

## ON PARAMETER ESTIMATION OF TWO-DIMENSIONAL POLYNOMIAL PHASE SIGNAL MODEL

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*Abstract:* Two-dimensional (2-D) polynomial phase signals occur in different areas of image processing. When the degree of the polynomial is two they are called chirp signals. In this paper, we consider the least squares estimators of the unknown parameters of the 2-D polynomial phase signal model in the presence of stationary noise, and derive their properties. The proposed least squares estimators are strongly consistent and we obtained their asymptotic distributions. It is observed that asymptotically the least squares estimators are normally distributed. We perform some simulation experiments to observe their behavior.

*Key words and phrases:* Asymptotic distribution, least squares estimators, linear processes, polynomial phase signals, strong consistency.

### 1. Introduction

One-dimensional polynomial phase signal models have received considerable attention in the statistical signal processing literature. The one-dimensional polynomial phase signal model has been used quite successfully in various areas of science and engineering, specifically in sonar, radar communications etc., see for example Barbarossa and Petrone (1997), Barbarossa, Scaglione and Giannakis (1998), and Wu, So and Liu (2008). Wu, So and Liu (2008) consider a specific case when the degree of polynomial is three, due to its applications in seismology. When the degree of polynomial is two, the polynomial phase signal model is known as chirp model, and it has also received considerable attention because of its wide scale applicability in sonar array processing. See for example Djuric and Kay (1990), Gini, Montanari and Verrazzani (2000), Kundu and Nandi (2008), and the references cited therein.

The two-dimensional (2-D) polynomial phase signal model also has received significant amount of attention as it has been used in modeling and analyzing magnetic resonance imaging (MRI), optical imaging, and different texture imaging. See for example Francos and Friedlander (1998, 1999), Hedley and Rosenfeld (1992), Peleg and Porat (1991), Cao, Wang and Wang (2006), Zhang and Liu

(2006), and Zhang, Wang and Cao (2008). Friedlander and Francos (1996) used the 2-D polynomial phase signal model to analyze finger print type data, and Djurovic et al. (2010) considered a specific 2-D cubic phase signal model due to its applications in modeling Synthetic Aperture Radar (SAR) data and, in particular, Interferometric SAR data.

Surprisingly, although extensive work has been done on estimating the parameters of different 2-D polynomial phase signal models least squares estimators (LSE's) of the 2-D polynomial phase signal have not been considered, nor their properties discussed. Many estimators have been proposed and their asymptotic variances compared with the Cramer-Rao lower bound. But unless it is established that the asymptotic variances of the maximum likelihood estimators attend the corresponding Cramer-Rao lower bound, this comparison may not be meaningful. When the error random variables  $X(m, n)$ 's are i.i.d. Gaussian, then the LSE's are the maximum likelihood estimators (MLE's).

We consider the most general 2-D polynomial (of degree  $r$ ) phase signal model which has the form

$$\begin{aligned} y(m, n) = & A^0 \cos \left( \sum_{p=1}^r \sum_{j=0}^p \alpha^0(j, p-j) m^j n^{p-j} \right) \\ & + B^0 \sin \left( \sum_{p=1}^r \sum_{j=0}^p \alpha^0(j, p-j) m^j n^{p-j} \right) \\ & + X(m, n); \quad m = 1, \dots, M; \quad n = 1, \dots, N. \end{aligned} \quad (1.1)$$

Here  $X(m, n)$  is stationary error,  $A^0$  and  $B^0$  are non zero amplitudes, and for  $j = 0, \dots, p$ ,  $p = 1, \dots, r$ ,  $\alpha^0(j, p-j)$ 's are distinct frequency rates of order  $(j, p-j)$ , respectively. They lie strictly between 0 and  $\pi$ ,  $\alpha^0(0, 1), \alpha^0(1, 0)$  are called frequencies. The explicit assumptions on the errors  $X(m, n)$  will be provided.

