Statistica Sinica: Supplement

PARAMETRIC BOOTSTRAP INFERENCE FOR STRATIFIED MODELS WITH HIGH-DIMENSIONAL NUISANCE SPECIFICATIONS

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Supplementary Material

Section S1 contains the proof of eq. (3.1) and (3.2) of the main text, namely it focuses on the main result of the paper for the case when $0 \leq \alpha < 1$. Section S2 reports the full set of results of the simulation studies described in the main text. In addition to the four models described in the main text (beta model, curved exponential family, truncated regression and logistic regression) two further models are considered, given by a gamma model and a Behrens-Fisher model, respectively. Finally, Section S3 reports the tail probabilities estimated by simulation for the statistics in Table 1 of the main text, for all models and all simulation settings.

S1 Proof of (3.1) and (3.2)

In the same setting as in Section 3 of the main text, we consider $q = O(m^{\alpha})$, $0 \leq \alpha < 1$. The case $\alpha = 0$ corresponds to the standard asymptotic setting, where $n = O(m)$.

The core results we prove here are

$$\mathrm{pr}_\theta\left(F_{\hat\theta_\psi}(S(\psi)) \le u\right) = u + O(m^{(\alpha-3)/2})\,. \tag{S1.1}$$

and

$$\mathrm{pr}_\theta\left(F_{\hat\theta}(S(\psi)) \le u\right) = u + O(m^{-1})\,. \tag{S1.2}$$

Hence, as with $\alpha = 0$ (Lee and Young, 2005), the order of error is different with constrained and unconstrained bootstrap.

In order to prove (S1.1) and (S1.2) we need some preliminary results about the distribution function $F_\theta(x)$ of $S(\psi)$. From Sartori (2003, formula (6)), $U_p = U_p(\psi)$ can be expanded as

$$U_p = U_{\psi|\lambda} + B + Re\,, \tag{S1.3}$$

where the terms on the right-hand side are of order $O_p(m^{(\alpha+1)/2})$, $O_p(m^\alpha)$ and $O_p(m^{\alpha-1})$, respectively. Indeed, using (A1) in the main text, we have $U_{\psi|\lambda} = \sum_{i=1}^q U_{\psi|\lambda_i} = O_p(m^{(\alpha+1)/2})$, with $U_{\psi|\lambda_i} = U_\psi^i - i_{\psi\lambda_i} i_{\lambda_i\lambda_i}^{-1} U_{\lambda_i}$ being $\mathrm{E}_\theta(U_{\psi|\lambda_i}) = 0$ and $\mathrm{Var}_\theta(U_{\psi|\lambda_i}) = i_{\psi\psi|\lambda_i} = O(m)$. Note that $i_{\psi\psi|\lambda} = \mathrm{Var}_\theta(U_{\psi|\lambda}) = \sum_{i=1}^q i_{\psi\psi|\lambda_i}$. Similarly, we have $B = \sum_{i=1}^q B^i(\psi, \lambda_i) = O_p(m^\alpha)$, where $B^i(\psi, \lambda_i)$ is the term of order $O_p(1)$ of the expansion of the profile score in the ith stratum, having both mean and variance of order $O(1)$. The same additivity property holds for $b(\theta) = \mathrm{E}_\theta(B)$, so that

$b(\theta) = \sum_{i=1}^q b^i(\psi, \lambda_i) = O(m^\alpha)$. Finally, the remainder term is $Re = \sum_{i=1}^q Re^i(\psi, \lambda_i)$, with $Re^i(\psi, \lambda_i)$ having mean and variance of order $O(m^{-1})$, so that $Re = O_p(m^{\max\{\alpha-1,(\alpha-1)/2\}}) = O_p(m^{(\alpha-1)/2})$ when $0 \leq \alpha < 1$. Therefore, the terms in (S1.3) are in descending order as opposed to what happens when $\alpha \geq 1$.

Let $M(\theta) = \mathrm{E}_\theta(S(\psi))$ and $\mathrm{Var}_\theta(S(\psi))$ be the expectation and variance of $S(\psi)$. We first show that

$$M(\theta) = \frac{b(\theta)}{i_{\psi\psi|\lambda}(\theta)^{1/2}} + M_1(\theta) + O(m^{-(\alpha+3)/2})\,, \tag{S1.4}$$

with $b(\theta) i_{\psi\psi|\lambda}(\theta)^{-1/2} = O(m^{(\alpha-1)/2})$ and

$$M_1(\theta) = \mathrm{E}_\theta\left(\frac{Re}{i_{\psi\psi|\lambda}^{1/2}}\right) = O(m^{(\alpha-3)/2})\,. \tag{S1.5}$$

Moreover,

$$\mathrm{Var}_\theta(S(\psi)) = 1 + v(\theta) + O(m^{-2})\,, \tag{S1.6}$$

where $v(\theta) = (\mathrm{Var}_\theta(B) + 2\,\mathrm{E}_\theta(U_{\psi|\lambda}B))/i_{\psi\psi|\lambda} = O(m^{-1})$.

In order to show (S1.4), as a first step, consider the expansion

$$i_{\psi\psi|\lambda}(\hat{\theta}_\psi) = i_{\psi\psi|\lambda} + C + O_p(m^\alpha)\,, \tag{S1.7}$$

with

$$C = \sum_{i=1}^q \frac{d}{d\lambda_i} i_{\psi\psi|\lambda_i}(\hat{\lambda}_{i\psi} - \lambda_i) = O_p(m^{(\alpha+1)/2})\,,$$

where the order is again determined using (A1) in the main text. Hence,

$$\{i_{\psi\psi|\lambda}(\hat{\theta}_\psi)\}^{-1/2} = i_{\psi\psi|\lambda}^{-1/2}\left\{1 - \frac{1}{2}\frac{C}{i_{\psi\psi|\lambda}} + O_p(m^{-1})\right\}, \tag{S1.8}$$

with $C/i_{\psi\psi|\lambda} = O_p(m^{-(\alpha+1)/2})$.

Using (S1.3) and (S1.8),

$$\begin{aligned} S(\psi) &= i_{\psi\psi|\lambda}^{-1/2}\{U_{\psi|\lambda} + B + Re\}\left\{1 - \frac{1}{2}\frac{C}{i_{\psi\psi|\lambda}} + O_p(m^{-1})\right\} \\ &= \frac{U_{\psi|\lambda}}{i_{\psi\psi|\lambda}^{1/2}} + \frac{B}{i_{\psi\psi|\lambda}^{1/2}} + \frac{Re}{i_{\psi\psi|\lambda}^{1/2}} - \frac{1}{2}\frac{U_{\psi|\lambda}C}{i_{\psi\psi|\lambda}^{3/2}} - \frac{1}{2}\frac{B\,C}{i_{\psi\psi|\lambda}^{3/2}} - \frac{1}{2}\frac{Re\,C}{i_{\psi\psi|\lambda}^{3/2}} \\ &\quad + O_p(m^{-1}) + O_p(m^{(\alpha-3)/2}), \end{aligned} \tag{S1.9}$$

where $O_p(m^{-1}) + O_p(m^{(\alpha-3)/2}) + O_p(m^{(\alpha-5)/2}) = O_p(m^{-1})$. The term of order $O_p(m^{(\alpha-3)/2})$ is given by $i_{\psi\psi|\lambda}^{-1/2}B$ times the term of order $O_p(m^{-1})$ in (S1.8). Its expectation is of order $O(m^{(\alpha-3)/2})$. The orders of terms in (S1.9) are as follows:

$$\begin{aligned} &\frac{U_{\psi|\lambda}}{i_{\psi\psi|\lambda}^{1/2}} = O_p(1), \quad \frac{B}{i_{\psi\psi|\lambda}^{1/2}} = O_p(m^{(\alpha-1)/2}), \quad \frac{Re}{i_{\psi\psi|\lambda}^{1/2}} = O_p(m^{-1}), \\ &\frac{1}{2}\frac{U_{\psi|\lambda}C}{i_{\psi\psi|\lambda}^{3/2}} = O_p(m^{-(\alpha+1)/2}) = o_p(1), \quad \frac{1}{2}\frac{B\,C}{i_{\psi\psi|\lambda}^{3/2}} = O_p(m^{-1}), \\ &-\frac{1}{2}\frac{Re\,C}{i_{\psi\psi|\lambda}^{3/2}} = O_p(m^{-(\alpha+3)/2}). \end{aligned}$$

Expansion (S1.4) for $\mathrm{E}_\theta(S(\psi))$ is obtained using (S1.9) and recalling that $b(\theta) = O(m^{\alpha})$. We have

$$\mathrm{E}_\theta\left(\frac{U_{\psi|\lambda}}{i^{1/2}_{\psi\psi|\lambda}}\right) = 0\,, \quad \mathrm{E}_\theta\left(\frac{B}{i^{1/2}_{\psi\psi|\lambda}}\right) = \frac{b(\theta)}{i^{1/2}_{\psi\psi|\lambda}} = O(m^{(\alpha-1)/2})\,,$$
$$\mathrm{E}_\theta\left(\frac{Re}{i^{1/2}_{\psi\psi|\lambda}}\right) = O(m^{(\alpha-3)/2})\,, \quad \mathrm{E}_\theta\left(\frac{1}{2}\frac{U_{\psi|\lambda}C}{i^{3/2}_{\psi\psi|\lambda}}\right) = O(m^{-(\alpha+3)/2}) = o(1)\,,$$
$$\mathrm{E}_\theta\left(\frac{1}{2}\frac{B\,C}{i^{3/2}_{\psi\psi|\lambda}}\right) = O_p(m^{-(\alpha+3)/2})\,, \quad \mathrm{E}_\theta\left(-\frac{1}{2}\frac{Re\,C}{i^{3/2}_{\psi\psi|\lambda}}\right) = O(m^{-(\alpha+3)/2})\,,$$

giving (S1.4).

Expansion (S1.6) for $\mathrm{Var}_\theta(S(\psi))$ is also obtained using (S1.9). In particular, the leading term has variance equal to 1, and, using a standard expansion for the stratum profile score $U^i_\psi(\psi, \hat{\lambda}_{i\psi})$ (see e.g Pace and Salvan, 1997, formula (8.88)), $\mathrm{Cov}_\theta(U_{\psi|\lambda}, B)$ and $\mathrm{Var}_\theta(B)$ are easily seen to be of order $O(m^{\alpha})$. Further terms of (S1.9) give contributions to the variance of order $O(m^{-2})$.

Higher order cumulants of $S(\psi)$, $r = 3, 4, \ldots$, have the form

$$\kappa_r(S(\psi)) = \frac{O(m^{\alpha+1})}{O(m^{r(\alpha+1)/2})} = O(m^{(\alpha+1)(1-r/2)}) = O(n^{1-r/2})$$

as in standard asymptotics.

As in Section 3 of the main text, the developments here rely on the assumption that the distribution function of $S(\psi)$ admits a valid Edgeworth expansion. In particular, in the continuous case,

$$F_\theta(x) = \mathrm{pr}_\theta\left(S(\psi) \le x\right) = \Phi\left(\frac{x - M(\theta)}{\sqrt{\mathrm{Var}_\theta(S(\psi))}}\right) + O(m^{-(\alpha+1)/2})\,, \quad \text{(S1.10)}$$

where the order of the remainder term is that of the third cumulant of $S(\psi)$. Using (S1.4) and (S1.6) we have

$$F_\theta(x) = \Phi(x) - \phi(x)M(\theta) + O(m^{-(\alpha+1)/2})\,. \tag{S1.11}$$

We first focus on constrained bootstrap. From (S1.11),

$$F_{\hat\theta_\psi}(x) = \Phi(x) - \phi(x)M(\hat\theta_\psi) + O_p\left(m^{-(\alpha+1)/2)}\right)\,. \tag{S1.12}$$

We, then, show the analogous of expression (3.18) of the main text in the case $0 \le \alpha < 1$, i.e.

$$M(\hat\theta_\psi) = M(\theta) + \Delta + O_p\left(m^{(\alpha-3)/2\}}\right)\,, \tag{S1.13}$$

with

$$\Delta = \frac{b_1(\theta)}{i^{1/2}_{\psi\psi|\lambda}} - \frac{C\, b(\theta)}{2\, i^{3/2}_{\psi\psi|\lambda}} \tag{S1.14}$$

that is of order $O_p(m^{-1})$ and where $b_1(\theta)$ is given in (S1.17) below. In order to show (S1.13) let $\overline{Re} = \mathrm{E}_\theta(Re)$. Then, from (S1.4) and (S1.5),

$$M(\hat\theta_\psi) = \left\{i_{\psi\psi|\lambda}(\hat\theta_\psi)\right\}^{-1/2}\left\{b(\hat\theta_\psi) + \overline{Re}(\hat\theta_\psi)\right\} + O_p(m^{-(\alpha+3)/2})\,, \tag{S1.15}$$

where $\overline{Re}(\hat\theta_\psi)$ is of order $O_p(m^{\alpha-1})$. Now,

$$b(\hat\theta_\psi) = b(\theta) + b_1(\theta) + O_p(m^{\alpha-1})\,, \tag{S1.16}$$

where

$$b_1(\theta) = \sum_{i=1}^{q} b^i_{\lambda_i}(\psi, \lambda_i)(\hat\lambda_{i\psi} - \lambda_i)\,, \tag{S1.17}$$

and $b^i_{\lambda_i}(\psi,\lambda_i) = \partial b^i(\psi,\lambda_i)/\partial\lambda_i$, and so on. Using (A1) in the main text, and being $b^i_{\lambda_i}(\psi,\lambda_i)$ of order $O(1)$,

$$\sum_{i=1}^{q} b^i_{\lambda_i}(\psi,\lambda_i)(\hat{\lambda}_{i\psi}-\lambda_i) = O_p(m^{(\alpha-1)/2}) + O_p(m^{\alpha-1})\,. \tag{S1.18}$$

The remainder in (S1.16) includes also the term

$$\frac{1}{2}\sum_{i=1}^{q} b^i_{\lambda_i\lambda_i}(\psi,\lambda_i)(\hat{\lambda}_{i\psi}-\lambda_i)^2$$

which is of order $O_p(m^{\alpha-1})$, being $b^i_{\lambda_i\lambda_i}(\psi,\lambda_i)$ of order $O(1)$. Moreover, $\overline{Re}(\hat{\theta}_\psi) = \overline{Re} + O_p(m^{(\alpha-3)/2})$.

Using (S1.8), we get

$$M(\hat{\theta}_\psi) = i^{-1/2}_{\psi\psi|\lambda} b(\theta) + \tilde{M}_1 + O_p\left(m^{(\alpha-3)/2}\right)\,, \tag{S1.19}$$

with

$$\tilde{M}_1 = i^{-1/2}_{\psi\psi|\lambda}\left\{b_1(\theta) - \frac{C\, b(\theta)}{2 i_{\psi\psi|\lambda}}\right\}\,,$$

which is of order $O_p(m^{-1})$ because both terms are of the same order and linear in $\hat{\lambda}_{i\psi}-\lambda_i$.

Therefore, (S1.19), (S1.4) and (S1.5) give (S1.13).

