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# ASYMPTOTICALLY OPTIMAL SEQUENTIAL POINT ESTIMATION OF THE MEAN OF AN EXPONENTIAL FAMILY

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*Abstract:* This paper provides a fully sequential procedure for estimating the mean of a one-parameter exponential family of probability distributions with a weighted squared error loss and a fixed cost for each observation. The procedure is shown to be asymptotically optimal to second order among all stopping times, for a general class of estimators of the mean.

*Key words and phrases:* One-parameter exponential family, regret, stopping time, uniformly integrable.

## 1. Introduction

Let $\Omega$ be an open interval and let $F_\omega$, $\omega \in \Omega$, denote a one-parameter exponential family of probability distributions on $\mathbf{R} = (-\infty, \infty)$; that is,

$$dF_\omega(x) = \exp\{\omega x - \psi(\omega)\}\lambda(dx)$$

for $x \in \mathbf{R}$ and $\omega \in \Omega$, where $\lambda$ is a non-degenerate $\sigma$-finite measure on $\mathbf{R}$, $\exp\{\psi(\omega)\} = \int e^{\omega x}\lambda(dx)$, and $\Omega$ is the set of all $\omega$ for which this integral is finite. Recall that the mean and variance of $F_\omega$ are $\theta = \psi'(\omega)$ and $\sigma^2 = \psi''(\omega)$, respectively, where the prime denotes differentiation.

Let $X_1, X_2, \ldots$ be independent random variables with common distribution function $F_\omega$, under a probability measure $P_\omega$, for some unknown $\omega \in \Omega$. Based on a fixed sample size $t$, the population mean $\theta$ can be estimated by its maximum likelihood estimator $\bar{X}_t = (X_1 + \cdots + X_t)/t$, which is an unbiased estimator of $\theta$. However, when $t$ is a stopping time, $\bar{X}_t$ may be biased for $\theta$. For instance, suppose that $F_\omega$ is the d.f. of the normal distribution with mean $\theta = \omega$ and unit variance, and define a stopping time $t$ by $t = \inf\{n \geq 1 : 2^{-1}n\bar{X}_n^2 > a\}$ for $a \geq 0$. Following the proof of Theorem 9.1 of Woodroofe (1982), it can be shown that $E_\omega[\bar{X}_t] = \theta(1 + 1/a) + o(1/a)$ as $a \to \infty$, for all $\omega \neq 0$. This suggests considering estimators of $\theta$ of the form $\bar{X}_t/(1+1/a)$ or $\hat{\theta}_t = \bar{X}_t - \bar{X}_t/(1+2^{-1}t\bar{X}_t^2)$.

More generally, let $b_n, n \geq 1$, be a sequence of bounded, continuous functions on $\mathbf{R}$, which converges to a continuously differentiable function $b$, uniformly on compact subsets of $\mathbf{R}$, and such that $\sup_{x\in\mathbf{R}} |b_n(x)| = o(\sqrt{n})$ as $n \to \infty$. Consider estimators of $\theta$ of the form

$$\hat{\theta}_n = \bar{X}_n - \frac{1}{n} b_n(\bar{X}_n)$$

for $n \geq 1$, and suppose that when observation is terminated at time $t = n$, the loss incurred is

$$L_a(n, \omega) = a^2\gamma^2(\omega)(\hat{\theta}_n - \theta)^2 + n \tag{1.1}$$

for $\omega \in \Omega$, where $a > 0$ and $\gamma$ is a positive, twice continuously differentiable function on $\Omega$. Then the risk is $R_a(n,\omega) = E_\omega[L_a(n,\omega)] = \frac{1}{n}[g(\theta)]^2 + n + o(1/n)$ for all sufficiently large $n$, where $E_\omega$ denotes expectation with respect to $P_\omega$ and