We provide the properties of the least squares estimators of the unknown parameters of the model (1.1). Deriving the exact distribution of the least squares estimators may well not be possible, and we rely on asymptotic results. The properties of 1-D chirp signal model have been discussed by Kundu and Nandi (2008). They established the strong consistency and asymptotic normality properties of the least squares estimators, but their results cannot be used directly here.

The present model does not satisfy the sufficient conditions of Jennrich (1969) and Wu (1981) for the least squares estimators to be consistent, and their results cannot be used directly here. We establish the strong consistency and

asymptotic normality properties of the least squares estimators of the unknown parameters of the model (1.1). The least squares estimators of  $\alpha^0(j, p-j)$  for  $j = 1, \dots, p$ ,  $p = 1, \dots, r$  have the convergence rates  $O_p(M^{-j-1/2}N^{-(p-j)-1/2})$ , and the least squares estimators of  $A^0$  and  $B^0$  have the convergence rate  $(MN)^{-1/2}$ . Thus the convergence rates of the estimators of  $\alpha^0(j, p-j)$  for  $j = 1, \dots, p$ ,  $p = 1, \dots, r$  are much faster than  $(MN)^{-1/2}$ , the usual convergence rate of an estimator for a general non-linear model. Moreover, when  $X(m, n)$ 's are i.i.d. random variables, then the asymptotic variances of the MLEs, the LSEs, attend the Cramer-Rao lower bound.

We have performed some simulation experiments to study the effectiveness of the least squares estimators for different sample sizes, models, error structures, and error random variables. We have considered independent and correlated error random variables that might be Gaussian or Laplace, and we have considered the polynomial phase with degrees two and three. In all the cases considered, the performances of the least squares estimators are quite satisfactory.

The rest of the paper is organized as follows. In Section 2, we provide the necessary assumptions, preliminary results, and the methodology for the least squares estimators. Strong consistency and asymptotic results are established in Section 3. Discussions on extensive simulation results and the analysis of a data set are presented in Section 4. We conclude the paper in Section 5. The proofs and the numerical results based on extensive simulations are provided in the Supplementary Section. See for example Djuric and Kay (1990); Gini, Montanari and Verrazzani (2000); Kundu and Nandi (2008, 2012); Lahiri, Kundu and Mitra (2013, 2015); Nandi and Kundu (2004), and the references cited therein.

## 2. Model Assumptions, Preliminary Results and Methodology

### 2.1. Model assumptions

**Assumption 1:** The error  $X(m, n)$  is

$$X(m, n) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} a(j, k)\varepsilon(m-j, n-k) \quad (2.1)$$

with

$$\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |a(j, k)| < \infty. \quad (2.2)$$

Here  $\varepsilon(m, n)$  is a double array sequences of independent and identically distributed (*i.i.d.*) random variables with zero mean, finite variance  $\sigma^2$ , and with

finite  $2r$ -th moment.

**Assumption 2:** The true parameter vector is  $\theta^0 = (A^0, B^0, \alpha^0(j, p-j), j = 0, \dots, p, p = 1, \dots, r)$  and the parameter space is  $\Theta = [-K, K] \times [-K, K] \times [0, \pi]^{\otimes r(r+3)/2}$ , where  $K > 0$  is an arbitrary constant and  $[0, \pi]^{\otimes r(r+3)/2}$  denotes the  $r(r+3)/2$  fold of  $[0, \pi]$ ,  $\theta^0$  is an interior point of  $\Theta$ .