As a result, the following Taylor expansion of (S1.12) holds

$$F_{\hat{\theta}_\psi}(x) = F_\theta(x) - \phi(x)\Delta + O_p(m^{(\alpha-3)/2})\,. \tag{S1.20}$$

In order to prove (S1.1), note that $F_{\hat{\theta}_\psi}(S(\psi)) \le u$ is equivalent to $S(\psi) \le s_u$, with s_u the u-quantile of $F_{\hat{\theta}_\psi}(\cdot)$, such that $F_{\hat{\theta}_\psi}(s_u) = u$. Let s^0_u

be the u-quantile of $F_\theta(\cdot)$. It is useful to express s_u in terms of s_u^0. Using (S1.20), we get

$$u = F_\theta(s_u^0) = F_{\hat\theta_\psi}(s_u) = F_\theta(s_u) - \phi(s_u)\Delta + O_p(m^{(\alpha-3)/2})\,.$$

Hence, $F_\theta(s_u) - F_\theta(s_u^0) = \phi(s_u)\Delta + O_p(m^{(\alpha-3)/2})$. On the other hand, letting $F'_\theta(x) = dF_\theta(x)/dx$, from

$$F_\theta(s_u^0) = F_\theta(s_u) + (s_u^0 - s_u)F'_\theta(s_u) + O_p((s_u^0 - s_u)^2)$$

and

$$F'_\theta(x) = \phi(x) - M(\theta)\phi'(x) + O(m^{\alpha-1}) = \phi(x) + M(\theta)x\phi(x) + O(m^{\alpha-1})$$

we get

$$\begin{aligned} s_u &= s_u^0 + \Delta + O_p(m^{(\alpha-3)/2}) + O_p(m^{\alpha-2}) \\ &= s_u^0 + \Delta + O_p(m^{(\alpha-3)/2})\,. \end{aligned}$$

Hence, $S(\psi) \le s_u$ is equivalent to $S(\psi) \le s_u^0 + \Delta + O_p(m^{(\alpha-3)/2})$, and

$$\mathrm{pr}_\theta\left(F_{\hat\theta_\psi}(S(\psi)) \le u\right) = \mathrm{pr}_\theta\left(\bar S(\psi) \le F_\theta^{-1}(u)\right)\,,$$

where $\bar S(\psi) = S(\psi) - \Delta + O_p(m^{(\alpha-3)/2})$, with Δ given by (S1.14), and such that $\mathrm{E}_\theta(\Delta) = O(m^{(\alpha-3)/2})$ (see (S1.18)). Moreover, we have

$$\begin{aligned}
\mathrm{E}_\theta(\bar{S}(\psi)) &= \mathrm{E}_\theta(S(\psi)) + O(m^{(\alpha-3)/2})\,, && \text{(S1.21)}\\
\mathrm{Var}_\theta(\bar{S}(\psi)) &= \mathrm{Var}_\theta(S(\psi) - \Delta) + O(m^{-2}) \\
&= \mathrm{Var}_\theta(S(\psi)) + \mathrm{Var}_\theta(\Delta) - 2\mathrm{Cov}_\theta(S(\psi), \Delta) + O(m^{-2}) \\
&= \mathrm{Var}_\theta(S(\psi)) + O(m^{-2})\,, && \text{(S1.22)}
\end{aligned}$$

since $\mathrm{Var}_\theta(\Delta) = O(m^{-2})$ and $\mathrm{Cov}_\theta(S(\psi), \Delta) = O(m^{-2})$, where the order of the latter is determined by the orthogonality between $U_{\psi|\lambda}$ and the leading term of $b_1(\theta)$. Finally, (S1.1) holds because

$$\begin{aligned}
\mathrm{pr}_\theta\left(\bar{S}(\psi) \le F_\theta^{-1}(u)\right) &= \mathrm{pr}_\theta\left(S(\psi) \le F_\theta^{-1}(u)\right) + O(m^{(\alpha-3)/2}) + O(m^{-2}) \\
&= \mathrm{pr}_\theta\left(S(\psi) \le F_\theta^{-1}(u)\right) + O(m^{(\alpha-3)/2}) \\
&= u + O(m^{(\alpha-3)/2})\,.
\end{aligned}$$

The proof of (S1.2) for unconstrained bootstrap is obtained along the same steps as above. In particular, an expansion for $F_{\hat{\theta}}(x)$ of the form (S1.20) holds with a different Δ term, which is still of order $O_p(m^{-1})$. However, while (S1.21) is still true, (S1.22) holds with an error of order $O(m^{-1})$, because there is no orthogonality between $U_{\psi|\lambda}$ and the leading terms of $b_2(\theta)$, given in (S1.23) below. Therefore, for unconstrained bootstrap we have

$$
\begin{aligned}
\mathrm{pr}_\theta\left(\bar{S}(\psi) \le F_\theta^{-1}(u)\right) &= \mathrm{pr}_\theta\left(S(\psi) \le F_\theta^{-1}(u)\right) + O(m^{(\alpha-3)/2}) + O(m^{-1}) \\
&= \mathrm{pr}_\theta\left(S(\psi) \le F_\theta^{-1}(u)\right) + O(m^{-1}) \\
&= u + O(m^{-1})\,.
\end{aligned}
$$

In order to obtain an expansion for $M(\hat{\theta})$, with $M(\theta)$ given in (S1.4), we follow the same steps as in (S1.15)–(S1.19), giving (S1.13). In particular, we have

$$
b(\hat{\theta}) = b(\theta) + b_2(\theta) + O_p(m^{\alpha-2})\,,
$$

where

$$
\begin{aligned}
b_2(\theta) = b_2(\psi,\lambda) &= \sum_{i=1}^{q} b_\psi^i\,(\hat{\psi}-\psi) + \sum_{i=1}^{q} b_{\lambda_i}^i(\hat{\lambda}_i-\lambda_i) + \frac{1}{2}\sum_{i=1}^{q} b_{\lambda_i\lambda_i}^i(\hat{\lambda}_i-\lambda_i)^2 \\
&\quad + \frac{1}{2}\sum_{i=1}^{q} b_{\psi\psi}^i(\hat{\psi}-\psi)^2 + \sum_{i=1}^{q} b_{\psi\lambda_i}^i(\hat{\lambda}_i-\lambda_i)(\hat{\psi}-\psi)\,. \qquad \text{(S1.23)}
\end{aligned}
$$

From Sartori (2003, below formula (9)), with $\alpha < 1$, $\hat{\psi}-\psi = O_p(m^{-(\alpha+1)/2})$, so that leading terms of $b_2(\theta)$ are the first two summands on the right-hand side of (S1.23), which are of order $O_p(m^{(\alpha-1)/2})$. The other terms give a contribution of order $O_p(m^{\alpha-1})$. This leads to

$$
M(\hat{\theta}) = M(\theta) + \Delta_1 + O_p(m^{(\alpha-3)/2})\,, \qquad \text{(S1.24)}
$$

where the term Δ_1 has form similar to (S1.14) and is of order $O_p(m^{-1})$, with expectation of order $O(m^{(\alpha-3)/2})$, because the leading term in (S1.23) are of the same order as $b_1(\theta)$ in (S1.16).

The remaining steps are analogous to those from (S1.20) with Δ replaced by Δ_1.

S2 Full set of simulation results

We report the full set of results of the simulation studies described in the main text. In addition to the models described there, which include also the logistic regression model described in the final section, two further models are considered, given by a gamma model and a Behrens-Fisher model, respectively. Some details about them are given as follows.

Gamma model

We take a stratified gamma model with common shape parameter, for which Y_{ij} has density function

$$g(y_{ij}; \alpha, \beta_i) = \frac{1}{\Gamma(\alpha)\, \beta_i^{\alpha}}\, y_{ij}^{\alpha-1} \exp\left\{-\frac{y_{ij}}{\beta_i}\right\} \quad (y_{ij} > 0)\,,$$

where $\Gamma(\cdot)$ is the gamma function. The parameter of interest is $\psi = \log(\alpha)$ and the stratum-specific nuisance parameters are given by $\lambda_i = \log(\beta_i)$. The simulation data sets where generated for $\psi_0 = \log(2)$ and the elements of λ_0 are fixed to the logarithm of random draws from an exponential distribution with rate $1/2$ and held fixed over all the replications.

The results for this model, reported in the following, are similar to those for the beta model.

Behrens-Fisher model

Suppose that Y_{ij} are normally distributed, with common mean ψ across all strata, and stratum-specific variances $\exp(\lambda_i)$. This example was studied in Young (2009), where it is illustrated that $R^*(\psi)$ is performing worse than constrained bootstrapping of $R(\psi)$ in a simulation study with $q = 20$ strata and the stratum sample size varying between 3 and 10. The simulation experiments are carried out for $\psi_0 = 0$ and λ_0 generated from a uniform distribution in (0, 1) and held fixed over all the replications.

The findings in Young (2009) are confirmed also in the more extensive simulation studies here, whose results are reported in the following. For the Behrens-Fisher model, the location-adjustment to $R(\psi)$ does not have much of an effect to the distribution of the statistic. On the other hand, the location-and-scale adjusted statistic recovers inferential performance. So, it seems that only a scale adjustment is sufficient to recover first-order inferential performance. Indeed, for this model the profile score bias is exactly equal to zero, so that the distribution of the first-order statistic $R(\psi)$ is essentially centred around zero. Also, both constrained and unconstrained bootstrap perform extremely well across all the simulation scenarios, and, along with R^c_{ls}, they outperform $R^*(\psi)$ in the most extreme settings.

A remarkable observation in this model is the excellent accuracy of the score statistic $S(\psi)$ for all combination of q and m considered. This can be explained by the fact that the leading term $U_{\psi|\lambda}$ of U_p in (S1.3) is exactly normally distributed. Moreover, $M(\theta)$ and $v(\theta)$, in (S1.4) and (S1.6) respectively, are equal to zero.

Results

For each model, simulation results for the 23 statistics are visualized. For plotting purposes, the statistics have been split in two groups. Table 1 lists the first group, and Table 2 the second one.

Table 1: Statistics computed in the simulation experiments (first group for plotting).

Statistic	Plotting Symbol	Description
$R(\psi)$	R	Signed likelihood root
$S(\psi)$	S	Score statistic
$T(\psi)$	T	Wald statistic
$\Phi^{-1}\{\hat{p}_1^T(\psi)\}$	T^u	Transformed p-value from unconstrained bootstrap of $T(\psi)$
$\Phi^{-1}\{\hat{p}_2^T(\psi)\}$	T^c	Transformed p-value from constrained bootstrap of $T(\psi)$
$\Phi^{-1}\{\hat{p}_1^S(\psi)\}$	S^u	Transformed p-value from unconstrained bootstrap of $S(\psi)$
$\Phi^{-1}\{\hat{p}_2^S(\psi)\}$	S^c	Transformed p-value from constrained bootstrap of $S(\psi)$
$\Phi^{-1}\{\hat{p}_1^R(\psi)\}$	R^u	Transformed p-value from unconstrained bootstrap of $R(\psi)$
$\Phi^{-1}\{\hat{p}_2^R(\psi)\}$	R^c	Transformed p-value from constrained bootstrap of $R(\psi)$
$R^*(\psi)$	R^*	Modified signed likelihood root
$R_m(\psi)$	R_m	Signed likelihood root computed from the modified profile likelihood

Table 2: Statistics computed in the simulation experiments (second group for plotting). The mean $\hat{\mu}^R$ and the standard deviation $\hat{\sigma}^R$ of $R(\psi)$ are estimated through unconstrained bootstrap, by simulating from the model at $\theta = \hat{\theta}$, and likewise for $S(\psi)$ and $T(\psi)$. The mean $\tilde{\mu}^R$ and the standard deviation $\tilde{\sigma}^R$ of $R(\psi)$ are estimated through constrained bootstrap, by simulating from the model at $\theta = \hat{\theta}_\psi$, and likewise for $S(\psi)$ and $T(\psi)$.

Statistic	Plotting Symbol	Description
$T(\psi) - \hat{\mu}^T$	T_l^u	Location adjusted $T(\psi)$, unconstrained bootstrap
$T(\psi) - \tilde{\mu}^T$	T_l^c	Location adjusted $T(\psi)$, constrained bootstrap
$S(\psi) - \hat{\mu}^S$	S_l^u	Location adjusted $S(\psi)$, unconstrained bootstrap
$S(\psi) - \tilde{\mu}^S$	S_l^c	Location adjusted $S(\psi)$, constrained bootstrap
$R(\psi) - \hat{\mu}^R$	R_l^u	Location adjusted $R(\psi)$, unconstrained bootstrap
$R(\psi) - \tilde{\mu}^R$	R_l^c	Location adjusted $R(\psi)$, constrained bootstrap
$(T(\psi) - \hat{\mu}^T)/\hat{\sigma}^T$	T_{ls}^u	Location-and-scale adjusted $T(\psi)$, unconstrained bootstrap
$(T(\psi) - \tilde{\mu}^T)/\tilde{\sigma}^T$	T_{ls}^c	Location-and-scale adjusted $T(\psi)$, constrained bootstrap
$(S(\psi) - \hat{\mu}^S)/\hat{\sigma}^S$	S_{ls}^u	Location-and-scale adjusted $S(\psi)$, unconstrained bootstrap
$(S(\psi) - \tilde{\mu}^S)/\tilde{\sigma}^S$	S_{ls}^c	Location-and-scale adjusted $S(\psi)$, constrained bootstrap
$(R(\psi) - \hat{\mu}^R)/\hat{\sigma}^R$	R_{ls}^u	Location-and-scale adjusted $R(\psi)$, unconstrained bootstrap
$(R(\psi) - \tilde{\mu}^R)/\tilde{\sigma}^R$	R_{ls}^c	Location-and-scale adjusted $R(\psi)$, constrained bootstrap

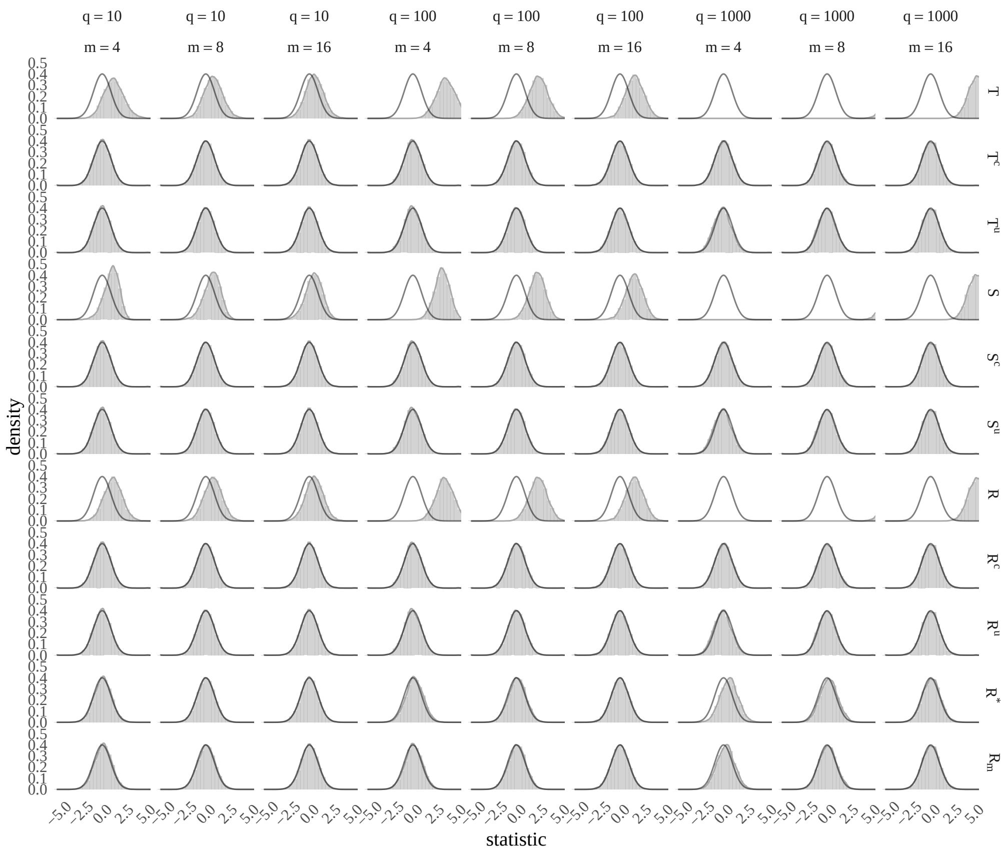

Figure 1: Gamma model. Estimated null distribution of statistics for the statistics in Table 1 for various combinations of q and m. The N(0,1) density function is superimposed.

