On Parameter Estimation of

Two-Dimensional Polynomial Phase Signal Model

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Supplementary Material

In this section we have provided all the tables based on the simulation experiments discussed in Section 4. In case of Model 1, for Gaussian errors the results are presented in Tables 1 - 4, and in case of Laplace errors the results are reported in Tables 5 - 8. For Model 2, the results are reported in Table 9. All the proofs also are presented in this section.

S1 Proofs

All the proofs are provided here.

Proof of Proposition-1 The proofs can be obtained using the results of Vinogradov (1954) for estimating Weyl (1916)'s sum for one and multi-dimensions. Suppose for a given $k > 0$, $f(n) = \alpha_1 n + \alpha_2 n^2 + \ldots \alpha_n n^k$ for $n = 1, 2, \cdots$, where $\alpha_1, \ldots, \alpha_k$ are real numbers, and for a positive integer N, $S = \sum_{n=1}^{N} e^{2\pi i f(n)}$. Then except for countable number of points $\alpha_1, \ldots, \alpha_k$,

$S = O(N^{1-\rho})$, for some $\rho > 0$. Hence

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i(\alpha_1 n+\alpha_2 n^2+\ldots+\alpha_k n^k)} = 0.$$

Therefore, we immediately have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \cos(\alpha_1 n + \alpha_2 n^2 + \ldots + \alpha_k n^k) = 0$$

and

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \sin(\alpha_1 n + \alpha_2 n^2 + \ldots + \alpha_k n^k) = 0.$$

Similarly, if we denote $f(m,n) = \sum_{p=1}^{r} \sum_{j=0}^{p} \alpha^0(j, p-j) m^j n^{p-j}$ and $S = \sum_{m=1}^{M} \sum_{n=1}^{N} e^{2\pi i f(m,n)}$, then except for countable number of points $\{\alpha^0(j, p-j), j = 0, 1, \ldots, p, p = 1, \ldots, r\}$, $S = O(N^{1-\rho_1} M^{1-\rho_2})$ for some $\rho_1 > 0$ and $\rho_2 > 0$. Hence, the first two limits of Proposition 1 follow as

$$\lim_{\min\{M,N\}\to\infty} \frac{1}{NM} \sum_{m=1}^{M} \sum_{n=1}^{N} e^{2\pi i f(m,n)} = 0.$$

For s, t positive integers, we also have

$$\lim_{\min\{M,N\}\to\infty} \frac{1}{N^{t+1} M^{s+1}} \sum_{m=1}^{M} \sum_{n=1}^{N} m^s n^t e^{2\pi i f(m,n)} = 0. \tag{S1.1}$$

Now for s, t positive integers, using (S1.1) and

$$\lim_{M\to\infty} \frac{1}{M^{s+1}} \sum_{m=1}^{M} m^s = \frac{1}{s+1} \quad \text{and} \quad \lim_{N\to\infty} \frac{1}{N^{t+1}} \sum_{n=1}^{M} n^t = \frac{1}{t+1},$$

the last two limits of Proposition 1 follow.

Proof of Lemma-1 First we will prove for $r = 2$ to observe how the proof works and then provide the proof for general r. We will use the following results to prove it.

Result 1: For fixed $j(k) = 1, 2, \cdots, M-1(N-1)$, and for fixed l, t such that $|j-l| < M,\ |k-t| < N$, using Holder's inequality we have

$$
\begin{aligned}
& E\left|\sum_{n=k+1}^{N-t+k}\sum_{m=j+1}^{M-l+j} \varepsilon(m,n)\varepsilon(m-j,n-k)\varepsilon(m+l,n+t)\varepsilon(m-j+l,n-k+t)\right| \\
& \leq \left[E\left(\sum_{n=k+1}^{N-t+k}\sum_{m=j+1}^{M-l+j} \varepsilon(m,n)\varepsilon(m-j,n-k)\varepsilon(m+l,n+t)\varepsilon(m-j+l,n-k+t)\right)^2\right]^{\frac{1}{2}} \\
& = \left[E\sum_{n=k+1}^{N-t+k}\sum_{m=j+1}^{M-l+j} \left(\varepsilon(m,n)\varepsilon(m-j,n-k)\varepsilon(m+l,n+t)\varepsilon(m-j+l,n-k+t)\right)^2\right]^{\frac{1}{2}} \\
& = O(MN)^{\frac{1}{2}}
\end{aligned}
$$

The equality at the third step of the above expression holds as contribution over cross product terms is zero. □