## 2.2. Preliminary results

**Proposition 1.** Suppose  $(\alpha^0(j, p-j), j = 0, \dots, p, p = 1, \dots, r) \in (0, \pi)^{\otimes r(r+3)/2}$ . Then, except for a countable number of points  $\alpha^0(j, p-j)$ , for  $s, t = 0, 1, 2, \dots$ ,

$$\lim_{\min\{M, N\} \rightarrow \infty} \frac{1}{MN} \sum_{n=1}^N \sum_{m=1}^M \cos \left( \sum_{p=1}^r \sum_{j=0}^p \alpha^0(j, p-j) m^j n^{p-j} \right) = 0, \quad (2.3)$$

$$\lim_{\min\{M, N\} \rightarrow \infty} \frac{1}{MN} \sum_{n=1}^N \sum_{m=1}^M \sin \left( \sum_{p=1}^r \sum_{j=0}^p \alpha^0(j, p-j) m^j n^{p-j} \right) = 0, \quad (2.4)$$

$$\begin{aligned} & \lim_{\min\{M, N\} \rightarrow \infty} \frac{1}{M^{(s+1)} N^{(t+1)}} \sum_{n=1}^N \sum_{m=1}^M m^s n^t \cos^2 \left( \sum_{p=1}^r \sum_{j=0}^p \alpha^0(j, p-j) m^j n^{p-j} \right) \\ &= \frac{1}{2(s+1)(t+1)}, \end{aligned} \quad (2.5)$$

$$\begin{aligned} & \lim_{\min\{M, N\} \rightarrow \infty} \frac{1}{M^{(s+1)} N^{(t+1)}} \sum_{n=1}^N \sum_{m=1}^M m^s n^t \sin^2 \left( \sum_{p=1}^r \sum_{j=0}^p \alpha^0(j, p-j) m^j n^{p-j} \right) \\ &= \frac{1}{2(s+1)(t+1)}. \end{aligned} \quad (2.6)$$

**Proof:** See the Supplementary Section.

**Lemma 1.** If  $X(m, n)$  satisfies Assumptions 1 and 2, then as  $\min\{M, N\} \rightarrow \infty$ ,

$$\sup_{\alpha(j, p-j), j=0, \dots, p, p=1, \dots, r} \left| \frac{1}{MN} \sum_{n=1}^N \sum_{m=1}^M X(m, n) e^{i \left( \sum_{p=1}^r \sum_{j=0}^p \alpha(j, p-j) m^j n^{p-j} \right)} \right| \rightarrow 0 \text{ a.s.}$$

**Proof:** See the Supplementary Section.

**Lemma 2.** If  $X(m, n)$  satisfies Assumptions 1 and 2, then as  $\min\{M, N\} \rightarrow \infty$ , for  $s, t = 0, 1, \dots$ ,

$$\begin{aligned} & \sup_{\alpha(j, p-j), j=0, \dots, p, p=1, \dots, r} \left| \frac{1}{M^{s+1} N^{t+1}} \sum_{n=1}^N \sum_{m=1}^M m^s n^t X(m, n) e^{i \left( \sum_{p=1}^r \sum_{j=0}^p \alpha(j, p-j) m^j n^{p-j} \right)} \right| \\ & \rightarrow 0 \text{ a.s.} \end{aligned}$$

**Proof:** See the Supplementary Section.

### 2.3. Methodology

We use the following notations;

$$\phi = \begin{bmatrix} A \\ B \end{bmatrix},$$

$$W(\alpha) = \begin{bmatrix} \cos \left( \sum_{p=1}^r \sum_{j=0}^p \alpha(j, p-j) \right) & \sin \left( \sum_{p=1}^r \sum_{j=0}^p \alpha(j, p-j) \right) \\ \cos \left( \sum_{p=1}^r \sum_{j=0}^p \alpha(j, p-j) 2^j \right) & \sin \left( \sum_{p=1}^r \sum_{j=0}^p \alpha(j, p-j) 2^j \right) \\ \vdots & \vdots \\ \cos \left( \sum_{p=1}^r \sum_{j=0}^p \alpha(j, p-j) M^j \right) & \sin \left( \sum_{p=1}^r \sum_{j=0}^p \alpha(j, p-j) M^j \right) \\ \vdots & \vdots \\ \cos \left( \sum_{p=1}^r \sum_{j=0}^p \alpha(j, p-j) N^{p-j} \right) & \sin \left( \sum_{p=1}^r \sum_{j=0}^p \alpha(j, p-j) N^{p-j} \right) \\ \cos \left( \sum_{p=1}^r \sum_{j=0}^p \alpha(j, p-j) 2^j N^{p-j} \right) & \sin \left( \sum_{p=1}^r \sum_{j=0}^p \alpha(j, p-j) 2^j N^{p-j} \right) \\ \vdots & \vdots \\ \cos \left( \sum_{p=1}^r \sum_{j=0}^p \alpha(j, p-j) M^j N^{p-j} \right) & \sin \left( \sum_{p=1}^r \sum_{j=0}^p \alpha(j, p-j) M^j N^{p-j} \right) \end{bmatrix},$$