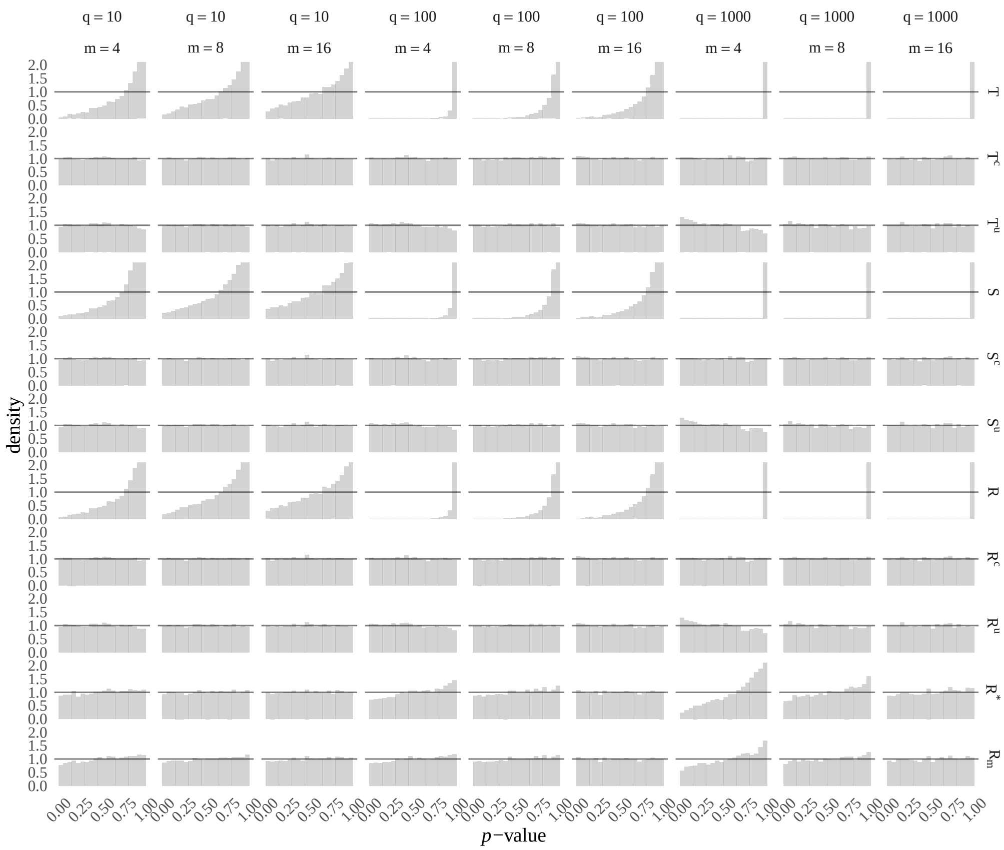

Figure 2: Gamma model. Estimated distribution of p-values for the statistics in Table 1 for various combinations of q and m. The Uniform(0,1) density function is superimposed.

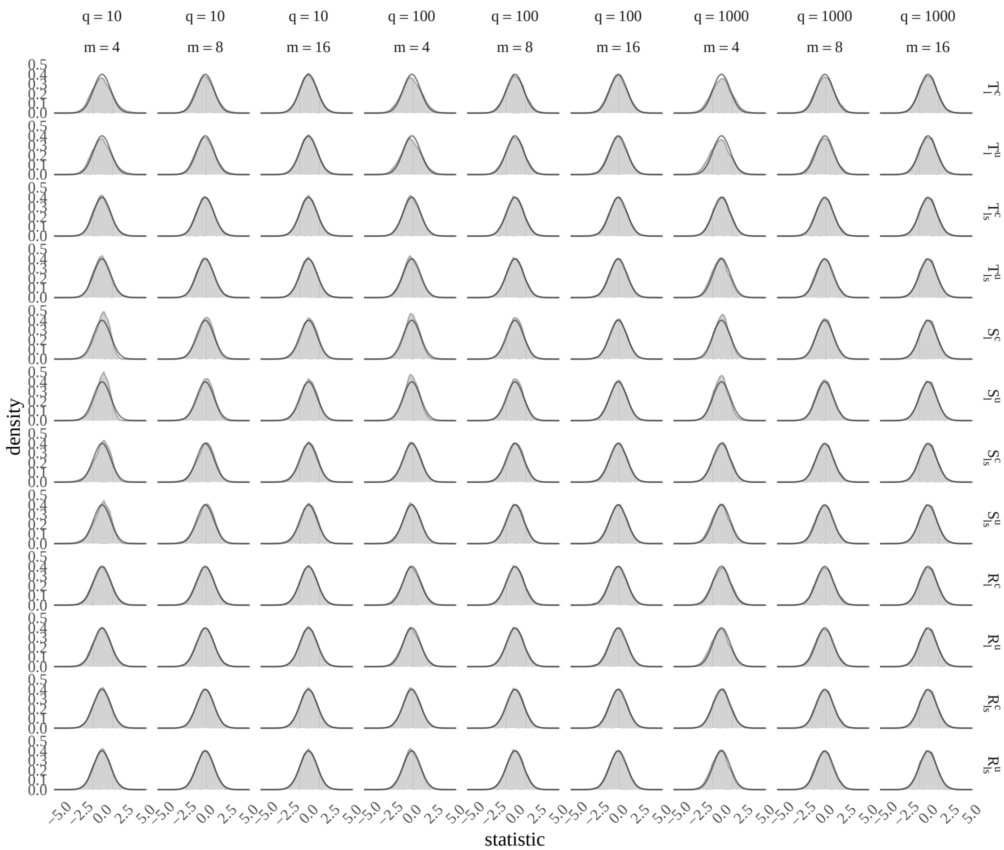

Figure 3: Gamma model. Estimated null distribution of statistics for the statistics in Table 2 for various combinations of q and m. The N(0,1) density function is superimposed.

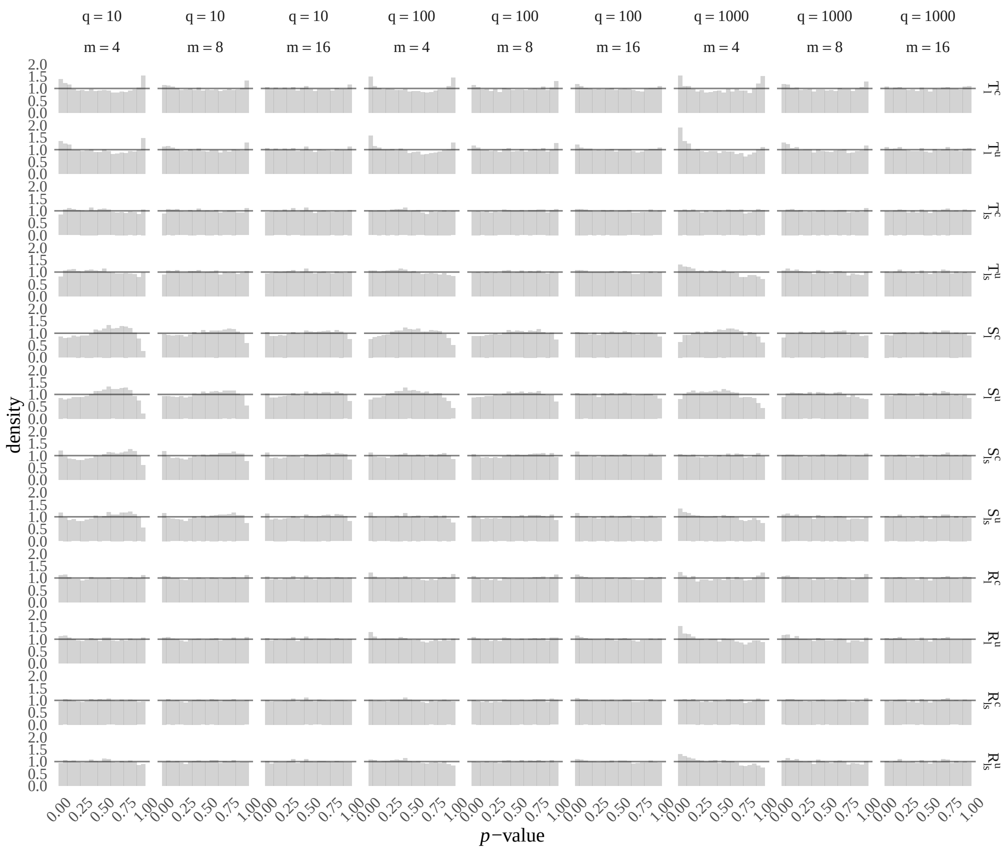

Figure 4: Gamma model. Estimated null distribution of p-values for the statistics in Table 2 for various combinations of q and m. The Uniform(0,1) density function is superimposed.

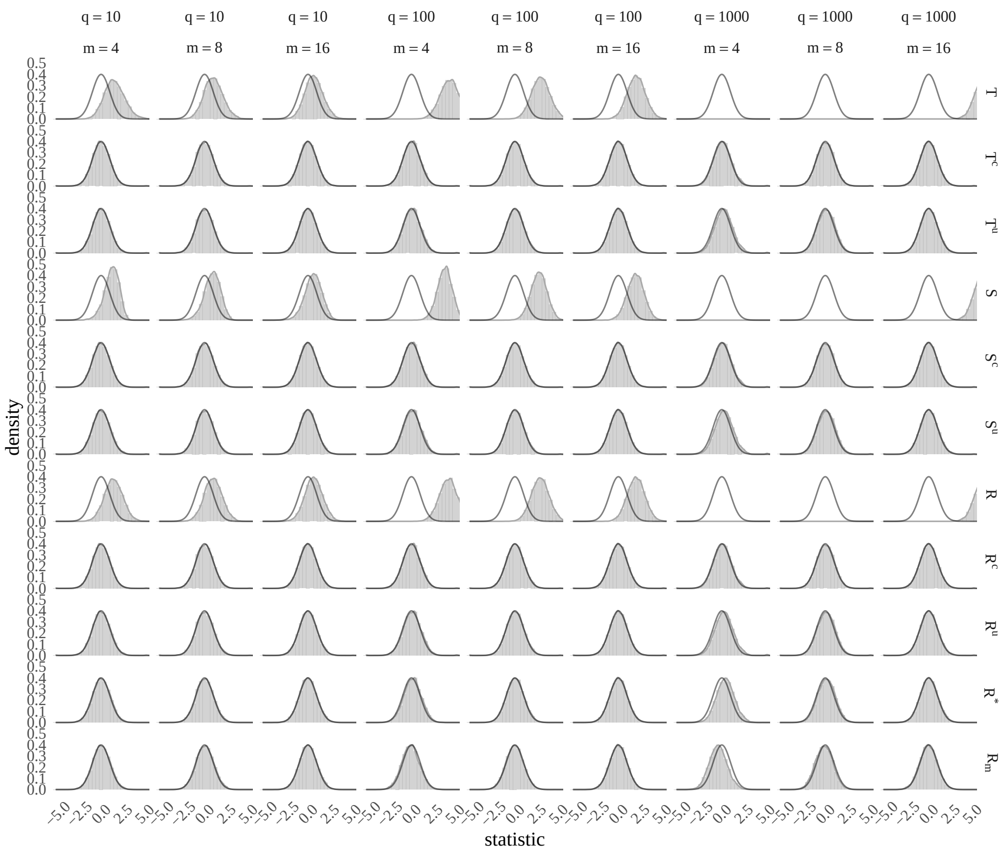

Figure 5: Beta model. Estimated null distribution of statistics for the statistics in Table 1 for various combinations of q and m. The N(0,1) density function is superimposed.

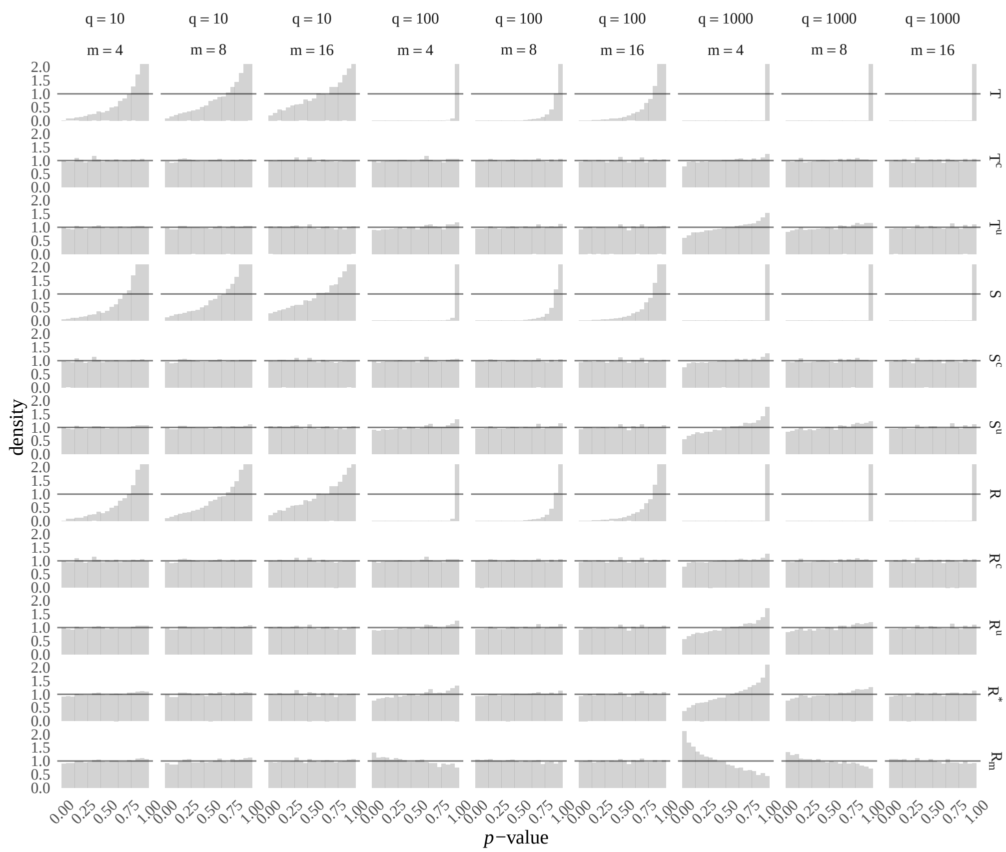

Figure 6: Beta model. Estimated distribution of p-values for the statistics in Table 1 for various combinations of q and m. The Uniform(0,1) density function is superimposed.