$$g(\theta) = \gamma(\omega)\sqrt{\psi''(\omega)}.$$

If $g(\theta)$ is known, then the approximate risk is minimized by letting $n$ be an integer adjacent to $n_0 = ag(\theta)$; in which case the minimum risk is $R_a(n_0, \omega) \approx 2ag(\theta)$ for all sufficiently large $a$ and all $\omega \in \Omega$. However, if $g(\theta)$ is unknown, then $n_0$ cannot be used and there is no fixed sample size procedure that achieves the risk $2ag(\theta)$. In view of $n_0$, let

$$t = \inf\{n \geq l : n > a\hat{g}_n\} \tag{1.2}$$

for $a \geq 0$, where $l \geq 1$ is an initial sample size and $\hat{g}_n$ is an estimator of $g(\theta)$ which is defined below. This type of stopping procedure was first considered by Robbins (1959) for the estimation of the mean of a normal distribution.

Recently, Tahir (1989) showed that

$$\liminf_{a\to\infty} \inf_s \sup_{\omega\in J} r_a(s;\omega) \geq \sup_{\omega\in J} G(\omega)$$

for every compact subinterval $J$ of $\Omega$, where

$$r_a(s;\omega) = E_\omega[L_a(s,\omega)] - 2ag(\theta) \tag{1.3}$$

is the regret of the stopping time $s$ and

$$G(\omega) = \psi''(\omega)\frac{[g'(\theta)]^2}{g^2(\theta)} + \gamma^2(\omega)\frac{d^2}{d\omega^2}\left[\frac{1}{g^2(\theta)}\right] - 2\gamma^2(\omega)\frac{d}{d\omega}\left[\frac{b(\theta)}{g^2(\theta)}\right] + \frac{b^2(\theta)}{\psi''(\omega)} \tag{1.4}$$

for $\omega \in \Omega$. The main result of this paper shows that $\lim_{a\to\infty} r_a(t;\omega) = G(\omega)$ uniformly with respect to $\omega$ on compact subintervals of $\Omega$, where $t$ is as in (1.2) with $\hat{g}_n = g_n(\bar{X}_n)$ and $g_n$, $n \geq 1$, is a sequence of continuous functions on $\mathbf{R}$ such that $g_n = g$ on $\Omega_n$, where $\Omega_n$, $n \geq 1$, is a sequence of compact subintervals of $\Omega$ for which $\frac{1}{n} \leq g \leq n$ on $\Omega_n$, for each $n \geq 1$. The result implies that $t$ has asymptotic local minimax regret (see Woodroofe (1985) for more details on asymptotic local minimaxity.) The estimators $\hat{g}_n$, $n \geq 1$, have been considered by Woodroofe (1987) who shows that asymptotic local minimax regret can be obtained by procedures that take observations in three stages when $\theta$ is estimated by $\bar{X}_n$, $n \geq 1$.

## 2. Preliminary Results

Let $t$ be defined by (1.2) and observe that $t \geq \sqrt{a}$ for all sufficiently large $a$.

**Lemma 2.1.** *If $J$ is any compact subinterval of $\Omega$, then*

$$P_\omega\{t \leq \epsilon a\} = o(1/a^p) \quad \textit{and} \quad \sum_{n > a/\epsilon} n^p P_\omega\{t > n\} \to 0$$

*uniformly in $\omega \in J$ as $a \to \infty$, for all sufficiently small $\epsilon > 0$ and all $p \geq 0$.*

**Proof.** The lemma follows from Lemma 2.1 of Woodroofe (1987) since $P_\omega\{t \leq \epsilon a\} \leq P_\omega\{|1/\hat{g}_n - 1/g(\theta)| > \epsilon, \exists n \geq \sqrt{a}\}$ for all $\omega \in J$ and all $\epsilon < 1/2$ such that $2\epsilon \leq \inf_{\omega \in J} g(\theta)$ and since, for any $n > a/\epsilon$, $P_\omega\{t > n\} \leq P_\omega\{\hat{g}_n > 1/\epsilon\} \leq P_\omega\{|\hat{g}_n - g(\theta)| > \epsilon\}$ for all $\omega \in J$ and all $\epsilon < 1/2$ such that $2\epsilon \leq \inf_{\omega \in J} g(\theta) \leq \sup_{\omega \in J} g(\theta) \leq 1/(2\epsilon)$.