and  $Y$  is the  $MN \times 1$  data vector

$$Y = (y(1, 1), \dots, y(M, 1), \dots, y(1, N), \dots, y(M, N))^T. \tag{2.7}$$

The least squares estimators of  $\theta = (A, B, \alpha(j, p-j), j = 0, \dots, p, p = 1, \dots, r)$ , can be obtained by minimizing

$$Q(\theta) = (Y - W(\alpha)\phi)^T(Y - W(\alpha)\phi) \tag{2.8}$$

with respect to  $\theta$ . Using the separable regression technique of Richards (1961), it can be seen that, for fixed  $(\alpha(j, p-j), j = 0, \dots, p, p = 1, \dots, r)$ , the

minimization of  $Q(\theta)$  with respect to  $A$  and  $B$  can be obtained as

$$\hat{\phi}(\alpha) = \begin{bmatrix} \hat{A}(\alpha) \\ \hat{B}(\alpha) \end{bmatrix} = (W(\alpha)^T W(\alpha))^{-1} W(\alpha)^T Y.$$

Therefore, the minimization of  $Q(\theta)$  can be obtained by minimizing

$$R(\alpha) = Y^T (I - P(\alpha)) Y$$

with respect to  $(\alpha(j, p-j), j = 0, \dots, p, p = 1, \dots, r)$ , where

$$P(\alpha) = W(\alpha)(W(\alpha)^T W(\alpha))^{-1} W(\alpha)^T$$

is the projection matrix on the column space of  $W(\alpha)$ .

If  $(\hat{\alpha}(j, p-j), j = 0, \dots, p, p = 1, \dots, r)$  minimizes  $R(\alpha)$ , the least squares estimates of  $A$  and  $B$  can be obtained as  $\hat{A} = \hat{A}(\hat{\alpha})$  and  $\hat{B} = \hat{B}(\hat{\alpha})$ . We write  $\hat{\theta} = (\hat{A}, \hat{B}, \hat{\alpha}(j, p-j), j = 0, \dots, p, p = 1, \dots, r)$ . By using the separable regression technique, the least squares estimators of the unknown parameters of the model (1.1) can be obtained by solving a  $r(r-3)/2$  dimensional optimization problem, rather than a  $2 + r(r-3)/2$  dimensional optimization problem.

### 3. Asymptotic Properties of the Least Squares Estimators

#### 3.1. Consistency of the least squares estimators

**Theorem 1.** *If the Assumptions 1 and 2 are satisfied, then  $\hat{\theta}$ , the least squares estimators of  $\theta^0$ , is a strongly consistent estimator of  $\theta^0$ .*

**Proof:** See the Supplementary Section.

The following result might be useful for error analysis of the model, and it may have some independent interest.

