\right) \left(|I - C_0 Y|^{-r} + r|I - C_0 Y|^{-r-1}(L+Q) + \binom{r+1}{2}|I - C_0 Y|^{-r-2} L^2\right) \tag{3.17}$$

with

$$L = \sum_{i,k=1}^{p} (-1)^{i+k} |(I - C_0 Y)_{i,k}| h_{ik} y_k,$$
$$Q = -\sum_{i<j,k<\ell} (-1)^{i+j+k+\ell} |(I - C_0 Y)_{ij,k\ell}| (h_{ik} h_{j\ell} - h_{i\ell} h_{jk}) y_k y_\ell,$$

where $(I - C_0 Y)_{ij,k\ell}$ is obtained from $I - C_0 Y$ by deleting the rows $i, j$ and the columns $k, \ell$. The inversion of (3.17), followed by integration, is a finite linear combination of terms of the type

$$E\left(\prod_{j=1}^{p} F_{r,N_j}(v_{0j} x_j, \frac{1}{2} b_0^j S_{2(r+n)} b_0^{j'})\right), \quad (n = 0, 1, 2), \tag{3.18}$$

which is expected to provide a good approximation to the d.f. $F_p(\cdot; q, R)$ if the deviations $h_{ij}$ are sufficiently small. For the corresponding Taylor expansion of the multivariate normal distribution see Royen (1987).

If there is only one element $h_{ij} = h_{ji} \neq 0$ (cf. end of Sec. 5) then the formal Taylor expansion of $\hat{f}$ is

$$c^r \left(\prod_{j=1}^{p} z_j^r\right) \sum_{N=0}^{\infty} \sum_{n_1 + 2n_2 = N} \frac{\Gamma(r + n_1 + n_2)}{\Gamma(r) n_1! n_2!} \frac{L^{n_1} Q^{n_2}}{|I - C_0 Y|^{r + n_1 + n_2}} \tag{3.19}$$

with simplified terms $L$ and $Q$. For approximations only Taylor polynomials of a low degree are applied with small values of $|h_{ij}|$. Nevertheless, conditions for the convergence of the inverted expansions are established in the following theorem.

**Theorem 3.2.** *Let $R$ and $R_0$ be non-singular $p \times p$-correlation matrices where $R_0 = W_0^{-2} + AA'$ is $m$-factorial with $m < p-1$. Furthermore let be $T = \mathrm{Diag}(t_1, \ldots, t_p)$ with real $t_j$, $Z = (I_p - 2iTW_0^{-2})^{-1}$ and $U = I - Z$.*
(a) *Let be $K = R - R_0$, $C = W_0RW_0 - I$, $C_0 = W_0R_0W_0 - I$ and $H = W_0KW_0$. If $R_0 - K > 0$, then the* ch.f. *$\hat{f}_p(t_1, \ldots, t_p; q, R) = (\prod_{j=1}^p z_j^r)|I_p + CU|^{-r}$ has a uniformly abs. convergent expansion*

$$\left(\prod_{j=1}^p z_j^r\right) |I_p + C_0U|^{-r} \sum_{n=0}^{\infty} \psi_n(H; U, C_0)$$

*with polynomials $\psi_n(H)$ of degree $n$ and it holds true for the functions $s_N$, obtained by Fourier inversion of the partial sums, that*

$$\|s_N - f_p\|_2 \to 0 \quad \text{and} \quad \|s_N - f_p\|_1 \to 0, \qquad (q = 2r > 1). \tag{3.20}$$

(b) *Let be $K = R_0^{-1} - R^{-1}$, $C = I - (W_0RW_0)^{-1}$, $C_0 = I - (W_0R_0W_0)^{-1}$ and $H = W_0^{-1}KW_0^{-1}$. If $R_0^{-1} + K > 0$, then $\hat{f}_p$ has the uniformly abs. convergent expansion*

$$|I - C|^r \left(\prod_{j=1}^p z_j^r\right) |I - C_0Z|^{-r} \sum_{n=0}^{\infty} \psi_n(-H; Z, -C_0)$$

*and* (3.20) *holds again.*

**Proof.** Only (b) is shown, since the proof of (a) is very similar.

We have $|I - CZ| = |I - C_0Z|\ |I - H(Z^{-1} - C_0)^{-1}| = |I - C_0Z|\ |I - K(R_0^{-1} - 2iT)^{-1}|$. For any char. root $\lambda \neq 0$ of $K(R_0^{-1} - 2iT)^{-1}$ we obtain with $\lambda^{-1} = \rho \exp(i\phi)$ the equation

$$|R_0^{-1} - \rho\cos(\phi)K - i(2T + \rho\sin(\phi)K)| = 0. \tag{3.21}$$

Because $R_0^{-1} \pm K > 0$ there exists an $\varepsilon > 0$ with $R_0^{-1} - \rho\cos(\phi)K > 0$ for all $\rho < (1-\varepsilon)^{-1}$. Thus, $\rho \geq (1-\varepsilon)^{-1}$ in (3.21) and $H(Z^{-1} - C_0)^{-1}$ has a spectral radius $|\lambda|_{\max} < 1$.

With $C_3 = I - 2(I + W_0R_0W_0)^{-1}$ (cf. (2.11)) it follows from (2.16) with $\Delta = 0$ that $|I - C_0|\ |I - C_0Z|^{-1} = |I - C_3|\ |I - C_3\Omega|^{-1}$. Since $\|C_3\Omega\| \leq \|C_3\| < 1, |I - C_0Z|^{-1}$ is uniformly bounded and, besides, we have $\int_{\mathbb{R}^p} \prod_{j=1}^p |z_j|^{2r} dt_j < \infty$ for $r > 1/2$. Thus, by Plancherel's theorem, we get $\|s_N - f_p\|_2 \to 0$ and $\int_{\mathcal{A}} |s_N - f_p| dx \to 0$ by Cauchy's inequality for any bounded $\mathcal{A} \subseteq \mathbb{R}_+^p$. Now, for a sufficiently small $\varepsilon$, $\|\exp(\varepsilon \sum x_j/2)(s_N - f_p)\|_2 \to 0$ can be shown as before, replacing $T$ by $T - (i/2)\varepsilon I$ and $Z^{-1}$ by $Z_\varepsilon^{-1} = I - \varepsilon W_0^{-2} - 2iTW_0^{-2}$. Then Cauchy's inequality implies $\|s_N - f_p\|_1 \to 0$.