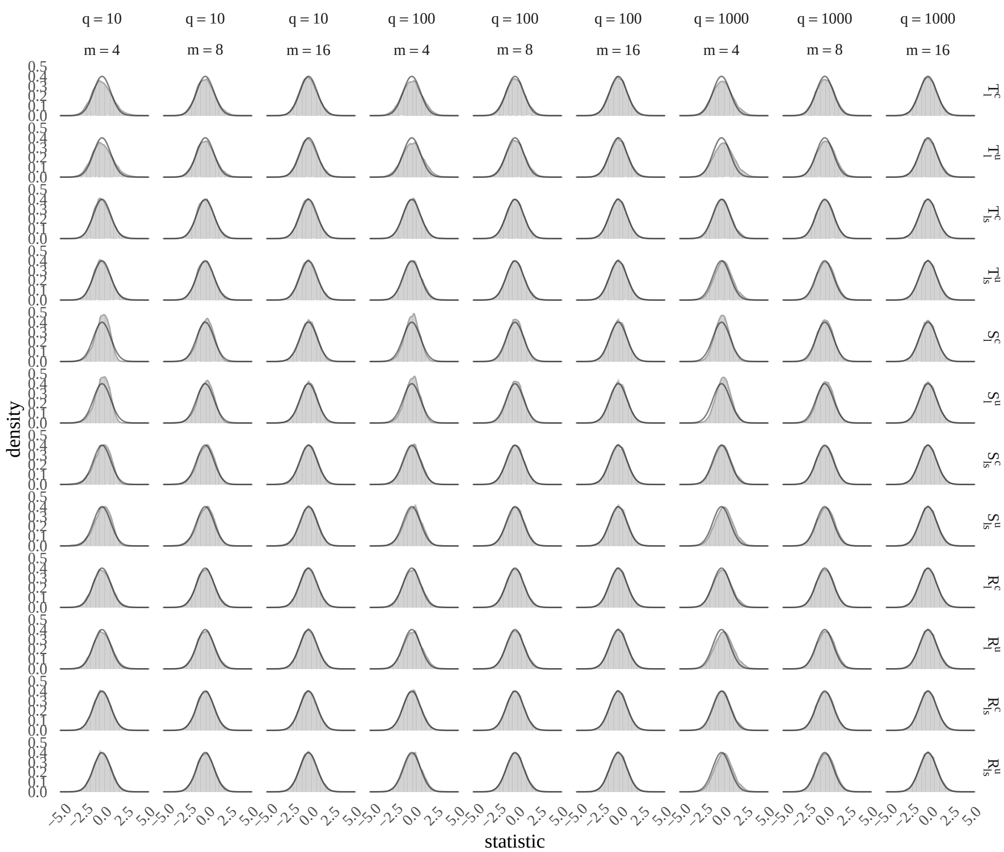

Figure 7: Beta model. Estimated null distribution of statistics for the statistics in Table 2 for various combinations of q and m. The N(0,1) density function is superimposed.

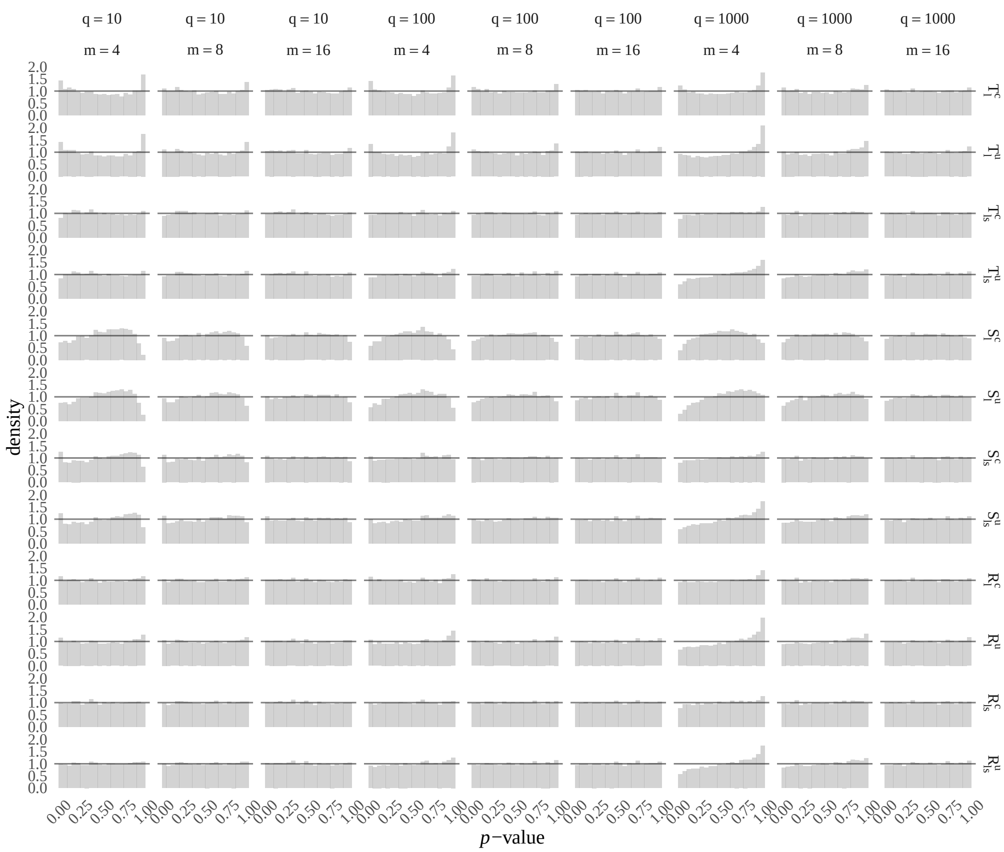

Figure 8: Beta model. Estimated null distribution of p-values for the statistics in Table 2 for various combinations of q and m. The Uniform(0,1) density function is superimposed.

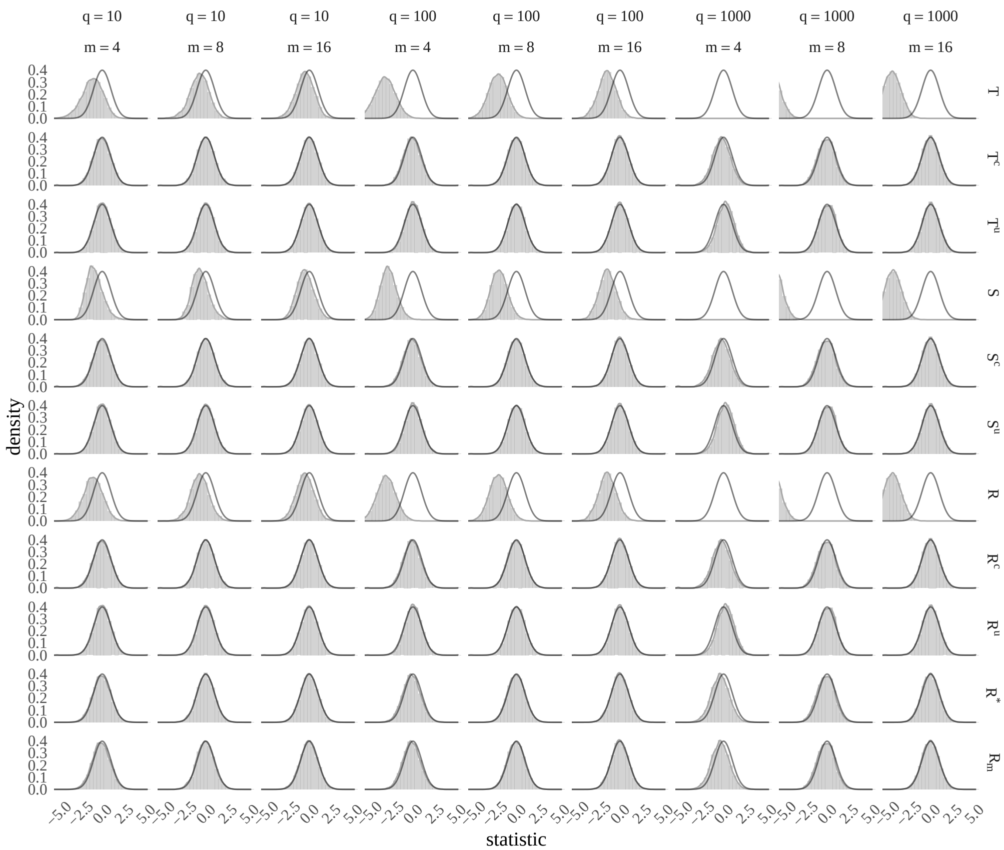

Figure 9: Curved exponential family. Estimated null distribution of statistics for the statistics in Table 1 for various combinations of q and m. The N(0,1) density function is superimposed.

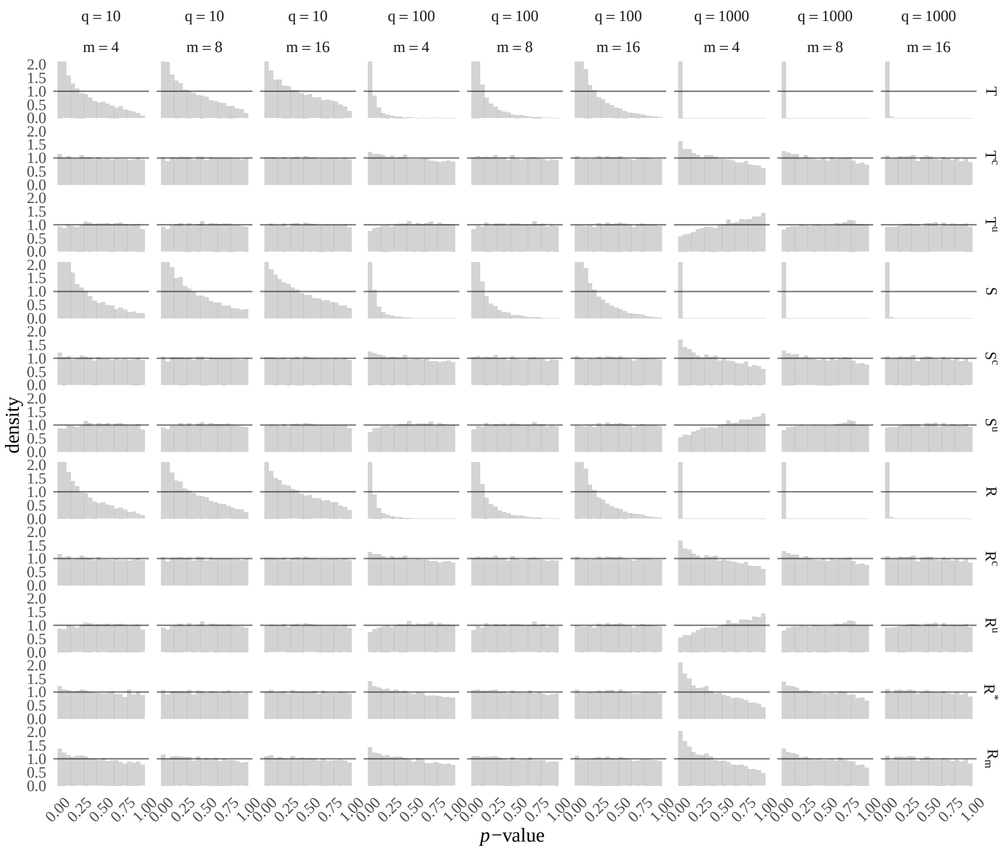

Figure 10: Curved exponential family. Estimated distribution of p-values for the statistics in Table 1 for various combinations of q and m. The Uniform(0,1) density function is superimposed.

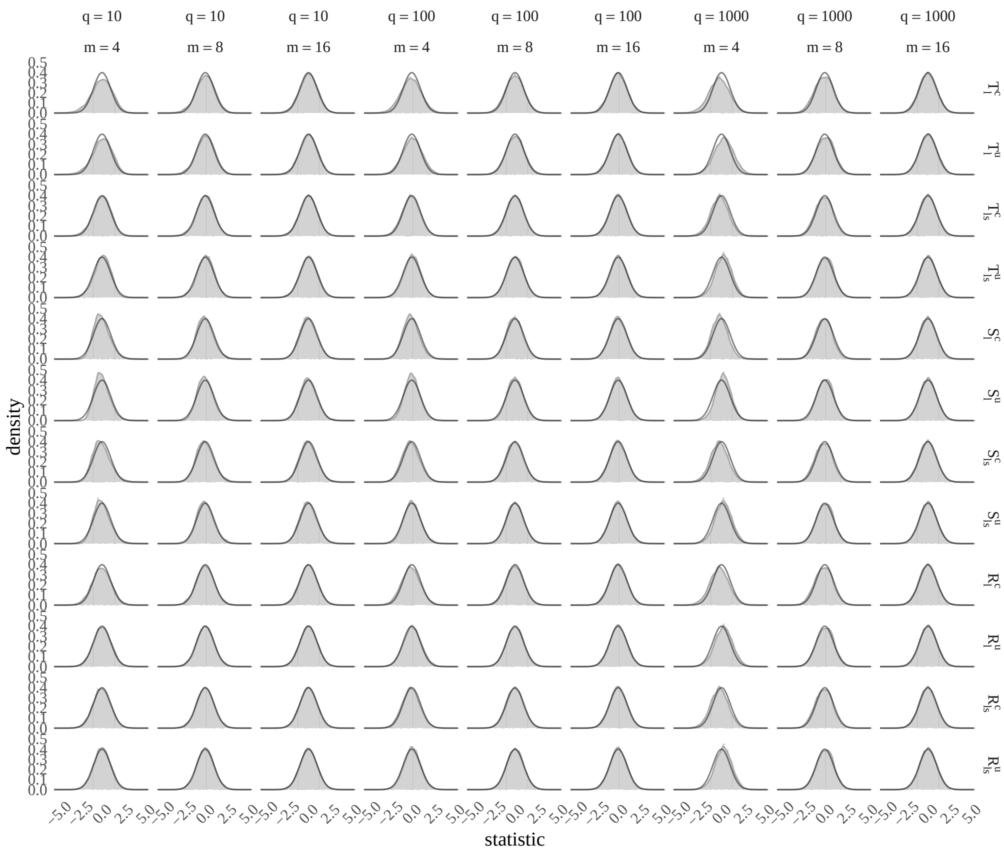

Figure 11: Curved exponential family. Estimated null distribution of statistics for the statistics in Table 2 for various combinations of q and m. The N(0,1) density function is superimposed.