**Lemma 3.** *If the Assumptions 1 and 2 are satisfied, then for  $j = 0, \dots, p, p = 1, \dots, r$ ,*

$$M^j N^{p-j} (\hat{\alpha}(j, p-j) - \alpha^0(j, p-j)) \rightarrow 0 \text{ a.s..}$$

**Proof:** See the Supplementary Section.

Using Lemma 3, we immediately obtain

$$\begin{aligned} \hat{A} &= A^0 + o(1) \text{ a.s.}, \quad \hat{B} = B^0 + o(1) \text{ a.s.}, \\ \hat{\alpha}(j, p-j) &= \alpha^0(j, p-j) + o(M^j N^{p-j}) \text{ a.s..} \end{aligned}$$

So we get

$$\hat{y}(m, n) = \hat{A} \cos\left(\sum_{p=1}^r \sum_{j=0}^p \hat{\alpha}(j, p-j) m^j n^{p-j}\right) + \hat{B} \sin\left(\sum_{p=1}^r \sum_{j=0}^p \hat{\alpha}(j, p-j) m^j n^{p-j}\right)$$

$$\begin{aligned}
 &= A^0 \cos\left(\sum_{p=1}^r \sum_{j=0}^p \alpha^0(j, p-j)m^j n^{p-j}\right) + B^0 \sin\left(\sum_{p=1}^r \sum_{j=0}^p \alpha^0(j, p-j)m^j n^{p-j}\right) \\
 &\quad + o(1) \quad a.s.
 \end{aligned}$$

which gives

$$y(m, n) - \hat{y}(m, n) = X(m, n) + o(1) \quad a.s.. \tag{3.1}$$

Therefore, (3.1) can be used for checking the error assumptions.

### 3.2. Asymptotic normality of the estimators

**Theorem 2.** *If the Assumptions 1 and 2 are satisfied, then  $(\hat{\theta} - \theta^0)D^{-1} \rightarrow N_d(0, 2c\sigma^2\Sigma^{-1})$  where the matrix  $D$  is the  $(2 + r(r + 3)/2) \times (2 + r(r + 3)/2)$  diagonal matrix*

$$\begin{aligned}
 D &= \text{diag}(M^{-1/2}N^{-1/2}, M^{-1/2}N^{-1/2}, M^{-j-1/2}N^{-(p-j)-1/2}, \\
 &\quad j = 0, \dots, p, p = 1, \dots, r), \\
 \Sigma &= \begin{bmatrix} 1 & 0 & V_1 \\ 0 & 1 & V_2 \\ V_1^T & V_2^T & M \end{bmatrix}. \tag{3.2}
 \end{aligned}$$

Here  $V_1 = (B^0/(j + 1)(p - j + 1), j = 0, \dots, p, p = 1, \dots, r)$ ,  $V_2 = (-A^0/(j + 1)(p - j + 1), j = 0, \dots, p, p = 1, \dots, r)$ , are vectors of order  $1 \times r(r + 3)/2$ ,  $M = ((A^{0^2} + B^{0^2})/(j + k + 1)(p + q - j - k + 1), j = 0, \dots, p, p = 1, \dots, r, k = 0, \dots, q, q = 1, \dots, r)$ , is a matrix of order  $r(r + 3)/2 \times r(r + 3)/2$ , and  $c = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} a(j, k)^2$ . Further,  $N_d(0, 2c\sigma^2\Sigma^{-1})$  denotes a  $d$ -variate normal distribution with the mean vector 0, and dispersion matrix  $2c\sigma^2\Sigma^{-1}$ , where  $d = 2 + r(r + 3)/2$ .

**Proof:** See the Supplementary Section.

**Comments:** When the  $X(m, n)$ 's are i.i.d. Gaussian, then the maximum likelihood estimator of  $\theta$  is the same as the least squares estimator. Hence, due to Theorem 2, it follows that  $(\hat{\theta} - \theta^0)D^{-1} \rightarrow N_d(0, 2\sigma^2\Sigma^{-1})$ . Now if  $l(\theta)$  denotes the log-likelihood function, then from the expressions of the elements of  $Q''(\theta)$ , see the Proof of Theorem 2, it follows that

$$E \left[ D \frac{\partial^2 l(\theta)}{\partial \theta \partial \theta'} D \right]_{\theta=\theta^0} \rightarrow \frac{1}{2\sigma^2} \Sigma.$$

Hence, it follows that the asymptotic variance of  $\hat{\theta}$  with proper normalization attains the Cramer-Rao lower bound.