## 4. Central Multivariate Chi-Square Distributions Related to Non-Central Ones with One-Factorial Correlation Matrices

Miller and Sackrowitz (1967) have found a close relation between a $\chi_p^2(q, R, \Delta)$ -distribution with a non-centrality matrix $\Delta = q\mu\mu'$ of rank 1 and a central $(p+1)$-dimensional chi-square distribution. A similar relation is given here in Theorem 4.1. The latter provides, in conjunction with (3.13), the result (4.3) in Theorem 4.2 concerning $\chi_{p+1}^2(q, R)$-distributions with a $(p+1) \times (p+1)$-correlation matrix

$$R = \mathrm{Diag}(w_1^{-2}, \ldots, w_p^{-2}, 0) + AA' > 0, \tag{4.1}$$

where $\mathrm{rank}(AA') = 2$. If the r.v. $(Y_1, \ldots, Y_{p+1})'$ has a $N_{p+1}(0, R)$-distribution with a one-factorial $p \times p$-correlation matrix $R^*$ of the conditional distribution of $(Y_1, \ldots, Y_p)' | Y_{p+1} = y$ then $R$ is of the type (4.1) with $A$ as given in Lemma 4.1.

**Theorem 4.1.** *Let be $R$ a non-singular $p \times p$-correlation matrix and $M$ any real $p \times q$-matrix of rank 1 with the columns $y_j\mu = y_j(\mu_1, \ldots, \mu_p)'$. With $x_{p+1} = \sum_{j=1}^q y_j^2$, the non-centrality matrix $\Delta = MM' = x_{p+1}\mu\mu'$ and*

$$\Sigma = \left(\begin{array}{c|c} R + \mu\mu' & \mu \\ \hline \mu' & 1 \end{array}\right)$$

*the $\chi_p^2(q, R, \Delta)$-density is given by*

$$f_p(x_1, \ldots, x_p; q, R, \Delta) = f_{p+1}(x_1, \ldots, x_{p+1}; q, \Sigma) / \Big(\frac{1}{2} g_r(x_{p+1}/2)\Big).$$

**Proof.** Note that $R > 0$ implies $\Sigma > 0$. Let $(Y_{1j}, \ldots, Y_{p+1j})'$ $(j = 1, \ldots, q)$ be independent $N(0, \Sigma)$-distributed column vectors and $X_i = \sum_{j=1}^q Y_{ij}^2$. The conditional distribution of $(Y_{1j}, \ldots, Y_{pj})' | Y_{p+1,j} = y_j$ is a $N(y_j\mu, R)$-distribution. With the joint density $f$ of $X_1, \ldots, X_p, Y_{p+1,1}, \ldots, Y_{p+1,q}$ it follows that

$$(2\pi)^{-q/2} \exp(-x_{p+1}/2) f_p(x_1, \ldots, x_p; q, R, \Delta) = f(x_1, \ldots, x_p, y_1, \ldots, y_q).$$

Since the left hand side depends on $y_1, \ldots, y_q$ only by $x_{p+1}$ (cf. (1.2)) the asserted result is obtained by integration over the sphere $\sum_j y_j^2 = x_{p+1}$.

The following lemma is used for Theorem 4.2:

**Lemma 4.1.** *In the representation of a correlation matrix $R = (r_{ij})$ of the type (4.1) the matrix $A = (a_{ij})$ can be chosen as a $(p+1) \times 2$-matrix with*

$$a_{p+1,1} = 0, \quad a_{p+1,2} = 1, \quad a_{i,2} = r_{i,p+1}, \quad (i = 1, \ldots, p).$$

**Proof.** The assumption $w_i^{-2} = 1 - r_{i,p+1}^2$ for all $i = 1, \ldots, p$ implies for all the determinants of the $3 \times 3$ submatrices with row and column indices $i < j \leq p$

the vanishing of $r_{ij} - r_{i,p+1}r_{j,p+1}$ since $\operatorname{rank}(AA') < 3$. But this would entail $\operatorname{rank}(AA') = 1$. Thus, possibly after a suitable permutation of the first $p$ rows and columns of $AA'$, we assume

$$a_{p,1}^2 = 1 - w_p^{-2} - r_{p,p+1}^2 \neq 0, \quad (a_{p,1}^2 < 0 \quad \text{admissible}).$$

Now we obtain for the non-singular lower right $2 \times 2$-submatrix in $AA'$ the decomposition

$$\begin{pmatrix} a_{p,1}^2 + r_{p,p+1}^2 & r_{p,p+1} \\ r_{p,p+1} & 1 \end{pmatrix} = \begin{pmatrix} a_{p,1} & r_{p,p+1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_{p,1} & 0 \\ r_{p,p+1} & 1 \end{pmatrix}$$

and from $\operatorname{rank}(AA') = 2$ it follows easily that $A$ has the asserted elements and

$$a_{i1} = (r_{i,p} - r_{i,p+1}r_{p,p+1})/a_{p,1}, \quad (i = 1, \ldots, p-1).$$

With $w_{p+1} = 1$, the $(p+1)$-column $e_{p+1} = (0, \ldots, 0, 1)'$ and the $(p+1) \times 2$-matrix $B = WA = (b_{j\ell})$ we get from (4.1):

$$WRW = I_{p+1} - e_{p+1}e_{p+1}' + BB', \quad (b_{p+1,1} = 0, b_{p+1,2} = 1). \tag{4.2}$$