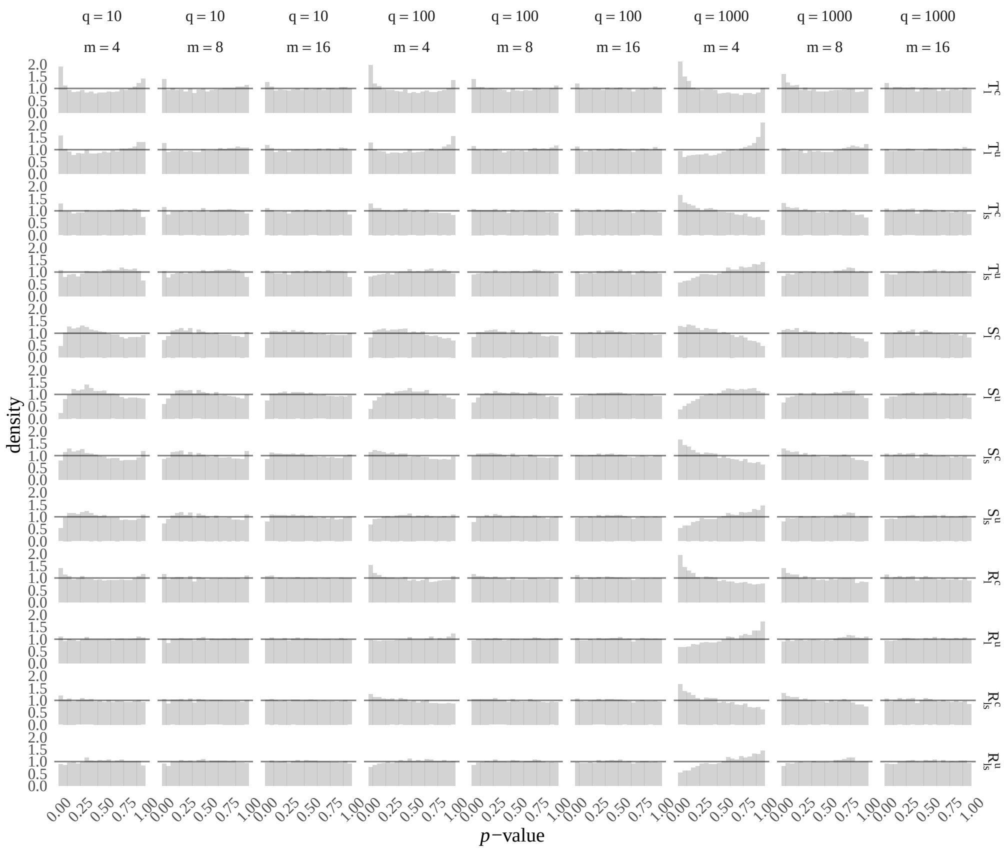

Figure 12: Curved exponential family. Estimated null distribution of p-values for the statistics in Table 2 for various combinations of q and m. The Uniform(0,1) density function is superimposed.

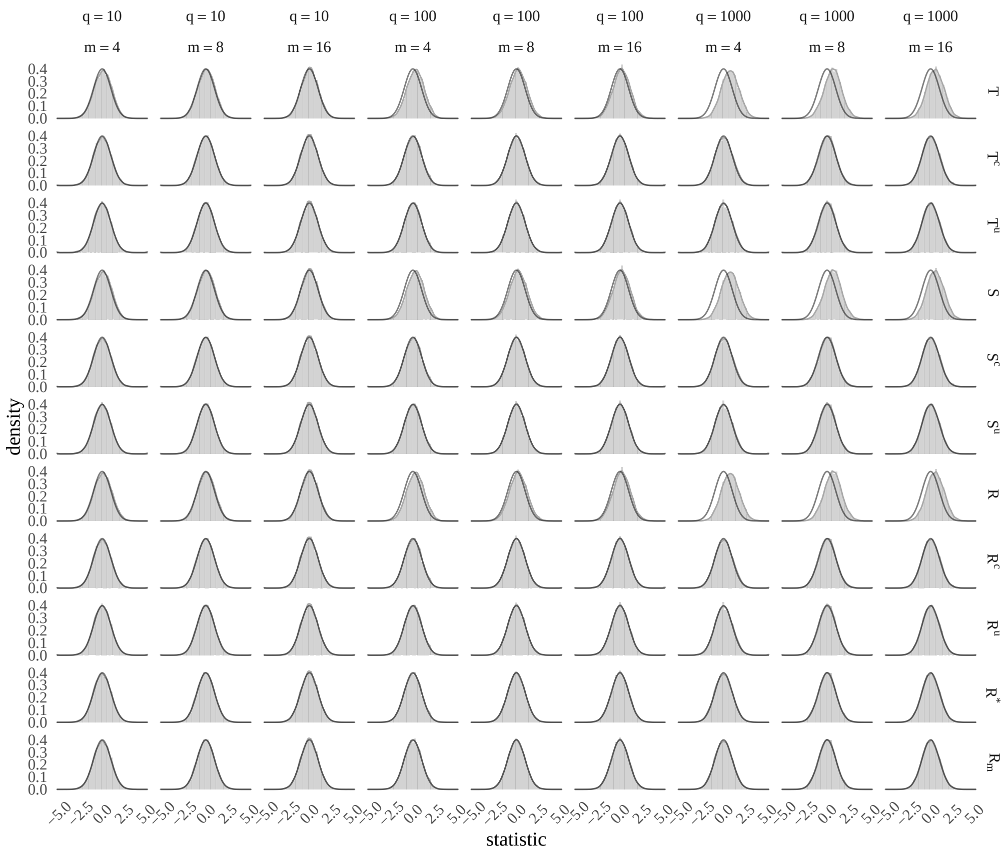

Figure 13: Truncated linear regression model. Estimated null distribution of statistics for the statistics in Table 1 for various combinations of q and m. The N(0,1) density function is superimposed.

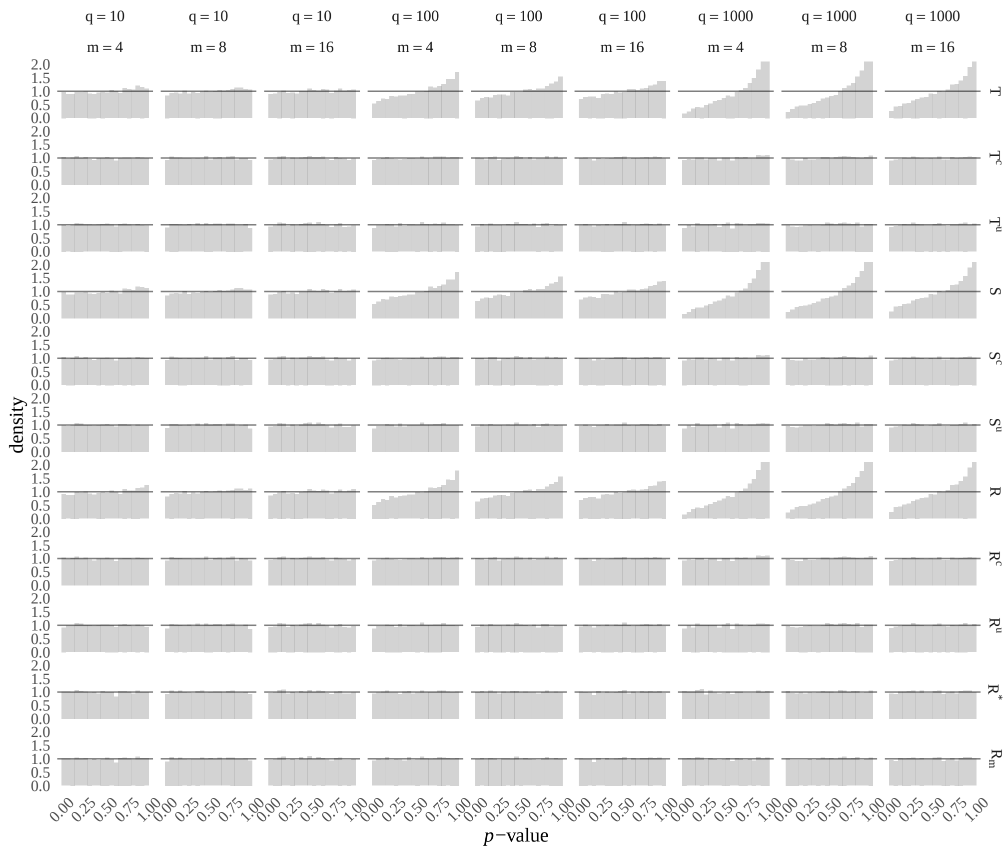

Figure 14: Truncated linear regression model. Estimated distribution of p-values for the statistics in Table 1 for various combinations of q and m. The Uniform(0,1) density function is superimposed.

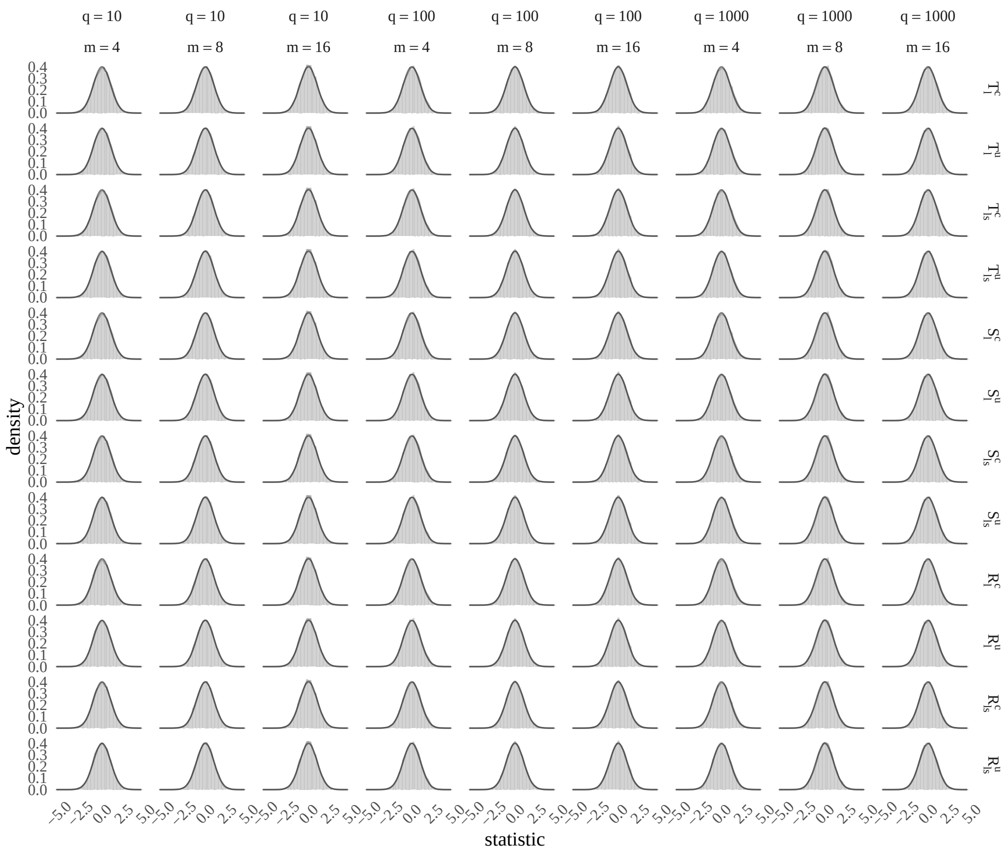

Figure 15: Truncated linear regression model. Estimated null distribution of statistics for the statistics in Table 2 for various combinations of q and m. The N(0,1) density function is superimposed.

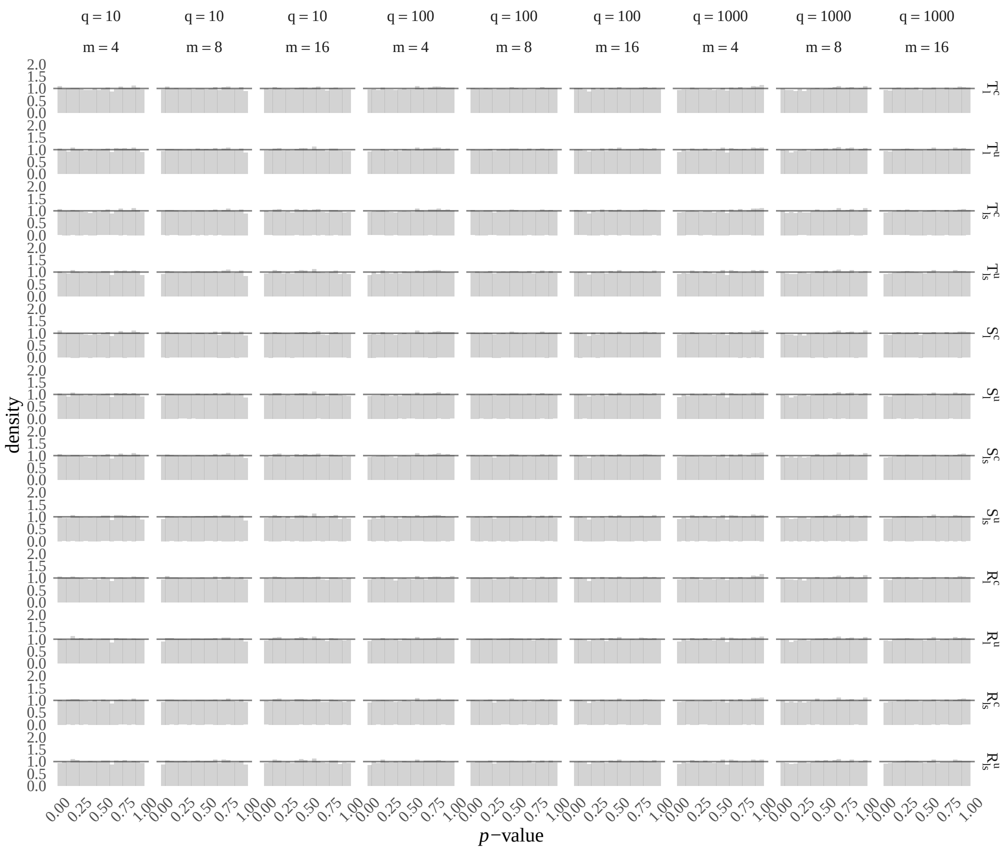

Figure 16: Truncated linear regression model. Estimated null distribution of p-values for the statistics in Table 2 for various combinations of q and m. The Uniform(0,1) density function is superimposed.

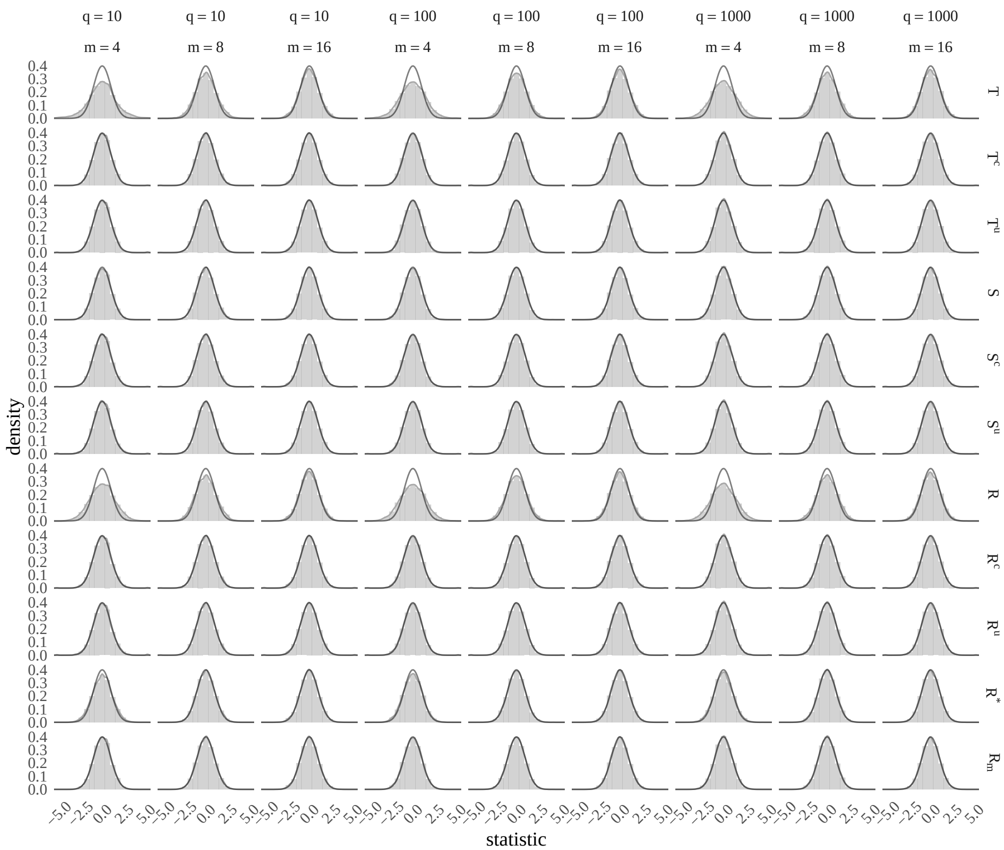

Figure 17: Behrens-Fisher model. Estimated null distribution of statistics for the statistics in Table 1 for various combinations of q and m. The N(0,1) density function is superimposed.