Result 2: If we denote $q(\phi, m, n) = t_1\alpha m + t_2\beta n$, i.e. $q(\phi, m, n)$ is a linear function of m and n, for some t_1, t_2, where $\phi = (\alpha, \beta)$ then again using

Holder's inequality, we have

$$\begin{aligned}
& E\sup_{\phi}\left|\sum_{n=1}^{N}\sum_{m=1}^{M}\varepsilon(m,n)\varepsilon(m-j,n-k)e^{iq(\phi,m,n)}\right| \\
\leq & \left[E\left(\sup_{\phi}\left|\sum_{n=1}^{N}\sum_{m=1}^{M}\varepsilon(m,n)\varepsilon(m-j,n-k)e^{iq(\phi,m,n)}\right|\right)^{2}\right]^{\frac{1}{2}} \\
= & \left[E\sup_{\phi}\left(\sum_{n=1}^{N}\sum_{m=1}^{M}\varepsilon(m,n)\varepsilon(m-j,n-k)e^{iq(\phi,m,n)}\right)\right. \\
& \left.\left(\sum_{n=1}^{N}\sum_{m=1}^{M}\varepsilon(m,n)\varepsilon(m-j,n-k)e^{-iq(\phi,m,n)}\right)\right]^{\frac{1}{2}}
\end{aligned}$$

$$\begin{aligned}
\leq & \left[E\sum_{n=1}^{N}\sum_{m=1}^{M}\varepsilon(m,n)^{2}\varepsilon(m-j,n-k)^{2}\right. \\
+ & 2E\left|\sum_{n=1}^{N-1}\sum_{m=1}^{M}\varepsilon(m,n)\varepsilon(m,n+1)\varepsilon(m-j,n-k)\varepsilon(m-j,n-k+1)\right| \\
+ & 2E\left|\sum_{n=1}^{N}\sum_{m=1}^{M-1}\varepsilon(m,n)\varepsilon(m+1,n)\varepsilon(m-j,n-k)\varepsilon(m-j+1,n-k)\right| \\
+ & \cdots+2E\left|\sum_{m=1}^{M}\varepsilon(m,1)\varepsilon(m,N)\varepsilon(m-j,1-k)\varepsilon(m-j,N-k)\right| \\
& +2E\left|\sum_{n=1}^{N}\varepsilon(1,n)\varepsilon(M,n)\varepsilon(1-j,n-k)\varepsilon(M-j,n-k)\right| \\
+ & 2E\left|\sum_{n=1}^{N-1}\sum_{m=1}^{M-1}\varepsilon(m,n)\varepsilon(m+1,n+1)\varepsilon(m-j,n-k)\varepsilon(m-j+1,n-k+1)\right| \\
+ & \cdots+2E\left|\varepsilon(M,1)\varepsilon(M,N)\varepsilon(M-j,1-k)\varepsilon(M-j,N-k)\right| \\
+ & \left.2E\left|\varepsilon(1,N)\varepsilon(M,N)\varepsilon(1-j,N-k)\varepsilon(M-j,N-k)\right|\right]^{\frac{1}{2}} \\
= & O(MN+MN.(MN)^{\frac{1}{2}})^{\frac{1}{2}}=O((MN)^{\frac{3}{4}}).
\end{aligned}$$

□

If $q(\xi, m, n) = \alpha m + \beta m^2 + \gamma n + \delta n^2 + \nu mn$, i.e. $q(\xi, m, n)$ is a quadratic function of m and n, where $\xi = (\alpha, \beta, \gamma, \delta, \nu)$ and let $m - j = m', n - k = n'$. Then