**Theorem 4.2.** *For a correlation matrix of the type* (4.1) *the* $\chi_{p+1}^2(q, R)$*-d.f. is given by*

$$\begin{aligned} &F_{p+1}(x_1, \ldots, x_{p+1}; q, R) \\ &= \int_0^{\frac{1}{2}x_{p+1}} \int_0^\infty \int_0^\pi \Big( \prod_{j=1}^p G_r(v_j x_j, b_{j1}^2 y_1 + b_{j2}^2 y_2 + 2b_{j1}b_{j2}(y_1 y_2)^{1/2}\cos(\phi)) \Big) \\ &\qquad \times f_r(\phi) g_r(y_1) g_r(y_2) \, d\phi \, dy_1 \, dy_2, \qquad (4.3) \\ &(v_j = w_j^2/2,\ r = q/2, w_j^2, b_{j\ell} \text{ from } (4.2),\ f_r \text{ from } (3.5)). \end{aligned}$$

**Proof.** For the density corresponding to (4.3) we have to verify

$$\begin{aligned} &f_{p+1}(x_1, \ldots, x_{p+1}; q, R) \\ &= \frac{1}{2} g_r\Big(\frac{1}{2}x_{p+1}\Big) \int_0^\infty \int_0^\pi \Big( \prod_{j=1}^p v_j g_r(v_j x_j, b_{j1}^2 y + \frac{1}{2} b_{j2}^2 x_{p+1} + b_{j1}b_{j2}(2x_{p+1}y)^{1/2}\cos(\phi)) \Big) \\ &\qquad \times f_r(\phi) g_r(y) \, d\phi \, dy. \qquad (4.4) \end{aligned}$$

With

$$\sigma_j^2 = (1 - a_{j2}^2)^{-1} \quad (j = 1, \ldots, p,\ a_{j2} = r_{j,p+1}), \qquad \sigma_{p+1}^2 = 1$$

we define

$$\Sigma = \mathrm{Diag}(\sigma_1, \ldots, \sigma_{p+1}) R \mathrm{Diag}(\sigma_1, \ldots, \sigma_{p+1}) = \left(\begin{array}{c|c} \widetilde{R} + \mu\mu' & \mu \\ \hline \mu' & 1 \end{array}\right)$$

with $\mu_j = a_{j2}\sigma_j$ $(j = 1, \ldots, p)$ and the $p \times p$-correlation matrix

$$\widetilde{R} = (\widetilde{r}_{ij}), \qquad \widetilde{r}_{ij} = \sigma_i a_{i1} a_{j1} \sigma_j, \quad (i \neq j).$$

Besides, we set

$$\Delta = (\delta_i \delta_j) \quad \text{with} \quad \delta_j = x_{p+1}^{1/2} \mu_j.$$

From the one-factorial $\widetilde{R}$ the following quantities $(j = 1, \ldots, p)$ are derived:

$$\begin{aligned}
\widetilde{v}_j &= \frac{1}{2}\widetilde{w}_j^2 = \frac{1}{2}(1 - a_{j1}^2 \sigma_j^2)^{-1} = \frac{1}{2}\frac{1 - a_{j2}^2}{1 - a_{j1}^2 - a_{j2}^2} = \frac{1}{2} w_j^2 / \sigma_j^2 = v_j / \sigma_j^2, \\
\widetilde{b}_j &= \widetilde{w}_j a_{j1} \sigma_j = w_j a_{j1} = b_{j1}, \\
d_j &= \delta_j \widetilde{v}_j^{1/2} = a_{j2}(v_j x_{p+1})^{1/2} = b_{j2}\Big(\frac{1}{2}x_{p+1}\Big)^{1/2}.
\end{aligned}$$

That, indeed, $\widetilde{R} > 0$ is recognized from the following implications:

$$\begin{aligned}
WRW > 0 &\Rightarrow I_{p+1} + BB' > 0 \Rightarrow I_2 + B'B > 0 \\
&\Rightarrow 1 + \sum_{j=1}^{p+1} b_{j1}^2 = 1 + \sum_{j=1}^{p} \widetilde{b}_j^2 \;=\; |\widetilde{W}\widetilde{R}\widetilde{W}| > 0.
\end{aligned}$$

Now we find with Theorem 4.1 that

$$\begin{aligned}
& f_{p+1}(x_1, \ldots, x_{p+1}; q, R) \\
&= \left(\prod_{j=1}^{p} \sigma_j^2\right) f_p(\sigma_1^2 x_1, \ldots, \sigma_p^2 x_p; q, \widetilde{R}, \Delta) g_r(x_{p+1}/2)/2
\end{aligned}$$

and according to (3.13) the factor $f_p$ is given by

$$\int_0^\infty \int_0^\pi \Big(\prod_{j=1}^{p} \widetilde{v}_j g_r(\widetilde{v}_j \sigma_j^2 x_j, b_{j1}^2 y + \frac{1}{2} b_{j2}^2 x_{p+1} + b_{j1} b_{j2} (2x_{p+1} y)^{1/2} \cos(\phi))\Big) \\ \times f_r(\phi) g_r(y) d\phi dy,$$

which yields (4.4).

A different verification of (4.4) by the L.t. is also possible.