## 4. Simulations and Data Analysis

### 4.1. Simulations

We performed some simulations for different models, sample sizes and error variances, mainly to see how the least squares estimators perform in practice based on the biases and mean squared errors (MSEs). We considered two models Model 1:

$$y(m, n) = A^0 \cos(\alpha^0 m + \beta^0 m^2 + \gamma^0 n + \delta^0 n^2) + B^0 \sin(\alpha^0 m + \beta^0 m^2 + \gamma^0 n + \delta^0 n^2) + X(m, n); \quad m = 1, \dots, M; \quad n = 1, \dots, N. \quad (4.1)$$

Here the model parameters were

$$A^0 = 5.0, B^0 = 5.0, \alpha^0 = 1.0, \beta^0 = 0.05, \gamma^0 = 1.5, \delta^0 = 0.5. \quad (4.2)$$

We took the sample sizes:  $50 \times 50$ ,  $75 \times 75$ ,  $100 \times 100$  and the two error structures

$$\text{Error-I: } X(m, n) = \varepsilon(m, n); \quad (4.3)$$

$$\text{Error-II: } X(m, n) = \varepsilon(m, n) + 0.5\varepsilon(m-1, n) + 0.33\varepsilon(m, n-1). \quad (4.4)$$

We took the  $\varepsilon(m, n)$ 's as *i.i.d.* Gaussian with mean 0, variance  $\sigma^2$  and  $\varepsilon(m, n)$ 's as *i.i.d.* Laplace with mean 0, and variance  $\sigma^2$ . We considered  $\sigma^2$  as 0.05 and 0.5.

Model 2:

$$y(m, n) = A^0 \cos(\alpha^0 m + \beta^0 m^2 + \eta^0 m^3 + \gamma^0 n + \delta^0 n^2 + \xi^0 n^3) + B^0 \sin(\alpha^0 m + \beta^0 m^2 + \eta^0 m^3 + \gamma^0 n + \delta^0 n^2 + \xi^0 n^3) + X(m, n); \quad m = 1, \dots, M; \quad n = 1, \dots, N. \quad (4.5)$$

Here the model parameters were

$$A^0 = 2.0, B^0 = 2.0, \alpha^0 = 1.0, \beta^0 = 0.05, \eta^0 = 0.01, \gamma^0 = 1.0, \delta^0 = 0.05, \xi^0 = 0.01. \quad (4.6)$$

Again sample sizes were  $50 \times 50$ ,  $75 \times 75$ ,  $100 \times 100$ , and the error structure only as defined in (4.4). The  $\varepsilon(m, n)$ 's were *i.i.d.* Gaussian with mean 0 and variance  $\sigma^2 = 0.5$

We used the random number generator RAN2 of Press et al. (1996) for generating the uniform random numbers. In each case the least squares estimators of the unknown parameters were obtained by using the *Downhill Simplex Algorithm*, see for example Press et al. (1996), whereas, the initial guesses were obtained by using a grid search method with grid size of 0.01 around the true parameter values.

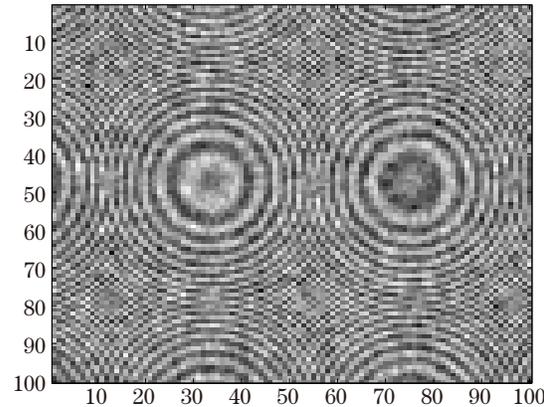


Figure 1. Noisy signal.