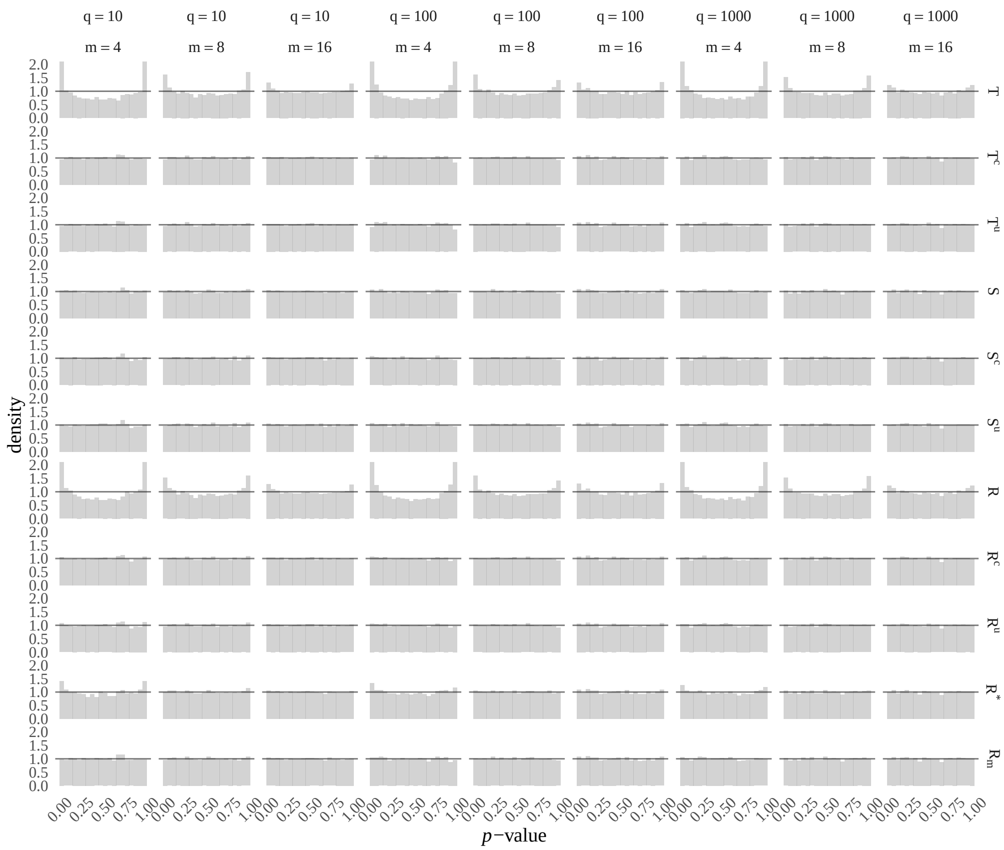

Figure 18: Behrens-Fisher model. Estimated distribution of p-values for the statistics in Table 1 for various combinations of q and m. The Uniform(0,1) density function is superimposed.

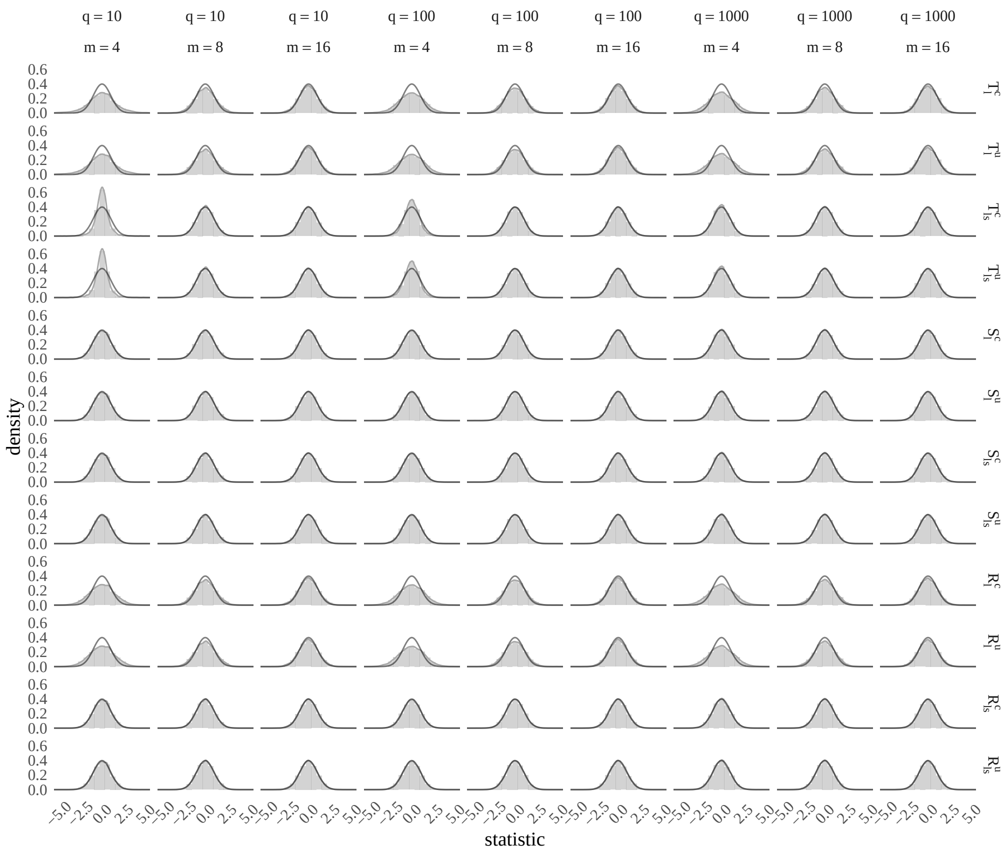

Figure 19: Behrens-Fisher model. Estimated null distribution of statistics for the statistics in Table 2 for various combinations of q and m. The N(0,1) density function is superimposed.

Figure 20: Behrens-Fisher model. Estimated null distribution of p-values for the statistics in Table 2 for various combinations of q and m. The Uniform(0,1) density function is superimposed.

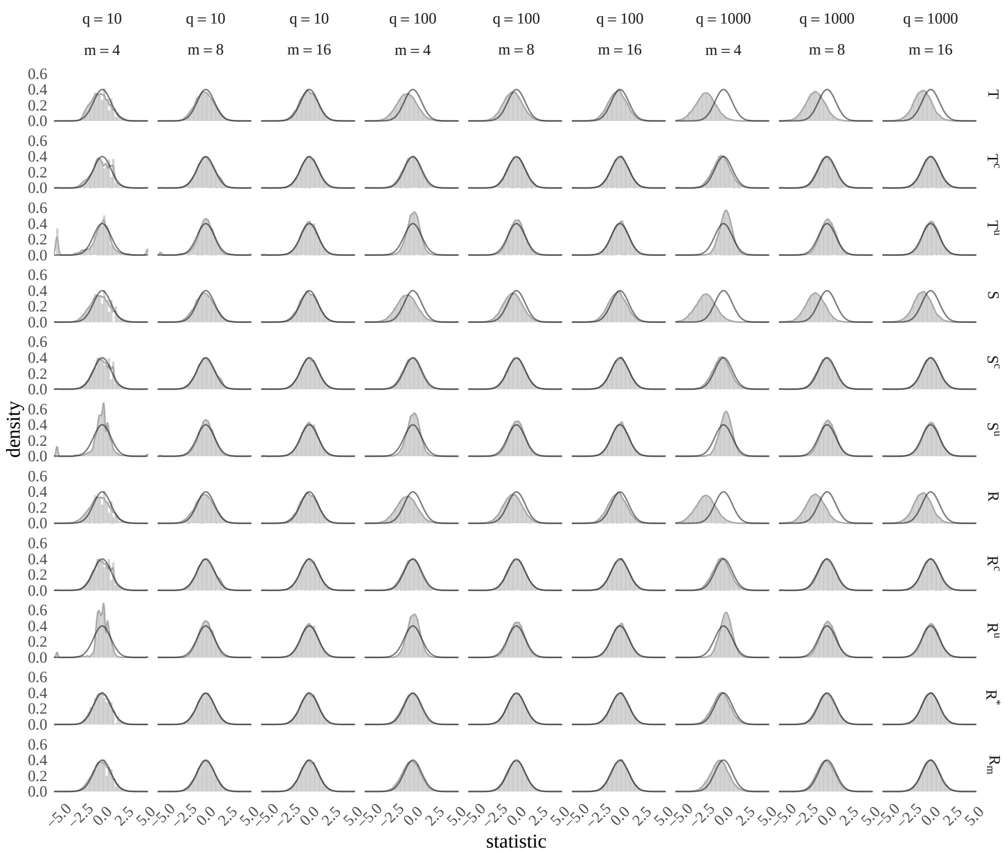

Figure 21: Logistic regression model. Estimated null distribution of statistics for the statistics in Table 1 for various combinations of q and m. The N(0,1) density function is superimposed.

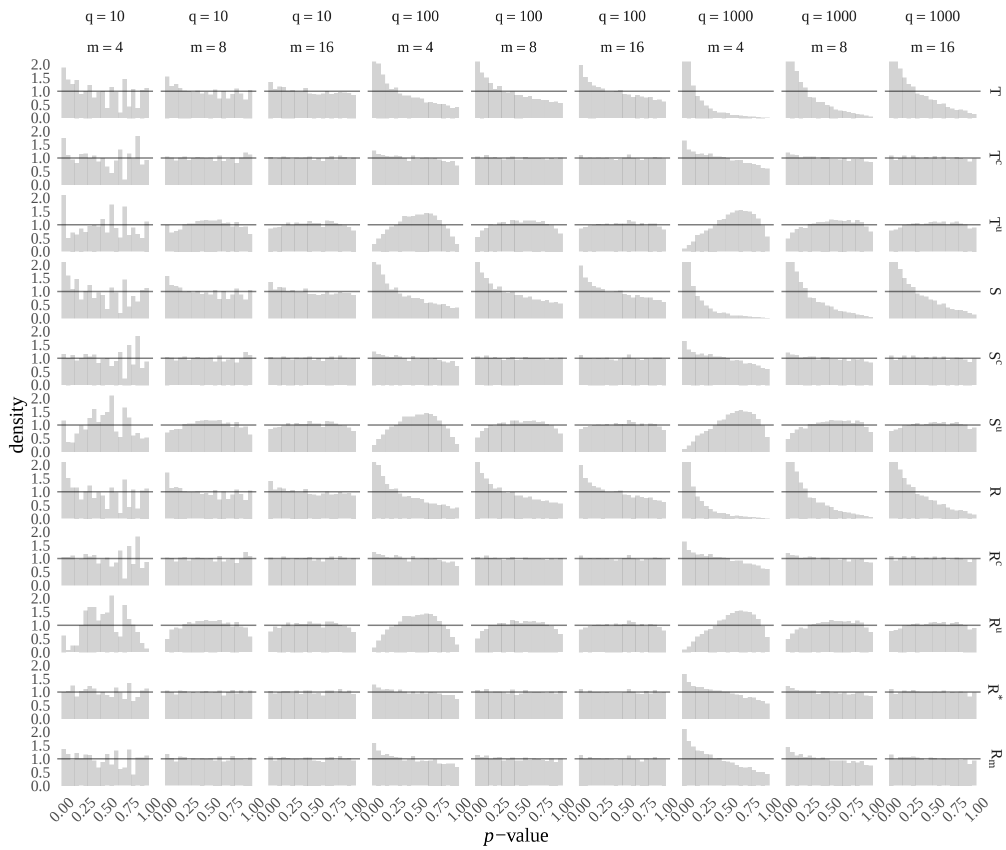

Figure 22: Logistic regression model. Estimated distribution of p-values for the statistics in Table 1 for various combinations of q and m. The Uniform(0,1) density function is superimposed.

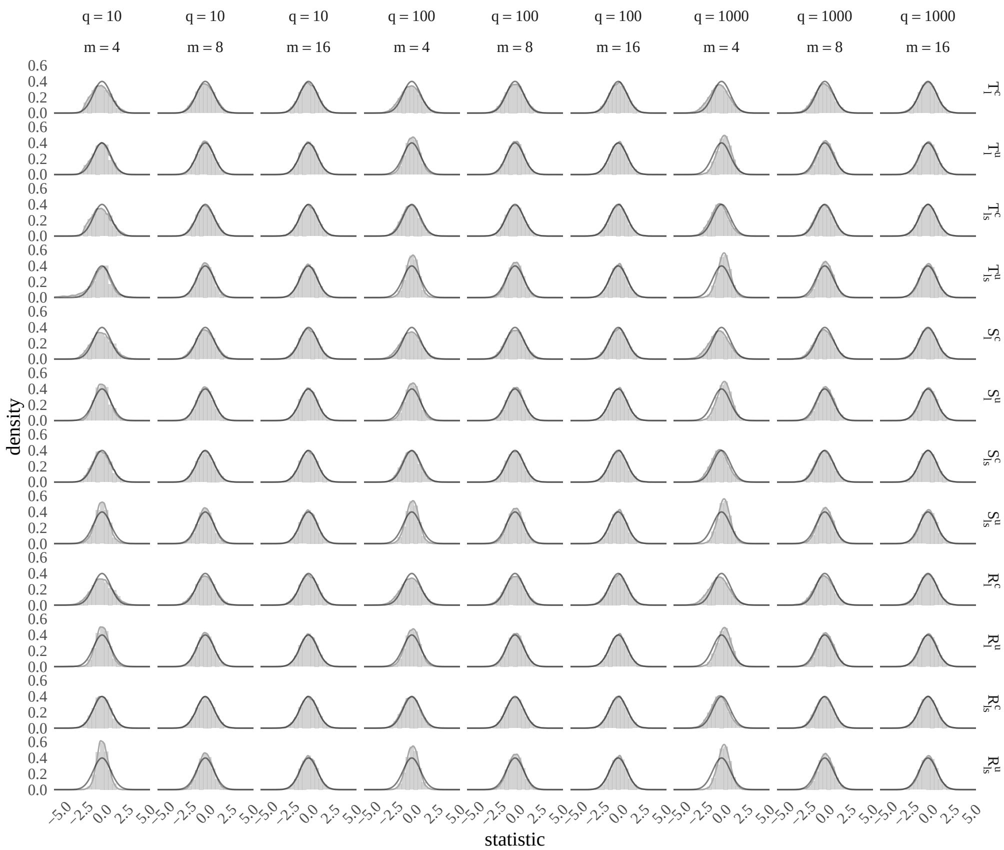

Figure 23: Logistic regression model. Estimated null distribution of statistics for the statistics in Table 2 for various combinations of q and m. The N(0,1) density function is superimposed.