$$
\begin{aligned}
& E \sup_{\xi} \left| \frac{1}{MN} \sum_{n=1}^{N} \sum_{m=1}^{M} X(m,n) e^{iq(\xi,m,n)} \right| \\
&= E \sup_{\xi} \left| \frac{1}{MN} \sum_{n=1}^{N} \sum_{m=1}^{M} \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} a(j,k) \varepsilon(m-j, n-k) e^{iq(\xi,m,n)} \right| \\
&= E \sup_{\xi} \left| \frac{1}{MN} \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} a(j,k) \sum_{n=1}^{N} \sum_{m=1}^{M} \varepsilon(m',n') e^{iq(\xi,m,n)} \right| \\
&\leq E \sup_{\xi} \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \left| a(j,k) \frac{1}{MN} \sum_{n=1}^{N} \sum_{m=1}^{M} \varepsilon(m',n') e^{iq(\xi,m,n)} \right| \\
&= E \sup_{\xi} \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} |a(j,k)| \frac{1}{MN} \left| \sum_{n=1}^{N} \sum_{m=1}^{M} \varepsilon(m',n') e^{iq(\xi,m,n)} \right| \\
&\leq \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} |a(j,k)| \left[E \sup_{\xi} \frac{1}{MN} \left| \sum_{n=1}^{N} \sum_{m=1}^{M} \varepsilon(m',n') e^{iq(\xi,m,n)} \right| \right] \\
&\leq \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} |a(j,k)| \left[E \sup_{\xi} \frac{1}{MN} \left| \sum_{n=1}^{N} \sum_{m=1}^{M} \varepsilon(m',n') e^{iq(\xi,m,n)} \right|^2 \right]^{\frac{1}{2}} \\
&= \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} |a(j,k)| \\
&\times \frac{1}{MN} \left[E \sup_{\xi} \left(\sum_{n=1}^{N} \sum_{m=1}^{M} \varepsilon(m',n') e^{iq(\xi,m,n)} \right) \left(\sum_{n=1}^{N} \sum_{m=1}^{M} \varepsilon(m',n') e^{-iq(\xi,m,n)} \right) \right]^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} |a(j,k)| \quad \times \frac{1}{MN} \Bigg[E \sum_{n=1}^{N} \sum_{m=1}^{M} \varepsilon(m', n')^2 \\
&+E \left| \sum_{n=1}^{N-1} \sum_{m=1}^{M} \varepsilon(m', n') \varepsilon(m', n'+1) e^{-i(2\delta n + \nu n)} \right| \\
&+E \left| \sum_{n=1}^{N} \sum_{m=1}^{M-1} \varepsilon(m', n') \varepsilon(m'+1, n') e^{-i(2\beta m + \nu m)} \right| \\
&+ \cdots + E \left| \sum_{m=1}^{M} \varepsilon(m', 1-k) \varepsilon(m', N-k) \right| \\
&+E \left| \sum_{n=1}^{N} \varepsilon(1-j, n') \varepsilon(M-j, n') \right| \\
&+ \cdots + E \left| \varepsilon(1-j, 1-k) \varepsilon(M-j, N-k) \right| \\
&+E \left| \varepsilon(1-j, 1-k) \varepsilon(M-j, N-k) \right| \Bigg]^{\frac{1}{2}} \\
&= \frac{1}{MN} O(MN + MN.(MN)^{\frac{3}{4}})^{\frac{1}{2}} = O((MN)^{-\frac{1}{8}}).
\end{aligned}
$$

Therefore,

$$
E \sup_{\xi} \left| \frac{1}{(MN)^9} \sum_{n=1}^{N^9} \sum_{m=1}^{M^9} X(m,n) e^{iq(\xi,m,n)} \right| \leq O((MN)^{-\frac{9}{8}}) \qquad \text{(S1.2)}
$$

Take

$$
Z(M,N) = \sup_{\xi} \left| \frac{1}{(MN)^9} \sum_{n=1}^{N^9} \sum_{m=1}^{M^9} X(m,n) e^{iq(\xi,m,n)} \right|
$$

and for $\epsilon \quad > 0$

$$
\sum_{N=1}^{\infty} \sum_{M=1}^{\infty} P(Z(M,N) > \epsilon) \leq \sum_{N=1}^{\infty} \sum_{M=1}^{\infty} \frac{EZ(M,N)}{\epsilon} \leq \sum_{N=1}^{\infty} \sum_{M=1}^{\infty} \frac{O((MN)^{-\frac{9}{8}})}{\epsilon} < \infty
$$

Therefore, by Borel Cantelli lemma we have

$$\sup_{\xi}\left|\frac{1}{(MN)^9}\sum_{n=1}^{N^9}\sum_{m=1}^{M^9}X(m,n)e^{iq(\xi,m,n)}\right| \rightarrow \quad 0 \quad a.s.$$