In each case we computed the least squares estimators, and obtained the average estimates, mean squared errors and variances over 1000 replications. We report the true parameter values (PARA), the average estimates (MEAN), the associated mean squared errors (MSE), and variances (VAR). For comparison purposes we report the asymptotic variances (ASYV) obtained using Theorem 2. For Model 1, in case of Gaussian errors, the results are in Tables 1 - 4, and in case of Laplace errors the results are in Tables 5 - 8. For Model 2, the results are reported in Table 9. The tables are provided in the Supplementary Section.

Some of the points are clear from these tables. As error variances decrease, the performance of the estimators in terms of MSEs improved. As the sample size increases, the variances and the mean squared errors decrease. The simulation results show that the least squares estimates are quite close to the true parameter values. For both the error structures, the mean squared errors of the least squares estimators match quite well with the corresponding asymptotic variances.

The MSEs for the LSEs of the model parameters are slightly lower when the errors are Gaussian than when the errors are Laplace, but the LSEs behave quite well even when the errors are Laplace. Even for Model 2, the LSEs of the unknown parameters behave quite satisfactorily compared to the asymptotic variances of the corresponding estimators of the unknown parameters. It seems that the asymptotic results work quite well even for moderate sample sizes for the cases considered here.

#### 4.2. Data analysis

For illustrative purposes, mainly to show how the proposed method can be implemented in practice, we have analyzed two simulated data sets obtained from

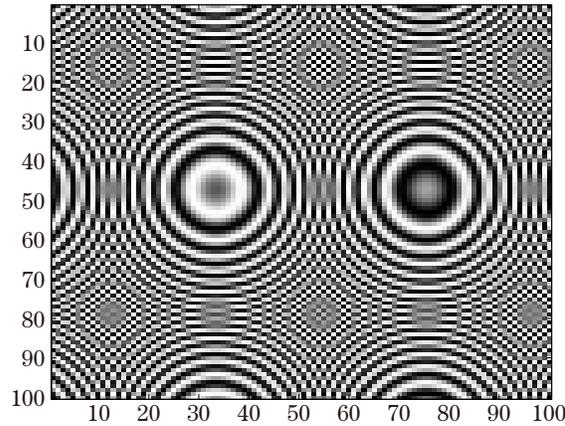


Figure 2. True signal.

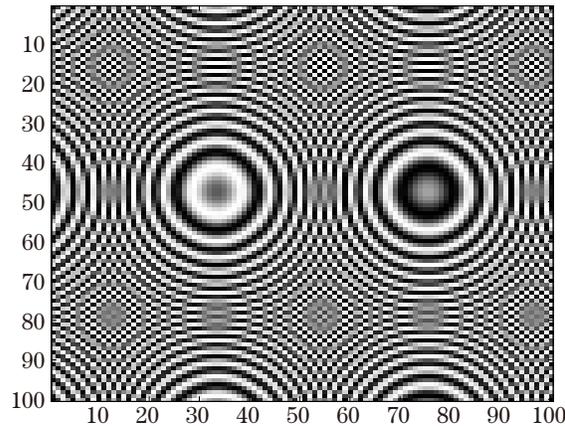


Figure 3. Estimated signal.

the model (1.1). We used the parameter values  $A^0 = 5.0$ ,  $B^0 = 1.0$ ,  $\alpha^0 = 1.55$ ,  $\beta^0 = 0.05$ ,  $\gamma^0 = 1.25$ ,  $\delta^0 = 0.075$ . The  $X(m, n)$ 's were

$$X(m, n) = \epsilon(m, n) + 0.5\epsilon(m-1, n) + 0.33\epsilon(m, n-1) + 0.2\epsilon(m-1, n-1),$$

with  $\epsilon(m, n)$ 's assumed to be *i.i.d.* Gaussian with mean 0 and variance  $\sigma^2 = 2.5$ . We plot one generated data set  $\{y(m, n); m = 1, \dots, 100, n = 1, \dots, 100\}$ , in Figure 1. Figure 1 represents the 2-D image plot of a simulated noise corrupted  $y(m, n)$ , whose gray level at  $(m, n)$  is proportional to the value of  $y(m, n)$ . The problem is to extract the true texture, see Figure 2, from the contaminated one.