## 5. The Four-Variate Chi-Square Distribution

Any irreducible $p \times p$-correlation matrix $R = (r_{ij})$ can be mapped to a connected graph $\mathbf{G}(R)$ with the $p$ vertices $1, \dots, p$ and containing the edge $[i,j]$ iff $r_{ij} \neq 0 \quad (i \neq j)$. Let $\mathcal{G}_{i_1,\dots,i_p}$ denote the class of non-singular correlation matrices corresponding to a graph with the vertex degrees $i_1 \geq \cdots \geq i_p$. In particular for $p = 4$ the classes $\mathcal{G}_{3111}$ and $\mathcal{G}_{2211}$ correspond to the spanning trees. Let $\mathcal{F}_m$ $(m < p)$ denote the class of the non-singular irreducible $m$-factorial $p \times p$-correlation matrices $R$ and $\mathcal{F}_m^0$ $(m < p-1)$ the set of the ($(m+1)$-factorial) $R$ allowing a representation $R = D + AA'$ with a diagonal $D \geq 0$ (but not $D > 0$) and $\operatorname{rank}(AA') = m$. Finally let $\mathcal{C}_0$ be the class of $4 \times 4$-correlation matrices $R$ with at least one vanishing element $r^{ij}$ in $R^{-1}$.

For an $N(0, R)$-distributed r.v. $(U_1, \dots, U_4)$ we obtain for the conditional covariances $\sigma_{ij|k\ell} = \operatorname{Cov}(U_i, U_j | u_k, u_\ell)$ from

$$r^{ij} = -|R|^{-1}(r_{ij} - r_{ij}r_{k\ell}^2 - r_{ik}r_{jk} - r_{i\ell}r_{j\ell} + r_{ik}r_{j\ell}r_{k\ell} + r_{i\ell}r_{jk}r_{k\ell}), \quad (i \neq j) \tag{5.1}$$

the relation

$$\sigma_{ij|k\ell} = -|R|r^{ij}/(1 - r_{k\ell}^2), \quad (i, j, k, \ell \quad \text{any permutation of} \quad 1, 2, 3, 4). \tag{5.2}$$

Thus, $\mathcal{C}_0$ corresponds to the class of $N(0, R)$-distributions with at least one pair of conditionally independent components, given the complementary pair. Consequently for any $R \in \mathcal{C}_0$ the $\chi_4^2(q, R)$-d.f. is given by

$$\begin{aligned}
&F_4(x_1, \dots, x_4; 2r, R) \\
&= ((1 - r_{k\ell}^2)^r \sqrt{\pi}\Gamma(r)\Gamma(r - 1/2))^{-1} \int_0^{\frac{1}{2}x_k} \int_0^{\frac{1}{2}x_\ell} \int_0^{\pi} \Big( \prod_{m=i,j} G_r(\tfrac{1}{2}x_m/\sigma_m^2, Q_m) \Big) \\
&\times (y_k y_\ell \sin^2\phi)^{r-1} \exp(-(y_k + y_\ell - 2r_{k\ell}(y_k y_\ell)^{1/2}\cos(\phi))/(1 - r_{k\ell}^2)) d\phi\, dy_\ell\, dy_k
\end{aligned} \tag{5.3}$$

with

$$\sigma_m^2 = \sigma_{mm|k\ell} = 1 - (r_{mk}^2 + r_{m\ell}^2 - 2r_{k\ell}r_{mk}r_{m\ell})/(1 - r_{k\ell}^2)$$

and

$$\begin{aligned}
Q_m = {} & (\sigma_m(1 - r_{k\ell}^2))^{-2}((r_{mk} - r_{m\ell}r_{k\ell})^2 y_k + (r_{m\ell} - r_{mk}r_{k\ell})^2 y_\ell \\
& + 2(r_{mk} - r_{m\ell}r_{k\ell})(r_{m\ell} - r_{mk}r_{k\ell})(y_k y_\ell)^{1/2}\cos(\phi)), \quad (m = i, j).
\end{aligned}$$

Let $\phi$ map $R$ to its standardized inverse, i.e. to the correlation matrix $Q = (q_{ij}) = (WRW)^{-1}$ with $W^2 = \operatorname{Diag}(r^{11}, \dots, r^{pp})$. It is $R = \phi(Q)$ and $\mathcal{F}_m^- \overset{\text{def}}{=} \phi(\mathcal{F}_m) = \mathcal{F}_m$. The classification in Table 5.1 of the non-singular irreducible $4 \times 4$-correlation matrices together with the criterion in Theorem 5.1

may be helpful for the computation of a $\chi^2_4(q,R)$-d.f.. The classes $\mathcal{F}^*_m \subset \mathcal{F}_m$ $(m=2,3)$ are by definition the relative complements of $\mathcal{F}^0_1 \cup \mathcal{G}_{3111} \cup \mathcal{G}_{2222}$ and $\mathcal{F}^0_2 \cup \mathcal{G}_{2211}$ respectively.

Table 5.1. Classification of irreducible $4 \times 4$-correlation matrices

| class | comment | d.f. $F_4(x_1,\dots,x_4;2r,R)$ |
|---|---|---|
| $\mathcal{F}_1$ | | $\int_0^\infty \Big(\prod_{j=1}^4 G_r(v_j x_j, b_j^2 y)\Big) g_r(y)dy$ |
| $\mathcal{F}_2$ containing | | |
| $\mathcal{F}^0_1$ | | $\int_0^{\frac{1}{2}x_\ell}\Big(\prod_{j\neq\ell} G_r\big(\frac{1}{2}x_j/(1-r^2_{j\ell}), r^2_{j\ell}y/(1-r^2_{j\ell})\big)\Big) g_r(y)dy$ |
| $\mathcal{G}_{3111}$ | $\mathcal{G}^-_{3111} = \mathcal{F}^0_1$ | $\int_0^\infty \Big(\prod_{j\neq\ell} G_r(\frac{1}{2}x_j, -r^2_{j\ell}y)\Big)(\frac{x_\ell}{2y})^{r/2} J_r((2x_\ell y)^{1/2}) g_r(y)dy$ (Royen (1994)) |
| $\mathcal{G}_{2222}$ | $\mathcal{G}^-_{2222} \subset \mathcal{C}_0$ | For densities with $R \in \mathcal{G}^-_{2222}$ see (4.2) in Miller and Sackrowitz (1967) |
| $\mathcal{F}^*_2$ | | See Corollary 1.b of Theorem 3.1, Theorem 4.2, Theorem 5.1 and Remark 1 below. |
| $\mathcal{F}_3$ containing | | |
| $\mathcal{F}^0_2$ | | See (5.3) or Theorem 4.2 |
| | $\mathcal{G}^-_{2211} \subset \mathcal{F}^0_2 \subset \mathcal{C}_0$ | |
| $\mathcal{G}_{2211}$ | | For bivariate integrals or series belonging to $\mathcal{G}_{2211}$, $\mathcal{G}^-_{2211}$ and more generally for correlation matrices $R$ or $Q$ corresponding to spanning trees see Royen (1994). For densities $f_p$ with a tridiagonal $R^{-1}$ see also Blumenson and Miller (1963). |
| $\mathcal{F}^*_3$ | | General series expansions (3.1) or Taylor approximation using (3.19) |