**Lemma 2.2.** *If $J$ is any compact subinterval of $\Omega$, then*

$$\lim_{a \to \infty} E_\omega\left[\left|\frac{t}{a} - g(\theta)\right|^p\right] = 0 \quad \textit{and} \quad \frac{a}{t} \to \frac{1}{g(\theta)} \quad \textit{in } P_\omega\textit{-probability}$$

*uniformly with respect to $\omega \in J$ as $a \to \infty$, for all $p \geq 1$.*

**Proof.** The first assertion follows from Lemma 2.2 of Woodroofe (1987) since $|t/a - g(\theta)|^p \leq 2^{p-1}a^{-p} + 2^{2p-2}|\hat{g}_{t-1} - g(\theta)|^p + 2^{2p-2}|\hat{g}_t - g(\theta)|^p$, by the $c_p$-inequality (Loève (1963)) and (1.2). The second assertion follows easily from Markov's inequality.

In the remainder of this paper let $S_n = X_1 + \cdots + X_n$ for each $n \geq 1$.

**Theorem 2.1.** *Let $W_n = (S_n - n\theta)/\sqrt{n\psi''(\omega)}$ for $\omega \in \Omega$ and $n \geq 1$. If $J$ is any compact subinterval of $\Omega$, then $W_t$ converges in distribution, under $P_\omega$, to a standard normal random variable, uniformly with respect to $\omega \in J$ as $a \to \infty$.*

**Proof.** The theorem can be established by following the proof of Theorem 1.4 of Woodroofe (1982) and using Lemma 2.2, Kolmogorov's inequality, and the fact that $\psi''$ is bounded on $J$.

**Lemma 2.3.** *If $J$ is any compact subinterval of $\Omega$, then there are positive constants $C = C(J)$ and $\delta = \delta(J)$ for which*

$$\sup_{\omega\in J} P_\omega\left\{ \max_{1\le n\le N} |S_n - n\theta| > x \right\} \le 2\max\left\{ e^{-x^2/(2NC)}, e^{-\frac{1}{2}\delta x} \right\}$$

*for all $x > 0$ and all integer $N \ge 1$.*

**Proof.** Since $S_n - n\theta$, $n \ge 1$, is a martingale with respect to $P_\omega$, for each $\omega \in \Omega$,

$$\begin{aligned} P_\omega\left\{ \max_{1\le n\le N} (S_n - n\theta) > x \right\} &\le e^{-rx} E_\omega[\exp\{r(S_N - N\theta)\}] \\ &= \exp\left\{ -rx + N[\psi(\omega + r) - \psi(\omega) - r\theta] \right\} \\ &= \exp\left\{ -rx + Nr^2\psi''(\omega^*)/2 \right\} \qquad (2.1) \end{aligned}$$

for all $r > 0$ and all $\omega \in \Omega$ for which $\omega + r \in \Omega$, and all $x > 0$, where $\omega^*$ is an intermediate point between $\omega$ and $\omega + r$. Let $J_\delta$ denote a $\delta$-neighborhood of $J$ and choose $\delta$ so small that $J_\delta \subset \Omega$. Let $C$ denote an upper bound for $\psi''$ on $J_\delta$. Then RHS(2.1) $\le \exp\{-rx + NCr^2/2\} = \exp\{Q(r,x)\}$, say. The lemma follows by minimizing $Q(r,x)$ with respect to $r > 0$ for $x \le NC\delta$ and for $x > NC\delta$, and applying the resulting inequality to both $S_n$ and $-S_n$.