We used the least squares technique and estimated the unknown parameters as  $\hat{A} = 5.003434$ ,  $\hat{B} = 0.965267$ ,  $\hat{\alpha} = 1.549228$ ,  $\hat{\beta} = 0.050006$ ,  $\hat{\gamma} = 1.250852$ , and  $\hat{\delta} = 0.074991$ . The estimated  $y(m, n)$  are plotted in Figure 3. The original and

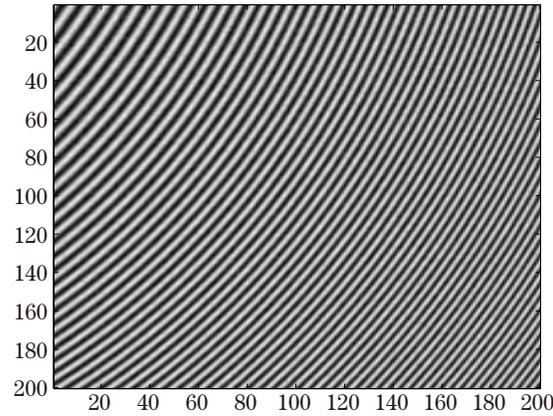


Figure 4. True image.

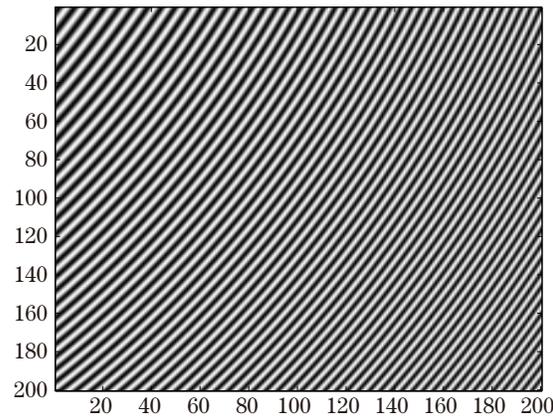


Figure 5. Estimated image.

the estimated plots match quite well.

We generated another data set with similar values as in Friedlander and Francos (1996) suitable for our model, with  $A^0 = 1.0$ ,  $B^0 = 1.0$ ,  $\alpha^0 = 0.45$ ,  $\beta^0 = 0.0015$ ,  $\gamma^0 = 0.82$ ,  $\delta^0 = 0.0022$ , and  $\sigma^2 = 0.005$ . The 2-D image plot of the generated sample is provided in Figure 4. We used our method as before to obtain the estimated image, and it is plotted in Figure 5. The two images, true and estimated, also look very similar. The estimates of the corresponding parameters were  $\hat{A} = 0.305698$ ,  $\hat{B} = -0.054401$ ,  $\hat{\alpha} = 0.586525$ ,  $\hat{\beta} = 0.000995$ ,  $\hat{\gamma} = 0.969553$ , and  $\hat{\delta} = 0.001654$ .

## 5. Conclusion

There are several open issues and generalizations which are of interest for

future work. For example, the least squares estimators can be obtained using a  $r(r+3)/2$  dimensional optimization problem. It would be interesting to develop a numerically efficient algorithm to find its solution. Moreover, although the least squares estimators are quite efficient, it is well known that they may not be very robust. Developing robust parameter estimation in this case would be of interest. More work is needed.

## Supplementary Materials

The numerical results are presented in Tables 1-9, and all proofs are given.

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