In the representation of an irreducible $R = D + aa' \in \mathcal{F}^0_1$ there is exactly one index $\ell$ with $a^2_\ell = 1$. It follows with $r_{ij} = a_i a_j$ from (5.1) that $r^{ij} = 0$ $(i \neq j,$ $i,j \neq \ell)$ and, besides, $r^{i\ell} \neq 0$ because of the irreducibility of $R^{-1}$. Therefore $\phi(\mathcal{F}^0_1) \subset \mathcal{G}_{3111}$, and $\mathcal{G}^-_{3111} \subset \mathcal{F}^0_1$ is also verified by (5.1). The remaining inclusions of Table 5.1 are verified in the proof of Theorem 5.1.

**Remark 1.** For any $p \geq 3$ the class $\mathcal{F}_2$ contains the class of the "formally one-factorial" $R = D + aa'$ with a non-singular indefinite $D = \mathrm{Diag}(d_1,\dots,d_p)$, where, without loss of generality, $d_j = 1 - a^2_j > 0$ $(j < p)$, and $d_p = 1 - a^2_p < 0$. With

$$w^2_j = |1 - a^2_j|^{-1}, \quad b_j = a_j w_j, \quad C = (c_{ij}) = I_p - 2(I_p + WRW)^{-1}$$

we obtain

$$c_{jk}=0,(j,k<p),c_{jp}=b_j/b_p,(j<p),c_{pp}=1-\Big(2+\sum_{j=1}^{p-1}b_j^2\Big)b_p^{-2},|c_{pp}|+\sum_{j=1}^{p-1}c_{jp}^2<1$$

and from (2.16) with $\Delta=0, Y=\Omega, v_j^2=w_j^2/2$ and $(r)_n=\Gamma(r+n)/\Gamma(r)$ the d.f.

$$\begin{aligned}&F_p(x_1,\dots,x_p;2r,R)\\&=(v_pa_p^2)^{-r}\sum_{n=0}^{\infty}(r)_nH_{r,n}(v_px_p)\sum_{(n)}\frac{c_{pp}^{n_p}}{n_p!}\prod_{j=1}^{p-1}\frac{c_{jp}^{2n_j}}{n_j!}H_{r,n_j}(v_jx_j).\end{aligned}$$

**Remark 2.** If $R\in\mathcal{G}_{2222}$ then $\|\dot{R}\|<1$ and the series derived from the L.t.

$$\left(\prod_{j=1}^{4}z_j^r\right)\left(1-\sum_{i<j}r_{ij}^2u_iu_j+|\dot{R}|u_1u_2u_3u_4\right)^{-r},\quad (z_j=(1+2t_j)^{-1},u_j=2t_jz_j)$$

also leads to a representation of the corresponding d.f..

**Remark 3.** From the proof of (d) in Theorem 5.1 also the following equivalence is recognized: $R\in\mathcal{G}_{2211}^{-1}\Leftrightarrow R\in\mathcal{F}_2^0$ and $\prod_{i<j}r_{ij}\neq 0$.

To decide if any $R$ belongs to $\mathcal{F}_2^*$ we define the following quantities:

$$\begin{aligned}\alpha_{k\ell}&=1-r_{ik}r_{jk}/r_{ij},\\ \beta_{k\ell}&=(r_{k\ell}-r_{ik}r_{j\ell}/r_{ij})(r_{k\ell}-r_{i\ell}r_{jk}/r_{ij})=\beta_{\ell k}\end{aligned}$$

and the functions

$$h_{k\ell}(x)=\alpha_{k\ell}+\beta_{k\ell}/(x-\alpha_{\ell k}),\quad (i,j,k,\ell \quad \text{any permutation of} \quad 1,2,3,4).$$

**Theorem 5.1.** *Let $R$ be a non-singular irreducible $4\times 4$-correlation matrix:*

(a) *$R\in\mathcal{F}_2^*$ is equivalent to the condition*
*There is an index $\ell$ with* $M_\ell=\bigcap_{k\neq\ell}\{d>0|h_{k\ell}(d)>0\}\neq\emptyset.$ (5.4)

(b) *If there is an index $\ell$ with $\alpha_{k\ell}>0$ for all $k\neq\ell$ (i.e. $R_{\ell\ell}$ is 1-factorial), then $R\in\mathcal{F}_1\cup\mathcal{F}_2^*$.*

(c) $\mathcal{G}_{2222}\subset\mathcal{F}_2$.

(d) $\mathcal{G}_{2211}^{-}\subset\mathcal{F}_2^0\subset\mathcal{C}_0$.

**Proof.** (a) If $R=D+AA'\in\mathcal{F}_2^*$, where $D>0$ is diagonal and $\text{rank}(AA')=2$, then there exists a $3\times 3$-submatrix $R_{\ell\ell}$ with $\prod_{i,j\neq\ell}r_{ij}\neq 0$. Solving the equations

$|(R-D)_{ij}| = 0$ for $d_k$ we obtain $d_k = h_{k\ell}(d_\ell)$. Thus, (5.4) is satisfied. Here and in the following $(i,j,k)$ is any permutation of the indices different from $\ell$.