**Theorem 2.2.** *Let $W_n$, $n \ge 1$, be as in Theorem 2.1. If $J$ is any compact subinterval of $\Omega$, then $W_t^p$, $a \ge 0$, are uniformly integrable with respect to $P_\omega$, uniformly in $\omega \in J$, for all $p \ge 1$.*

**Proof.** Let $\epsilon > 0$ be as in the proof of the second assertion of Lemma 2.1, and set $N_1 = [\epsilon a] + 1$ and $N_2 = [a/\epsilon]$, where $[b]$ denotes the greatest integer that does not exceed $b$. Then

$$\begin{aligned} \int_{t\le\epsilon a} \left|(S_t - t\theta)/\sqrt{t}\right|^p dP_\omega &\le (\epsilon a)^{p/2} \int_{t\le\epsilon a} \sup_{1\le n\le N_1} |\bar{X}_n - \theta|^p dP_\omega \\ &\le C\epsilon^{p/2}\sqrt{a^p P_\omega\{t \le \epsilon a\}}\sqrt{E_\omega[|X_1 - \theta|^{2p}]} \\ &\to 0 \end{aligned}$$

uniformly in $\omega \in J$ as $a \to \infty$, for all $p \ge 1$, for some constant $C > 0$, by Schwarz's inequality, Doob's maximal inequality (applied to the reverse martingale $\bar{X}_n$, $n \ge 1$), and the first assertion of Lemma 2.1. Next, Let $C_a = \{\epsilon a < t \le a/\epsilon\}$. Then an integration by parts shows that for each $\omega \in \Omega$

$$\begin{aligned} \int_{C_a} \left|\frac{S_t - t\theta}{\sqrt{t}}\right|^p dP_\omega &\le p(\epsilon a)^{-p/2} \int_0^\infty x^{p-1} P_\omega\{C_a, |S_t - t\theta| > x\} dx \\ &\le \frac{p}{(\epsilon a)^{p/2}} \int_0^\infty x^{p-1} P_\omega\left\{ \max_{N_1\le n\le N_2} |S_n - n\theta| > x \right\} dx. \qquad (2.2) \end{aligned}$$

Furthermore, by Lemma 2.3, there are positive constants $C$ and $\delta$ for which

$$\sup_{\omega\in J} \mathrm{LHS}(2.2) \le 2p(\epsilon a)^{-p/2}\int_0^\infty x^{p-1}\Big[e^{-x^2/(2CN_2)} + e^{-\frac{1}{2}\delta x}\Big]dx \to I_p$$

as $a \to \infty$, where $I_p = 2p\epsilon^{-p}C^{-p/2}\int_0^\infty y^{p-1}e^{-\frac{1}{2}y^2}dy < \infty$, for all $p \ge 1$. Finally,

$$\int_{t>a/\epsilon}\Big|(S_t - t\theta)/\sqrt{t}\Big|^p dP_\omega \le \sqrt{\sum_{n>a/\epsilon} n^p P_\omega\{t>n\}}\sqrt{E_\omega \sup_{n>N_2}|\bar{X}_n - \theta|^{2p}} \to 0$$

uniformly in $\omega \in J$ as $a \to \infty$, by Schwarz's inequality and the facts that the term under the first square root tends to zero uniformly in $\omega \in J$ as $a \to \infty$ by the second assertion of Lemma 2.1; and the other term is $O(N_2^{-p})$ uniformly in $\omega \in J$ as $a \to \infty$, by Doob's maximal inequality and Von Bahr's theorem.

**Lemma 2.4.** *Let $t^* = (t - ag(\theta))/\sqrt{ag(\theta)}$. If $J$ is any compact subinterval of $\Omega$, then $t^*$ converges in distribution, under $P_\omega$, to a normal random variable with mean 0 and variance $\tau^2(\omega) = [g'(\theta)]^2[g(\theta)]^{-2}\psi''(\omega)$, uniformly in $\omega \in J$ as $a \to \infty$.*

**Proof.** The lemma can be established by following the proof of Lemma 4.2 of Woodroofe (1982) and using Theorem 2.2.

**Corollary 2.1.** *If $J$ is any compact subinterval of $\Omega$, then*

$$\lim_{a\to\infty} E_\omega\left[\frac{(t - ag(\theta))^2}{t}\right] = \tau^2(\omega)$$