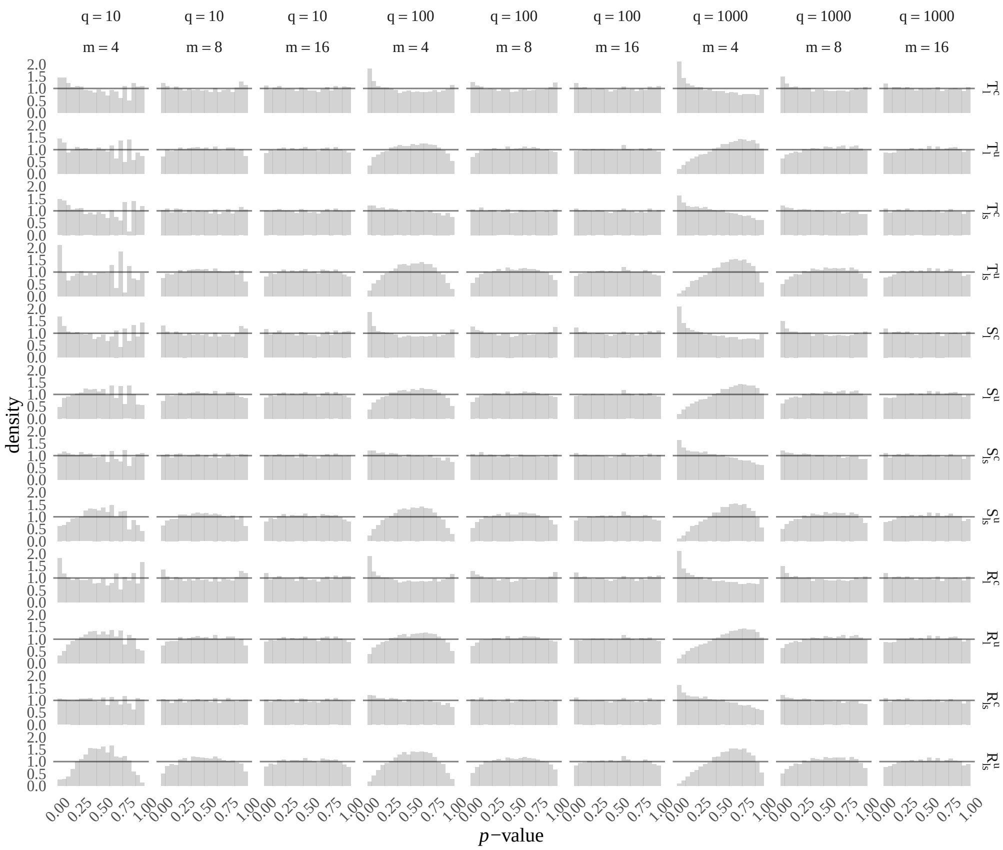

Figure 24: Logistic regression model. Estimated null distribution of p-values for the statistics in Table 2 for various combinations of q and m. The Uniform(0,1) density function is superimposed.

S3 Empirical tail probabilities

Tables 3-11 report empirical tail probabilities for the statistics in Table 1 of the main text, for all models and all simulation settings. The rows of each table are subdivided into 6 different blocks, each one corresponding to a different model.

Table 3: Empirical tail probabilities $\times 100$ for the statistics in Table 1 of the main text and all models that have been considered in the simulation experiments. The figures shown have been rounded to 1 decimal and are for $q = 10$ and $m = 4$.

		Nominal					
Model	Statistic	1.0	2.5	5.0	95.0	97.5	99.0
Gamma	R	0.1	0.1	0.3	65.2	75.6	85.3
	R^*	0.8	2.2	4.3	94.5	97.1	98.8
	R^u	1.0	2.5	4.7	95.6	97.7	99.2
	R^c	1.0	2.5	4.8	95.3	97.7	99.0
	R^c_l	1.5	3.1	5.6	94.4	96.9	98.7
	R^c_{ls}	1.0	2.5	4.8	95.3	97.5	99.0
Beta	R	0.0	0.0	0.1	58.5	69.3	80.1
	R^*	0.9	2.1	4.5	94.6	97.1	98.7
	R^u	1.0	2.4	5.0	94.7	97.2	98.8
	R^c	0.9	2.4	4.9	95.1	97.5	99.0
	R^c_l	1.2	2.8	5.8	94.2	96.8	98.5
	R^c_{ls}	0.9	2.3	4.9	95.1	97.5	98.9
Curved exponential family	R	11.8	19.4	28.1	99.4	99.8	99.9
	R^*	1.4	3.2	6.1	95.6	97.6	99.0
	R^u	0.8	2.1	4.4	95.8	98.0	99.2
	R^c	1.4	3.0	5.9	95.3	97.6	99.0
	R^c_l	1.9	4.1	7.1	94.2	96.9	98.6
	R^c_{ls}	1.4	3.1	6.0	95.4	97.6	99.0
Truncated regression	R	1.0	2.4	4.6	93.4	96.8	98.7
	R^*	1.2	2.6	5.1	95.1	97.5	99.1
	R^u	1.0	2.4	4.6	95.3	97.7	99.1
	R^c	1.1	2.6	5.2	95.0	97.5	99.0
	R^c_l	1.2	2.8	5.3	94.8	97.3	99.0
	R^c_{ls}	1.2	2.8	5.1	94.9	97.4	99.0
Behrens-Fisher	R	4.9	8.0	12.1	88.0	91.8	95.1
	R^*	1.9	4.0	7.0	92.9	96.0	98.1
	R^u	1.4	2.9	5.4	94.4	97.0	98.8
	R^c	1.1	2.7	5.2	94.7	97.3	98.9
	R^c_l	4.9	7.9	12.1	87.9	91.8	95.1
	R^c_{ls}	1.0	2.6	5.3	94.7	97.4	99.0
Logistic regression	R	3.8	7.1	12.5	94.4	97.0	98.4
	R^*	0.5	2.2	5.0	94.4	97.1	98.7
	R^u	2.1	2.6	3.1	99.4	99.6	99.7
	R^c	0.8	2.4	5.2	95.6	97.6	99.0
	R^c_l	2.9	5.5	9.1	91.7	94.5	97.2
	R^c_{ls}	1.0	2.6	5.3	94.8	97.3	99.0

Table 4: Empirical tail probabilities ×100 for the statistics in Table 1 of the main text and all models that have been considered in the simulation experiments. The figures shown have been rounded to 1 decimal and are for $q = 10$ and $m = 8$.

		Nominal					
Model	Statistic	1.0	2.5	5.0	95.0	97.5	99.0
Gamma	R	0.1	0.3	0.9	78.6	86.5	92.8
	R^*	0.9	2.3	4.6	94.7	97.4	98.8
	R^u	1.0	2.4	4.9	95.2	97.7	99.1
	R^c	1.0	2.5	4.9	95.0	97.5	99.0
	R^c_l	1.2	2.8	5.4	94.4	97.2	98.7
	R^c_{ls}	1.0	2.5	4.9	95.0	97.5	98.9
Beta	R	0.1	0.2	0.5	73.8	82.8	90.5
	R^*	0.9	2.3	4.9	94.7	97.1	98.9
	R^u	0.9	2.4	4.9	94.5	97.1	98.8
	R^c	0.9	2.4	4.8	94.8	97.4	99.0
	R^c_l	1.0	2.6	5.2	94.4	96.9	98.8
	R^c_{ls}	0.9	2.3	4.9	94.8	97.2	98.9
Curved exponential family	R	5.8	10.5	17.4	98.8	99.5	99.8
	R^*	1.1	3.0	5.3	95.1	97.5	99.0
	R^u	0.9	2.3	4.5	95.4	97.7	99.3
	R^c	1.1	2.9	5.2	95.0	97.4	99.1
	R^c_l	1.3	3.3	5.8	94.5	97.1	98.8
	R^c_{ls}	1.1	3.0	5.3	95.0	97.4	99.1
Truncated regression	R	0.8	1.9	4.1	94.4	97.2	98.8
	R^*	0.9	2.2	4.6	95.4	97.8	99.1
	R^u	0.8	1.9	4.3	95.7	97.9	99.2
	R^c	0.9	2.2	4.7	95.4	97.7	99.1
	R^c_l	0.9	2.3	4.7	95.3	97.6	99.0
	R^c_{ls}	0.9	2.2	4.7	95.4	97.6	99.0
Behrens-Fisher	R	2.0	4.3	7.6	92.0	95.2	97.6
	R^*	1.0	2.5	5.0	94.3	97.0	99.0
	R^u	1.0	2.3	4.8	94.5	97.3	99.1
	R^c	1.0	2.3	4.8	94.5	97.3	99.2
	R^c_l	2.0	4.3	7.6	91.9	95.2	97.6
	R^c_{ls}	0.9	2.4	4.9	94.5	97.2	99.0
Logistic regression	R	2.2	4.7	8.6	94.8	97.0	99.1
	R^*	0.9	2.7	5.3	94.8	97.0	99.1
	R^u	0.1	0.7	2.4	97.2	98.9	99.6
	R^c	0.9	2.6	5.2	94.5	97.9	99.1
	R^c_l	1.6	3.6	6.7	94.0	96.9	98.4
	R^c_{ls}	0.9	2.6	5.2	94.8	97.4	99.0

Table 5: Empirical tail probabilities ×100 for the statistics in Table 1 of the main text and all models that have been considered in the simulation experiments. The figures shown have been rounded to 1 decimal and are for $q = 10$ and $m = 16$.

		Nominal					
Model	Statistic	1.0	2.5	5.0	95.0	97.5	99.0
Gamma	R	0.2	0.6	1.5	85.7	91.6	96.0
	R^*	1.1	2.5	4.9	95.0	97.4	98.9
	R^u	1.0	2.6	5.0	95.2	97.5	99.0
	R^c	1.1	2.6	5.1	95.1	97.5	99.0
	R^c_l	1.2	2.8	5.3	94.9	97.3	98.9
	R^c_{ls}	1.2	2.6	5.1	95.1	97.4	98.9
Beta	R	0.2	0.4	1.1	83.4	89.8	94.8
	R^*	1.0	2.4	5.0	94.9	97.5	99.1
	R^u	1.1	2.6	5.1	94.8	97.4	99.0
	R^c	1.0	2.4	5.0	95.0	97.5	99.1
	R^c_l	1.0	2.5	5.1	94.9	97.3	99.0
	R^c_{ls}	1.0	2.4	5.0	95.0	97.5	99.1
Curved exponential family	R	3.7	7.7	13.1	98.4	99.2	99.8
	R^*	1.1	2.7	5.1	95.3	97.8	99.1
	R^u	1.0	2.4	4.9	95.5	97.9	99.2
	R^c	1.1	2.6	5.2	95.3	97.8	99.1
	R^c_l	1.3	3.3	5.8	94.5	97.1	98.8
	R^c_{ls}	1.1	3.0	5.3	95.0	97.4	99.1
Truncated regression	R	0.9	1.9	4.3	94.5	97.3	99.0
	R^*	1.0	2.2	4.8	95.1	97.7	99.2
	R^u	0.8	2.1	4.7	95.2	97.8	99.2
	R^c	0.9	2.2	4.7	95.1	97.7	99.2
	R^c_l	1.0	2.2	4.9	95.0	97.7	99.1
	R^c_{ls}	0.9	2.2	4.8	95.1	97.6	99.2
Behrens-Fisher	R	1.8	3.5	6.4	93.7	96.7	98.6
	R^*	1.2	2.9	5.3	94.8	97.4	99.0
	R^u	1.2	2.8	5.2	94.9	97.5	99.1
	R^c	1.2	2.8	5.1	94.9	97.5	99.1
	R^c_l	1.8	3.5	6.4	93.8	96.6	98.6
	R^c_{ls}	1.2	2.9	5.1	94.9	97.4	99.1
Logistic regression	R	1.7	3.7	7.0	95.7	98.0	99.3
	R^*	1.0	2.7	5.2	95.4	97.7	99.3
	R^u	0.6	1.8	3.8	96.3	98.4	99.5
	R^c	1.0	2.6	5.1	94.9	97.5	99.0
	R^c_l	1.2	3.1	6.0	94.6	97.2	98.9
	R^c_{ls}	1.0	2.7	5.2	95.1	97.6	99.1

Table 6: Empirical tail probabilities ×100 for the statistics in Table 1 of the main text and all models that have been considered in the simulation experiments. The figures shown have been rounded to 1 decimal and are for $q = 100$ and $m = 4$.

		Nominal					
Model	Statistic	1.0	2.5	5.0	95.0	97.5	99.0
Gamma	R	0.0	0.0	0.0	2.7	5.2	10.0
	R^*	0.6	1.6	3.5	92.8	96.2	98.4
	R^u	1.0	2.8	5.4	95.9	98.0	99.3
	R^c	0.9	2.6	5.2	95.2	97.6	99.1
	R^c_l	1.3	3.4	6.1	94.2	96.9	98.7
	R^c_{ls}	1.0	2.6	5.2	95.2	97.5	99.1
Beta	R	0.0	0.0	0.0	0.7	1.4	3.4
	R^*	0.8	2.0	3.8	93.4	96.7	98.7
	R^u	0.9	2.3	4.5	93.7	97.0	98.7
	R^c	0.9	2.5	4.9	94.8	97.4	99.1
	R^c_l	1.3	2.9	5.7	93.8	96.9	98.7
	R^c_{ls}	1.0	2.5	4.9	94.7	97.5	99.0
Curved exponential family	R	75.1	84.2	90.3	100.0	100.0	100.0
	R^*	1.6	3.7	7.1	96.0	97.9	99.2
	R^u	0.7	1.7	3.7	95.0	97.5	99.0
	R^c	1.3	3.2	6.2	95.7	97.8	99.1
	R^c_l	1.8	4.2	7.7	94.7	97.2	98.7
	R^c_{ls}	1.5	3.3	6.3	95.7	97.7	99.1
Truncated regression	R	0.4	1.2	2.5	91.1	95.1	98.2
	R^*	0.9	2.2	4.8	95.0	97.9	99.3
	R^u	0.8	2.0	4.4	95.1	97.8	99.3
	R^c	0.9	2.2	4.6	94.8	97.6	99.2
	R^c_l	0.9	2.3	4.7	94.6	97.5	99.2
	R^c_{ls}	0.9	2.2	4.6	94.8	97.6	99.2
Behrens-Fisher	R	5.2	8.8	12.8	88.5	92.2	95.2
	R^*	1.7	3.5	6.7	94.2	96.8	98.6
	R^u	1.2	2.8	5.3	95.1	97.7	98.9
	R^c	1.1	2.8	5.3	95.2	97.7	99.0
	R^c_l	5.3	8.9	12.8	88.4	92.2	95.3
	R^c_{ls}	1.1	2.8	5.3	95.2	97.7	99.0
Logistic regression	R	8.1	14.0	20.5	98.0	99.0	99.5
	R^*	1.4	3.2	6.4	96.4	98.1	99.2
	R^u	0.1	0.2	0.8	98.6	99.6	99.9
	R^c	1.3	3.2	6.2	96.4	98.2	99.2
	R^c_l	2.8	5.7	9.5	94.2	96.8	98.3
	R^c_{ls}	1.3	3.2	6.1	96.4	98.1	99.2

Table 7: Empirical tail probabilities $\times 100$ for the statistics in Table 1 of the main text and all models that have been considered in the simulation experiments. The figures shown have been rounded to 1 decimal and are for $q = 100$ and $m = 8$.