Conversely, if (5.4) holds, then we can choose $d = d_\ell \neq 1$ from the open set $M_\ell$. Besides, $r_{ij} \neq 0$ and $d_\ell \neq \alpha_{\ell k}$ since $h_{k\ell}$ has to be well-defined. With

$$a_{\ell 1} = \pm\,(1-d_\ell)^{1/2}, \qquad a_{i1} = r_{i\ell}/a_{\ell 1},$$
$$\rho_{ij} = r_{ij} - r_{i\ell}r_{j\ell}/a_{\ell 1}^2 \quad (\neq 0 \text{ since } d_\ell \neq \alpha_{\ell k}), \qquad s_{ij} = \operatorname{sgn}(\rho_{ij}), \qquad s = s_{ij}s_{jk}s_{ki}$$

the 2nd column of $A$ is defined by $a_{\ell 2} = 0$, $a_{i2} = s_{jk}\sqrt{s}(|\rho_{ij}\rho_{ik}/\rho_{jk}|)^{1/2}$, and $d_i(d_\ell) = 1 - a_{i1}^2 - a_{i2}^2 = 1 - r_{i\ell}^2/(1-d_\ell) - \rho_{ij}\rho_{ik}/\rho_{jk} = h_{i\ell}(d_\ell)$ is verified. Hence $R = D + AA' \in \mathcal{F}_2^*$.
(b) If $R \notin \mathcal{F}_1$ the assertion is recognized with $d_\ell \to \infty$.
(c) If $R \in \mathcal{G}_{2222}$ with $r_{ij}r_{jk}r_{k\ell}r_{\ell i} \neq 0$ then for a diagonal $D = I - X$ it can be verified that $\operatorname{rank}(R-D) = \operatorname{rank}(\dot{R}+X) = 2$ with

$$x_i = r_{i\ell}^2/x_\ell + r_{ij}^2/x_j, \quad x_j = -x_\ell r_{ij}r_{jk}/(r_{i\ell}r_{k\ell}), \quad x_k = r_{k\ell}^2/x_\ell + r_{jk}^2/x_j$$

and $D > 0$ can always be satisfied by a suitable $x_\ell < 1$.
(d) First, show $\mathcal{F}_2^0 \subset \mathcal{C}_0$:

If $R = D + AA' \in \mathcal{F}_2^0$ with $\operatorname{rank}(AA') = 2$ then $D$ contains at most two zeros on its diagonal. If e.g. $d_1 = d_2 = 0$, then $\operatorname{rank}(R-D) =$

$$\operatorname{rank}\begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} - D_2 \end{pmatrix} = 2,$$

which implies $R_{22} - D_2 = R_{21}R_{11}^{-1}R_{12}$. Thus $\sigma_{34|12} = 0$ and $R \in \mathcal{C}_0$ because of (5.2). Now let $D \geq 0$ contain exactly three positive elements. Since $\mathcal{F}_2^0 \cap (\mathcal{G}_{3111} \cup \mathcal{G}_{2211} \cup \mathcal{G}_{2222}) = \emptyset$ there is always an index $\ell$ with $\prod_{i,j\neq\ell} r_{ij} \neq 0$. Then $d_k = h_{k\ell}(d_\ell) = 0$, $d_i, d_j, d_\ell > 0$ imply $\beta_{k\ell} = \alpha_{k\ell} = 0$. Otherwise, a change of $d_\ell$ would lead to $D > 0$, which is impossible for $R \in \mathcal{F}_2^0$. For the same reason $d_\ell = 0$, $d_i, d_j, d_k > 0$ is impossible. The identity

$$r_{ij}(\beta_{k\ell} - \alpha_{k\ell}\alpha_{\ell k}) = |R|r^{ij} \tag{5.5}$$

implies again $R \in \mathcal{C}_0$.

If $R \in \mathcal{G}_{2211}^-$ then $Q = \phi(R) \in \mathcal{G}_{2211}$ with $q_{ij}q_{jk}q_{k\ell} \neq 0$. Formula (5.1) applied to $Q$ shows that $r_{ij} = q^{ij}(q^{ii}q^{jj})^{-1/2} \neq 0$ for all $i \neq j$ and $\alpha_{j\ell} = \alpha_{jk} = \alpha_{ki} = \alpha_{kj} = 0$. Then, also $\beta_{j\ell} = \beta_{jk} = \beta_{ki} = \beta_{kj} = 0$ because of (5.5). Hence, for every index $\ell$, one of the expressions $d_j(d_\ell) = h_{j\ell}(d_\ell)$ must vanish identically. Since $\alpha_{j\ell} \leq 0$ for at most one index $j \neq \ell$ we find $d_i, d_k, d_\ell > 0, d_j \equiv 0$ for all sufficiently large $d_\ell$ and consequently $R \in \mathcal{F}_2^0$.