*uniformly in $\omega \in J$, where $\tau^2(\omega)$ is given in Lemma 2.4.*

**Proof.** The proof is similar to that of Corollary 4.1 of Woodroofe (1987).

**Lemma 2.5.** *If $J$ is any compact subinterval of $\Omega$, then*

$$\int_{|\bar{X}_t-\theta|>\epsilon} \frac{a^2}{t}\Big|[b_t(\bar{X}_t) - b(\theta)](\bar{X}_t - \theta)\Big|dP_\omega = o(1)$$

*and*

$$\int_{|\bar{X}_t-\theta|>\epsilon} \frac{a^2}{t^2}[b_t(\bar{X}_t)]^2 dP_\omega = o(1)$$

*uniformly in $\omega \in J$ as $a \to \infty$, for all sufficiently small $\epsilon > 0$.*

**Proof.** The lemma follows by Schwarz's inequality and Lemmas 2.1 and 2.2 of Woodroofe (1987) (and Lemma 2.2 of Tahir (1989) to obtain the second assertion).

## 3. Asymptotic Local Minimaxity

The main result is presented in the following theorem:

**Theorem 3.1.** *Let* $m = [\sqrt{a}]$ *and let* $\mathcal{F}_m$ *denote the* $\sigma$*-algebra generated by* $X_1, \ldots, X_m$. *Also, let* $t$ *be defined by* (1.2) *and let* $J$ *be any compact subinterval of* $\Omega$. *If there is a constant* $C > 0$ *for which* $m^4 |E_\omega\{t - a\hat{g}_m|\mathcal{F}_m\}| \le C$ *w.p.1* $(P_\omega)$, *for all* $\omega \in J$ *and all sufficiently large* $a$, *then* $r_a(t;\omega) = G(\omega) + o(1)$ *uniformly in* $\omega \in J$ *as* $a \to \infty$, *where* $r_a(t;\omega)$ *is defined by* (1.3) *and* $G(\omega)$ *is given in* (1.4).

**Proof.** Write

$$\bar{X}_t - \theta = \frac{1}{t}\Big[m(\bar{X}_m - \theta) + (X_{m+1} - \theta + \cdots + X_t - \theta)\Big] \tag{3.1}$$

on $\{t \ge m\}$, expand the square, and use the independence of $X_1, X_2, \ldots$ to obtain

$$E_\omega\{(\bar{X}_t - \theta)^2|\mathcal{F}_m\} = \frac{m^2}{t^2}(\bar{X}_m - \theta)^2 + \frac{\psi''(\omega)}{t} - \frac{m}{t^2}\psi''(\omega)$$

for all $\omega \in \Omega$. Combine this observation with (1.1) and (1.3) to get

$$\begin{aligned} r_a(t;\omega) &= E_\omega\left[\frac{(t - ag(\theta))^2}{t}\right] + \gamma^2(\omega)E_\omega\left[\frac{a^2}{t^2}\Big[(S_m - m\theta)^2 - m\psi''(\omega)\Big]\right] \\ &\quad - 2\gamma^2(\omega)E_\omega\left[\frac{a^2}{t}b_t(\bar{X}_t)(\bar{X}_t - \theta)\right] + \gamma^2(\omega)E_\omega\left[\frac{a^2}{t^2}b_t^2(\bar{X}_t)\right] \\ &= E_\omega[T_1] + \gamma^2(\omega)E_\omega[T_2] - 2\gamma^2(\omega)E_\omega[T_3] + \gamma^2(\omega)E_\omega[T_4], \end{aligned} \tag{3.2}$$

say. Furthermore,

$$E_\omega[T_1] \to \frac{[g'(\theta)]^2}{[g(\theta)]^2}\psi''(\omega) \tag{3.3}$$

uniformly in $\omega \in J$ as $a \to \infty$, by Corollary 2.1. To evaluate $E_\omega[T_2]$, write