		Nominal					
Model	Statistic	1.0	2.5	5.0	95.0	97.5	99.0
Gamma	R	0.0	0.0	0.0	20.5	29.8	43.5
	R^*	0.8	2.0	4.4	93.8	97.1	98.9
	R^u	0.9	2.4	5.0	95.2	97.8	99.1
	R^c	1.1	2.4	5.0	94.9	97.5	99.1
	R^c_l	1.1	2.7	5.3	94.3	97.2	99.0
	R^c_{ls}	1.0	2.5	5.0	94.9	97.5	99.1
Beta	R	0.0	0.0	0.0	10.7	17.5	28.6
	R^*	0.8	2.2	4.6	94.4	97.2	99.0
	R^u	0.9	2.3	4.7	94.3	97.1	98.9
	R^c	0.9	2.4	4.9	94.7	97.2	99.0
	R^c_l	1.1	2.6	5.2	94.4	97.1	98.9
	R^c_{ls}	0.9	2.4	4.9	94.7	97.4	99.0
Curved exponential family	R	40.6	54.4	66.0	100.0	100.0	100.0
	R^*	1.1	2.8	5.3	95.3	97.7	99.1
	R^u	0.8	2.1	4.1	95.2	97.6	99.0
	R^c	1.1	2.7	5.1	95.4	97.6	99.0
	R^c_l	1.2	3.0	5.8	94.9	97.4	98.8
	R^c_{ls}	1.1	2.7	5.2	95.3	97.6	99.0
Truncated regression	R	0.5	1.5	3.2	92.2	95.9	98.2
	R^*	1.0	2.5	5.0	95.1	97.4	99.0
	R^u	0.9	2.4	4.7	95.1	97.5	99.0
	R^c	0.9	2.5	5.1	95.0	97.4	98.9
	R^c_l	1.0	2.6	5.1	94.8	97.3	98.8
	R^c_{ls}	1.0	2.5	5.1	95.0	97.3	98.9
Behrens-Fisher	R	2.1	4.6	8.0	92.9	96.0	97.8
	R^*	1.1	2.6	5.2	95.2	97.5	99.1
	R^u	1.0	2.5	5.1	95.4	97.6	99.1
	R^c	1.0	2.5	5.1	95.4	97.6	99.1
	R^c_l	2.1	4.5	8.0	92.8	95.9	97.9
	R^c_{ls}	1.0	2.5	5.2	95.4	97.5	99.1
Logistic regression	R	4.0	7.7	12.9	97.2	98.7	99.4
	R^*	1.1	2.5	5.3	94.9	97.5	99.0
	R^u	0.3	1.2	2.5	96.7	98.7	99.6
	R^c	1.1	2.5	5.3	94.9	97.5	99.0
	R^c_l	1.6	3.5	6.4	93.8	96.6	98.5
	R^c_{ls}	1.1	2.6	5.3	94.9	97.5	99.0

Table 8: Empirical tail probabilities ×100 for the statistics in Table 1 of the main text and all models that have been considered in the simulation experiments. The figures shown have been rounded to 1 decimal and are for $q = 100$ and $m = 16$.

		Nominal					
Model	Statistic	1.0	2.5	5.0	95.0	97.5	99.0
Gamma	R	0.0	0.0	0.1	48.7	61.2	73.7
	R^*	1.2	2.7	5.3	94.9	97.4	99.0
	R^u	1.2	2.8	5.5	95.1	97.7	99.0
	R^c	1.2	2.7	5.4	95.0	97.6	99.1
	R^c_l	1.3	2.9	5.7	94.8	97.5	99.0
	R^c_{ls}	1.3	2.8	5.5	95.1	97.6	99.1
Beta	R	0.0	0.0	0.0	34.8	47.2	61.1
	R^*	0.9	2.3	4.7	94.7	97.4	98.9
	R^u	1.0	2.3	4.6	94.7	97.3	98.9
	R^c	0.9	2.2	4.7	94.8	97.4	98.9
	R^c_l	1.0	2.4	4.9	94.5	97.4	98.9
	R^c_{ls}	1.0	2.4	4.8	94.8	97.4	98.9
Curved exponential family	R	19.0	30.3	42.2	99.9	100.0	100.0
	R^*	1.0	2.6	5.5	95.3	97.5	99.0
	R^u	0.8	2.3	4.9	95.2	97.6	99.0
	R^c	0.9	2.6	5.4	95.2	97.5	99.0
	R^c_l	1.1	2.8	5.6	95.0	97.4	98.9
	R^c_{ls}	1.0	2.6	5.4	95.2	97.5	99.0
Truncated regression	R	0.7	1.5	3.5	93.0	96.2	98.3
	R^*	1.0	2.4	5.1	95.2	97.4	98.9
	R^u	1.0	2.3	4.8	95.2	97.5	99.0
	R^c	0.8	2.5	5.0	95.2	97.4	98.9
	R^c_l	1.0	2.5	5.1	95.0	97.3	98.9
	R^c_{ls}	0.9	2.4	5.1	95.0	97.3	98.9
Behrens-Fisher	R	1.7	3.6	6.6	93.4	96.4	98.4
	R^*	1.2	2.8	5.4	94.6	97.4	98.9
	R^u	1.1	2.8	5.3	94.6	97.4	98.9
	R^c	1.1	2.7	5.3	94.6	97.4	98.9
	R^c_l	1.7	3.6	6.7	93.3	96.4	98.4
	R^c_{ls}	1.1	2.9	5.4	94.5	97.4	98.9
Logistic regression	R	2.7	5.9	10.0	97.0	98.6	99.4
	R^*	1.1	2.8	5.5	95.0	97.6	99.1
	R^u	0.7	1.9	4.2	96.0	98.2	99.3
	R^c	1.1	2.8	5.6	95.1	97.6	99.0
	R^c_l	1.4	3.2	6.2	94.5	97.2	98.9
	R^c_{ls}	1.1	2.8	5.5	95.0	97.6	99.1

Table 9: Empirical tail probabilities $\times 100$ for the statistics in Table 1 of the main text and all models that have been considered in the simulation experiments. The figures shown have been rounded to 1 decimal and are for $q = 1000$ and $m = 4$.

		Nominal					
Model	Statistic	1.0	2.5	5.0	95.0	97.5	99.0
Gamma	R	0.0	0.0	0.0	0.0	0.0	0.0
	R^*	0.2	0.5	1.2	83.7	90.3	95.3
	R^u	1.3	3.2	6.4	96.4	98.2	99.2
	R^c	1.0	2.5	5.1	94.8	97.4	98.9
	R^c_l	1.4	3.1	6.2	93.9	96.7	98.5
	R^c_{ls}	1.0	2.5	5.1	95.0	97.4	98.9
Beta	R	0.0	0.0	0.0	0.0	0.0	0.0
	R^*	0.3	0.8	1.8	88.6	93.5	96.7
	R^u	0.5	1.3	2.8	91.4	94.9	97.6
	R^c	0.7	1.9	3.8	93.7	96.4	98.6
	R^c_l	1.0	2.3	4.6	92.9	95.7	98.1
	R^c_{ls}	0.8	1.8	3.9	93.7	96.5	98.5
Curved exponential family	R	100.0	100.0	100.0	100.0	100.0	100.0
	R^*	3.2	6.3	10.6	97.8	99.0	99.6
	R^u	0.4	1.3	2.7	92.8	96.4	98.5
	R^c	2.2	4.8	8.3	96.9	98.5	99.4
	R^c_l	3.1	6.0	9.8	96.2	98.0	99.2
	R^c_{ls}	2.4	4.9	8.3	96.9	98.6	99.4
Truncated regression	R	0.1	0.4	0.8	80.5	87.9	93.6
	R^*	1.0	2.3	5.2	94.8	97.5	98.9
	R^u	0.8	2.0	4.4	94.8	97.5	98.9
	R^c	0.9	2.2	4.6	94.4	97.2	98.8
	R^c_l	0.9	2.1	4.7	94.2	97.1	98.7
	R^c_{ls}	0.9	2.1	4.6	94.4	97.2	98.8
Behrens-Fisher	R	5.3	8.7	12.3	87.8	92.0	95.1
	R^*	1.5	3.3	6.3	94.1	96.8	98.6
	R^u	1.0	2.6	5.2	95.1	97.5	99.0
	R^c	1.0	2.7	5.1	95.1	97.5	99.1
	R^c_l	5.3	8.7	12.3	87.8	92.0	95.1
	R^c_{ls}	1.1	2.7	5.1	95.1	97.4	99.0
Logistic regression	R	40.2	53.2	64.3	99.9	100.0	100.0
	R^*	2.0	4.5	8.3	97.1	98.5	99.5
	R^u	0.0	0.1	0.5	97.2	98.9	99.8
	R^c	1.8	4.2	8.1	97.0	98.5	99.4
	R^c_l	3.7	7.5	11.8	95.2	97.3	98.7
	R^c_{ls}	1.8	4.2	8.2	97.0	98.5	99.5

Table 10: Empirical tail probabilities ×100 for the statistics in Table 1 of the main text and all models that have been considered in the simulation experiments. The figures shown have been rounded to 1 decimal and are for $q = 1000$ and $m = 8$. The table includes all the results of Table 2 of the main text.

		Nominal					
Model	Statistic	1.0	2.5	5.0	95.0	97.5	99.0
Gamma	R	0.0	0.0	0.0	0.0	0.0	0.0
	R^*	0.7	1.5	3.3	92.1	95.5	98.2
	R^u	1.1	2.7	5.3	95.1	97.7	99.2
	R^c	1.0	2.5	5.0	94.6	97.2	99.0
	R^c_l	1.1	2.8	5.4	94.2	96.8	98.9
	R^c_{ls}	1.1	2.5	4.9	94.5	97.2	99.1
Beta	R	0.0	0.0	0.0	0.0	0.0	0.0
	R^*	0.7	1.8	3.8	93.7	96.8	98.8
	R^u	0.8	1.9	4.1	94.0	97.0	98.7
	R^c	1.0	2.3	4.8	95.0	97.4	99.1
	R^c_l	1.1	2.5	5.1	94.7	97.3	98.9
	R^c_{ls}	0.9	2.3	4.8	95.1	97.5	99.0
Curved exponential family	R	100.0	100.0	100.0	100.0	100.0	100.0
	R^*	1.4	3.5	6.9	96.6	98.3	99.4
	R^u	0.6	1.8	4.0	95.0	97.7	99.2
	R^c	1.2	3.3	6.4	96.2	98.2	99.4
	R^c_l	1.5	3.6	7.1	95.8	98.0	99.2
	R^c_{ls}	1.3	3.2	6.5	96.3	98.2	99.4
Truncated regression	R	0.2	0.5	1.1	84.2	90.4	95.1
	R^*	1.0	2.5	5.1	94.8	97.3	98.9
	R^u	0.9	2.3	4.8	94.9	97.2	98.9
	R^c	0.9	2.4	4.9	94.5	97.2	98.7
	R^c_l	0.9	2.4	5.0	94.4	97.0	98.8
	R^c_{ls}	0.9	2.4	5.0	94.4	97.0	98.8
Behrens-Fisher	R	2.4	4.6	7.6	92.1	95.6	98.0
	R^*	1.2	2.8	5.3	94.8	97.5	99.0
	R^u	1.1	2.7	5.2	95.0	97.6	99.1
	R^c	1.1	2.7	5.1	94.9	97.6	99.1
	R^c_l	2.4	4.7	7.6	92.1	95.5	98.0
	R^c_{ls}	1.1	2.6	5.1	95.0	97.6	99.1
Logistic regression	R	18.1	28.4	39.3	99.7	99.9	100.0
	R^*	1.4	3.1	6.1	95.8	97.9	99.1
	R^u	0.4	1.1	2.4	96.3	98.4	99.4
	R^c	1.5	3.1	6.0	95.8	97.9	99.2
	R^c_l	1.9	4.1	7.5	94.7	97.1	98.8
	R^c_{ls}	1.4	3.2	6.0	95.8	97.9	99.1

Table 11: Empirical tail probabilities ×100 for the statistics in Table 1 of the main text and all models that have been considered in the simulation experiments. The figures shown have been rounded to 1 decimal and are for $q = 1000$ and $m = 16$.

		Nominal					
Model	Statistic	1.0	2.5	5.0	95.0	97.5	99.0
Gamma	R	0.0	0.0	0.0	0.0	0.0	0.2
	R^*	0.9	2.1	4.3	94.3	96.9	98.5
	R^u	1.1	2.5	5.0	95.2	97.6	98.9
	R^c	1.0	2.5	4.9	95.0	97.5	98.9
	R^c_l	1.1	2.6	5.1	94.8	97.3	98.6
	R^c_{ls}	1.1	2.5	5.0	95.0	97.5	98.8
Beta	R	0.0	0.0	0.0	0.0	0.0	0.0
	R^*	1.0	2.3	4.5	94.3	97.2	98.8
	R^u	1.1	2.3	4.7	94.4	97.2	98.8
	R^c	1.0	2.4	4.8	94.8	97.4	99.0
	R^c_l	1.2	2.5	5.0	94.7	97.3	98.9
	R^c_{ls}	1.1	2.4	4.9	94.8	97.4	98.9
Curved exponential family	R	98.2	99.3	99.7	100.0	100.0	100.0
	R^*	1.0	2.8	5.5	95.9	98.1	99.3
	R^u	0.7	2.1	4.5	95.3	97.9	99.3
	R^c	1.1	2.6	5.4	95.7	98.1	99.3
	R^c_l	1.1	2.8	5.7	95.5	97.9	99.2
	R^c_{ls}	1.0	2.6	5.4	95.7	98.0	99.3
Truncated regression	R	0.2	0.5	1.2	86.6	92.4	96.4
	R^*	0.8	2.2	4.8	95.0	97.7	98.9
	R^u	0.8	2.0	4.5	94.8	97.7	99.0
	R^c	0.8	2.1	4.6	94.9	97.6	99.0
	R^c_l	0.9	2.2	4.7	94.8	97.5	98.9
	R^c_{ls}	0.8	2.1	4.6	94.9	97.5	98.9
Behrens-Fisher	R	1.5	3.4	6.1	93.9	96.6	98.5
	R^*	1.1	2.6	5.0	94.9	97.5	98.9
	R^u	1.1	2.6	5.0	95.0	97.5	99.1
	R^c	1.1	2.6	5.0	95.1	97.5	99.1
	R^c_l	1.5	3.4	6.2	93.8	96.7	98.5
	R^c_{ls}	1.1	2.6	4.9	95.0	97.5	99.0
Logistic regression	R	8.5	15.2	23.7	99.3	99.7	100.0
	R^*	1.3	2.9	5.5	95.1	97.7	99.0
	R^u	0.7	1.9	3.9	95.5	98.0	99.2
	R^c	1.3	2.9	5.5	95.1	97.6	99.0
	R^c_l	1.6	3.4	6.0	94.7	97.2	98.9
	R^c_{ls}	1.3	2.9	5.5	95.1	97.6	99.1

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