The free choice of a $d_\ell \in M_\ell$ in the representation $D + AA'$ of a two-factorial $R$ can be used to minimize the spectral norm of $C = C_k = \widetilde{B}_k \widetilde{B}_k'$ ($k = 1, 2, 3$; cf. (2.11)), which accelerates the rate of convergence in (3.8) (cf. remarks after Corollary 1b). The equation $\operatorname{tr}(C) = 0$ implies always $\|C\| < 1$. Therefore a solution $d_\ell \in M_\ell$ of one of the following equations (with $d_j = h_{j\ell}(d_\ell), \beta_1, \beta_2$ from (1.4)) would be an appealing choice if it exists.

$$\frac{1}{4}\sum_{j=1}^{4}\frac{1}{d_j} = 1 \qquad (\Longleftrightarrow (\beta_1 + \beta_2)/2 = 1 \Longleftrightarrow \operatorname{tr}(C_1) = 0)$$

$$\frac{1}{4}\sum_{j=1}^{4} d_j r^{jj} = 1 \quad (\Longleftrightarrow (1/\beta_1 + 1/\beta_2)/2 = 1 \Longleftrightarrow \operatorname{tr}(C_2) = 0)$$

$$\prod_{j=1}^{4} d_j = |R| \qquad (\Longleftrightarrow \beta_1\beta_2 = 1 \Longleftrightarrow \operatorname{tr}(C_3) = 0)\,.$$

If $R$ (and consequently $Q$) belongs to $\mathcal{F}_3^*$ then look for an $R_0$ (or $Q_0$) $\in \mathcal{F}_2$ differing from $R$ (or $Q$) only by a single element. E.g. the condition (b) in Theorem 5.1 can always be satisfied by increasing the absolute value of the absolutely smallest correlation in $R_{\ell\ell}$ but the condition $R_0 > 0$ has to be observed. If such a 2-factorial approximation to $R$ (or $Q$) is available then the Taylor approximation (3.19) is applicable besides the general expansions in (3.1).

## 6. Applications to Some Multivariate Multiple Comparisons

In multiple test procedures, based on union intersection tests, $\alpha$-level bounds (or "$p$-values") are needed for the maxima of several stochastically dependent test statistics. Conservative bounds are frequently obtained by Bonferroni's inequality. However, for a larger number of correlated statistics, more accurate bounds are desirable. Better conservative bounds are found by Bonferroni inequalities of third order using three-variate marginal distributions of higher dimensional statistics. E.g. the bounds for the multivariate maximum range test for all pairwise comparisons of the parameter vectors of several univariate linear models with identical design matrices – originally computed by large simulations (Royen (1989, 1990)) – can be approximated by this method. Here several three-variate F-distributions (or studentized $\chi^2$-distributions) occur. The error terms are bounded by a sum of probabilities from four-variate F-distributions.

By studentizing formulas from Sec. 5 exact $\alpha$-level bounds ($p$-values) are computable for simultaneous comparisons of $p = 4$ stochastically dependent linear contrasts $z_i = \sum_k \gamma_{ik} x_k$ of independent $N_q(\xi_k, \sigma^2\tau_k V)$-distributed random vectors $x_k$ with covariance matrices $\sigma^2\tau_k V$, known except for $\sigma^2$, which is estimated by $s^2$ ($\operatorname{rank}(\gamma_{ik}) = p$, test statistics $z_i' V^{-1} z_i/(s^2 \sum_k \gamma_{ik}^2 \tau_k), (i = 1, \ldots, p)$ and correlations $\rho_{ij} = (\sum_k \gamma_{ik}^2\tau_k \sum_k \gamma_{jk}^2 \tau_k)^{-1/2} \sum_k \gamma_{ik}\gamma_{jk}\tau_k)$.

Furthermore, for investigations of the power of several competing multiple test procedures, analytical methods are preferable to simulations if they are available. As an example the application of (3.9) to simultaneous comparisons with a control under classical linear model assumptions is given below.

For identical correlations $\rho_{ij} = \rho$ we obtain from (3.9)–(3.12) with $k = 2$ and $w_j^2 = 1/(1-\rho)$, $b_j^2 = \rho/(1-\rho)$, $\beta = 1 + p\rho/(1-\rho) = (1+(p-1)\rho)/(1-\rho)$ the d.f.

$$F_p(x_1,\dots,x_p;q,\rho,\Delta) = \exp\Big(-\frac{\rho}{1-\rho}\sum_{i\le j} d'_{ij}\Big)\sum_{n=0}^{\infty}\Big(\frac{\rho}{1-\rho}\Big)^n$$
$$\times\int_0^\infty\Big(\sum_{(n)}\prod_{1\le i\le j\le p} d_{ij}'^{n_{ij}}/n_{ij}!\prod_{j=1}^{p} G_{r+n_j}\Big(\frac{x_j}{2(1-\rho)}, d_{jj}+\frac{\rho}{1-\rho}y\Big)\Big)g_{r+n}(y)dy \qquad (6.1)$$

with $r = q/2$, $n_j = n_{jj} + \sum_{i=1}^{p} n_{ij}$ $(n_{ji} = n_{ij})$, $\sum_{j=1}^{p} n_j = 2n$,

$$D = (d_{ij}) = \frac{1}{2}(1-\rho)R^{-1}\Delta R^{-1}$$
$$= \frac{1}{2}(1-\rho)^{-1}\left(I - \rho(1+(p-1)\rho)^{-1}\mathbf{1}\mathbf{1}'\right)\Delta\left(I-\rho(1+(p-1)\rho)^{-1}\mathbf{1}\mathbf{1}'\right),$$

$$\mathbf{1} = (1,\dots,1)' \quad\text{and}\quad d'_{ij} = \begin{cases} d_{jj}, & i = j,\\ 2d_{ij}, & i\ne j.\end{cases}$$

The coefficients

$$d(n_1,\dots,n_p) = \sum_{n_{jj}+n_{\cdot j}=n_j}\ \prod_{1\le i\le j\le p} d_{ij}'^{n_{ij}}/n_{ij}!$$

of the $\prod_{j=1}^{p} G_{r+n_j}$ are recursively computed as the coefficients of the polynomials $(\sum_{i\le j} d'_{ij}x_ix_j)^n/n!$. If rank$(\Delta) = 1$, $\Delta = (\delta_i\delta_j)$, then (6.1) is simplified according to (3.13).