$$\begin{aligned} T_2 &= \left[\frac{a^2}{t^2} - \frac{1}{\hat{g}_m^2}\right]\Big[(S_m - m\theta)^2 - m\psi''(\omega)\Big] \\ &\quad + \left[\frac{1}{\hat{g}_m^2} - \frac{1}{g^2(\theta)}\right]\Big[(S_m - m\theta)^2 - m\psi''(\omega)\Big] \\ &\quad + \frac{1}{g^2(\theta)}\Big[(S_m - m\theta)^2 - m\psi''(\omega)\Big] \\ &= T_{21} + T_{22} + T_{23}, \end{aligned}$$

say. Then, $E_\omega[T_{23}] = 0$ for all $\omega \in \Omega$ and all $a > 0$. For $T_{21}$, expand $1/t^2$ about $a\hat{g}_m$ to obtain $T_{21} = -2a^2 g_*^{-3}(t - a\hat{g}_m)[(S_m - m\theta)^2 - m\psi''(\omega)]$ on $\{t \ge m\}$, where $g_*$ is an intermediate point between $t$ and $a\hat{g}_m$. Thus, $E_\omega[T_{21}] \to 0$ uniformly in

$\omega \in J$ as $a \to \infty$, by the assumption of the theorem. Let $u(\theta) = 1/g^2(\theta)$ and $u_n = 1/g_n^2$ for $n \geq 1$. Then $T_{22} = [u'(\theta)(\bar{X}_m - \theta) + \frac{1}{2}u''(\theta_m^*)(\bar{X}_m - \theta)^2][(S_m - m\theta)^2 - m\psi''(\omega)]$ for $a > 0$, where $\theta_m^*$ is an intermediate point between $\bar{X}_m$ and $\theta$. Hence, after some algebra,

$$\lim_{a\to\infty} E_\omega[T_2] = \lim_{a\to\infty} E_\omega[T_{22}] = u'(\theta)\psi'''(\omega) + [\psi''(\omega)]^2 u''(\theta) = \frac{d^2}{d\omega^2}\left[\frac{1}{g^2(\theta)}\right] \quad (3.4)$$

uniformly in $\omega \in J$. To estimate $E_\omega[T_3]$, write

$$\begin{aligned} T_3 = \frac{a^2}{t} b_t(\bar{X}_t)(\bar{X}_t - \theta) &= \frac{b(\theta)}{g^2(\theta)} t(\bar{X}_t - \theta) + b(\theta)\left[\frac{a^2}{t^2} - \frac{1}{g^2(\theta)}\right] t(\bar{X}_t - \theta) \\ &\quad + \frac{a^2}{t^2}[b_t(\bar{X}_t) - b(\theta)] t(\bar{X}_t - \theta) \\ &= Q_1 + b(\theta)Q_2 + Q_3, \end{aligned}$$

say. By Wald's lemma, $E_\omega[Q_1] = 0$ for all $\omega$ and all $a$. For the term $Q_2$, write

$$Q_2 = \left[\frac{a^2}{t^2} - \frac{1}{\hat{g}_m^2}\right] t(\bar{X}_t - \theta) + \left[\frac{1}{\hat{g}_m^2} - \frac{1}{g^2(\theta)}\right] t(\bar{X}_t - \theta) = Q_{21} + Q_{22},$$

say. Then a Taylor's expansion for $1/t^2$ about $a\hat{g}_m$, followed by (3.1), yields $E_\omega\{Q_2|\mathcal{F}_m\} = -2a^2 E_\omega\{g_*^{-3}(t - a\hat{g}_m)(S_m - m\theta)|\mathcal{F}_m\}$ on $\{t \geq m\}$; so that $E_\omega[Q_{21}] \to 0$ uniformly in $\omega \in J$ as $a \to \infty$, by the assumption of the theorem. To estimate $E_\omega[Q_{22}]$, let $R_n = u_n(\bar{X}_n) - u(\theta) - u'(\theta)(\bar{X}_n - \theta) - \frac{1}{2}u''(\theta)(\bar{X}_n - \theta)^2$ for $n \geq 1$ and $\omega \in \Omega$. Condition on $\mathcal{F}_m$, use (3.1), and take expectation to obtain

$$\begin{aligned} E_\omega\{Q_{22}|\mathcal{F}_m\} &= u'(\theta)m(\bar{X}_m - \theta)^2 + \frac{1}{2}u''(\theta)m(\bar{X}_m - \theta)^3 + mR_m(\bar{X}_m - \theta) \\ &= Q_{221} + Q_{222} + Q_{223}, \end{aligned}$$

say. Furthermore, $E_\omega[Q_{221}] = u'(\theta)\psi''(\omega)$ for all $\omega$ and all $a > 0$. Also, $E_\omega[Q_{222}] \to 0$ and $E_\omega[Q_{223}] \to 0$ uniformly in $\omega \in J$ as $a \to \infty$, by Schwarz's inequality and the fact that $\sup_{\omega\in J} E_\omega[R_m^2] = o(1/m^2)$ as $a \to \infty$, as in Lemma 4.1 of Woodroofe (1987). Collecting terms yields

$$\lim_{a\to\infty} E_\omega[Q_2] = \lim_{a\to\infty} E_\omega[Q_{221}] = u'(\theta)\psi''(\omega) = -2\frac{g'(\theta)}{g^3(\theta)}\psi''(\omega)$$

uniformly in $\omega \in J$. To evaluate $E_\omega[Q_3]$, use the first assertion of Lemma 2.5 and the mean value theorem to obtain

$$E_\omega[Q_3] = \int_{|\bar{X}_t - \theta| \leq \epsilon} b'(\theta_t^*)\frac{a^2}{t}(\bar{X}_t - \theta)^2 dP_\omega + o(1)$$

uniformly in $\omega \in J$ as $a \to \infty$, for all sufficiently small $\epsilon > 0$, where $\theta_t^*$ is an intermediate point between $\bar{X}_t$ and $\theta$. Furthermore, given $\eta > 0$, let $\delta > 0$ be so small that $|b'(\theta_t^*) - b'(\theta)| < \eta$ whenever $|\theta_t^* - \theta| < \delta$, for all sufficiently large $a$. Then $E_\omega[Q_3] \leq [b'(\theta) + \eta]E_\omega[a^2(\bar{X}_t - \theta)^2/t] + o(1)$ uniformly in $\omega \in J$ as $a \to \infty$; so that $\limsup_{a\to\infty} E_\omega[Q_3] \leq [b'(\theta) + \eta]\psi''(\omega)[g(\theta)]^{-2}$ uniformly in $\omega \in J$, by Theorem 2.1, Lemma 2.2, and Theorem 2.2. A similar argument yields $\liminf_{a\to\infty} E_\omega[Q_3] \geq [b'(\theta) - \eta]\psi''(\omega)[g(\theta)]^{-2}$ uniformly in $\omega \in J$. Since $\eta > 0$ is arbitrary, it follows that $E_\omega[Q_3] = b'(\theta)\psi''(\omega)[g(\theta)]^{-2} + o(1)$ uniformly in $\omega \in J$. Therefore, collecting terms yields

$$\lim_{a\to\infty} E_\omega[T_3] = -2\frac{g'(\theta)}{g^3(\theta)}b(\theta)\psi''(\omega) + \frac{b'(\theta)}{g^2(\theta)}\psi''(\omega) = \frac{d}{d\omega}\left[\frac{b(\theta)}{g^2(\theta)}\right] \tag{3.5}$$

uniformly in $\omega \in J$. Finally,

$$\lim_{a\to\infty} E_\omega[T_4] = \frac{b^2(\theta)}{g^2(\theta)} \tag{3.6}$$

uniformly in $\omega \in J$ as $a \to \infty$, by the second assertion of Lemma 2.5, Lemma 2.2, the conditions imposed on $b_n$, $n \geq 1$, and Lemma 2.2 of Tahir (1989). Take the limit as $a \to \infty$ in (3.2) and use (3.3)-(3.6) to complete the proof of the theorem.

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