Now let $p+1$ independent observation vectors $y_i = X\beta_i + e_i$ $(i = 0,\dots,p)$ be given with identical $n\times q$-design matrices $X$ of rank $q$ and $N(0,\sigma^2 I_n)$-distributed columns $e_i$. Furthermore let $s_\nu^2$ be the usual $\sigma^2\chi_\nu^2/\nu$-distributed pooled estimate of $\sigma^2$ with $\nu = (p+1)(n-q)$ degrees of freedom. For the tests of the hypotheses $H_i : \beta_i = \beta_0$ against $\overline{H}_i : \beta_i \ne \beta_0$ $(i = 1,\dots,p)$ at the simultaneous level $\alpha$ we use the test statistics $\sigma^2\chi_i^2/s_\nu^2 = \frac{1}{2}(y_i - y_0)'X(X'X)^{-1}X'(y_i-y_0)/s_\nu^2$.

With the solution $c = c(\alpha;p,q = 2r,\nu;\rho)$ of the equation

$$\int_0^\infty\int_0^\infty\left(G_r\Big(\frac{cz}{2(1-\rho)},\frac{\rho}{1-\rho}y\Big)\right)^p g_r(y)f_\nu(z)dydz = 1-\alpha,$$

where $f_\nu(z) = \frac{1}{2}\nu(\nu z/2)^{\nu/2-1}\exp(-\nu z/2)/\Gamma(\nu/2)$ is the density of $s_\nu^2/\sigma^2$, we find for the probability of at least one rejection with $\rho = 1/2$ the formula

$$1-\exp\left(-\sum_{i\le j} d'_{ij}\right)$$
$$\times\sum_{n=0}^{\infty}\int_0^{\infty}\int_0^{\infty}\Big(\sum_{(n)}\prod_{i\le j} d_{ij}^{\prime n_{ij}}/n_{ij}!\times\prod_{j=1}^{p}G_{r+n_j}(cz,d_{jj}+y)\Big)g_{r+n}(y)f_\nu(z)dydz, \qquad (6.2)$$
$$(d_{ij})=\left(I-\frac{1}{p+1}\mathbf{1}\mathbf{1}'\right)\Delta\left(I-\frac{1}{p+1}\mathbf{1}\mathbf{1}'\right),$$
$$\Delta=(\delta_{ij})\quad\text{with}\quad \delta_{ij}=\frac{1}{2}(\beta_i-\beta_0)'X'X(\beta_j-\beta_0)/\sigma^2.$$

Similar formulas are available for further probabilities related to power.

For large values of $\nu$ the double integrals over $\prod_j G_{r+n_j}$ in (6.2) can also be approximated by the asymptotic expansion

$$\left(1+\nu^{-1}\frac{d^2}{dz^2}+\nu^{-2}\left(\frac{4}{3}\frac{d^3}{dz^3}+\frac{1}{2}\frac{d^4}{dz^4}\right)+\nu^{-3}\left(2\frac{d^4}{dz^4}+\frac{4}{3}\frac{d^5}{dz^5}+\frac{1}{6}\frac{d^6}{dz^6}\right)\right)$$
$$\int_0^{\infty}\Big(\prod_{j=1}^{p}G_{r+n_j}(cz,d_{jj}+y)\Big)g_{r+n}(y)dy\Big|_{z=1}+O(\nu^{-4}),$$

obtained from the expectation over a Taylor polynomial of powers of $Z-1$ with a $\chi^2_\nu/\nu$-distributed r.v. $Z$. At least the term of order $\nu^{-1}$ is easily computed by numerical differentiation.

## References

Abramowitz, M. and Stegun, I. A. (1965). Handbook of Mathematical Functions. Dover, New York.

Anderson, T. W. (1984). An Introduction to Multivariate Statistical Analysis, 2nd edition. John Wiley, New York.

Blumenson, L. E. and Miller, K. S. (1963). Properties of generalized Rayleigh distributions. Ann. Math. Statist. **34**, 903-910.

Jensen, D. R. (1969). Limit properties of non-central multivariate Rayleigh and chi-square distributions. SIAM J. Appl. Math. **17**, 802-814.

Jensen, D. R. (1970). A generalization of the multivariate Rayleigh distribution. Sankhyā Ser. A **32**, 193-208.

Johnson, N. L. and Kotz, S. (1972). Distributions in Statistics: Continuous Multivariate Distributions. John Wiley, New York.

Khatri, C. G., Krishnaiah, P. R. and Sen, P. K. (1977). A note on the joint distribution of correlated quadratic forms. J. Statist. Plann. Inference **1**, 299-307.

Krishnamoorthy, A. S. and Parthasarathy, M. (1951). A multivariate gamma-type distribution. Ann. Math. Statist. **22**, 549-557. Correction (1960). Ann. Math. Statist. **31**, 229.

Miller, K. S. (1964). Multidimensional Gaussian Distributions. John Wiley, New York.

Miller, K. S. and Sackrowitz, H. (1967). Relationships between biased and unbiased Rayleigh distributions. SIAM J. Appl. Math. **15**, 1490-1495.

Royen, T. (1987). An approximation for multivariate normal probabilities of rectangular regions. Statistics **18**, 389-400.

Royen, T. (1989). Generalized maximum range tests for pairwise comparisons of several populations. Biometrical J. **31**, 905-929.

Royen, T. (1990). Tables for studentized multivariate maximum ranges and their application by maximum range tests. Biometrical J. **32**, 643-680.

Royen, T. (1991a). Multivariate gamma distributions with one-factorial accompanying correlation matrices and applications to the distribution of the multivariate range. Metrika **38**, 299-315.

Royen, T. (1991b). Expansions for the multivariate chi-square distribution J. Multivariate Anal. **38**, 213-232.

Royen, T. (1992). On representation and computation of multivariate gamma distributions. Data Analysis and Statistical Inference – Festschrift in Honour of Friedhelm Eicker (Edited by Schach, S. and Trenkler, G.), 201-216. Eul-Verlag, Bergisch Gladbach.

Royen, T. (1994). On some multivariate gamma distributions connected with spanning trees. Ann. Inst. Statist. Math. **46**, 361-371.

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