Denote, $\{(J,K) : N^9 < K \leq (N+1)^9,\ M^9 < J \leq (M+1)^9\} \ = S_{JK}$

Define,

$$\begin{aligned}
U(M,N) \quad &= \sup_{\xi}\sup_{S_{JK}}\left|\frac{1}{(MN)^9}\sum_{n=1}^{N^9}\sum_{m=1}^{M^9}X(m,n)e^{iq(\xi,m,n)} - \frac{1}{JK}\sum_{n=1}^{K}\sum_{m=1}^{J}X(m,n)e^{iq(\xi,m,n)}\right| \\
&= \sup_{\xi}\sup_{S_{JK}}\left|\frac{1}{(MN)^9}\sum_{n=1}^{N^9}\sum_{m=1}^{M^9}X(m,n)e^{iq(\xi,m,n)} - \frac{1}{(MN)^9}\sum_{n=1}^{K}\sum_{m=1}^{J}X(m,n)e^{iq(\xi,m,n)}\right. \\
&\qquad \left. + \frac{1}{(MN)^9}\sum_{n=1}^{K}\sum_{m=1}^{J}X(m,n)e^{iq(\xi,m,n)} - \frac{1}{JK}\sum_{n=1}^{K}\sum_{m=1}^{J}X(m,n)e^{iq(\xi,m,n)}\right| \\
&\leq \sup_{\xi}\sup_{S_{JK}}\left[\frac{1}{(MN)^9}\left|\sum_{n=1}^{N^9}\sum_{m=1}^{M^9}X(m,n)e^{iq(\xi,m,n)} - \sum_{n=1}^{K}\sum_{m=1}^{J}X(m,n)e^{iq(\xi,m,n)}\right|\right. \\
&\qquad \left. + \left(\frac{1}{(MN)^9} - \frac{1}{(M+1)^9(N+1)^9}\right)\left|\sum_{n=1}^{K}\sum_{m=1}^{J}X(m,n)e^{iq(\xi,m,n)}\right|\right] \\
&\leq V(M,N) + W(M,N),
\end{aligned}$$

where

$$\begin{aligned}
V(M,N) = \quad &\sup_{\xi}\left|\sum_{n=N^9+1}^{(N+1)^9}\sum_{m=1}^{M^9}X(m,n)e^{iq(\xi,m,n)}\right| + \sup_{\xi}\left|\sum_{n=1}^{N^9}\sum_{m=M^9+1}^{(M+1)^9}X(m,n)e^{iq(\xi,m,n)}\right| \\
&+ \sup_{\xi}\left|\sum_{n=N^9+1}^{(N+1)^9}\sum_{m=M^9+1}^{(M+1)^9}X(m,n)e^{iq(\xi,m,n)}\right|,
\end{aligned}$$

$$W(M,N) = \left(\frac{1}{(MN)^9} - \frac{1}{(M+1)^9(N+1)^9}\right)\sup_{\xi}\left|\sum_{n=1}^{(N+1)^9}\sum_{m=1}^{(M+1)^9}X(m,n)e^{iq(\xi,m,n)}\right|.$$

We have

$$E\sup_{\xi}\left|\sum_{n=n_0+1}^{N}\sum_{m=m_0+1}^{M}X(m,n)e^{iq(\xi,m,n)}\right|\leq O((M-m_0)(N-n_0))^{\frac{7}{8}} \tag{S1.3}$$

and

$$P(V(M,N)>\epsilon)\leq\frac{EV(M,N)}{\epsilon}\leq\frac{O(\frac{(M^9N^8)^{\frac{7}{8}}+(M^8N^9)^{\frac{7}{8}}}{(MN)^9})}{\epsilon}. \tag{S1.4}$$

Therefore,

$$\sum_{N=1}^{\infty}\sum_{M=1}^{\infty}P(V(M,N)>\epsilon)\leq\sum_{N=1}^{\infty}\sum_{M=1}^{\infty}\frac{O((MN)^{-\frac{9}{8}})}{\epsilon}<\infty.$$

We also have

$$P(W(M,N)>\epsilon)\leq\frac{EW(M,N)}{\epsilon}\leq\frac{O((\frac{1}{M}+\frac{1}{N})(M+1)^{\frac{-9}{8}}(N+1)^{\frac{-9}{8}})}{\epsilon}$$

Therefore,

$$\sum_{N=1}^{\infty}\sum_{M=1}^{\infty}P(W(M,N)>\epsilon)\leq\sum_{N=1}^{\infty}\sum_{M=1}^{\infty}\frac{O((MN)^{-\frac{9}{8}})}{\epsilon}<\infty$$

Now by Borel Cantelli lemma we get $U(M,N)\rightarrow\ \ 0\ \ a.s.$

Now we will analyze the previous proof. To show almost sure convergence the tool used was Borel Cantelli lemma for which we were required Markov inequality. To get probability bound in Markov inequality, we tried to calculate corresponding expectation as follows:

$$E\sup_{\xi}\left|\frac{1}{MN}\sum_{n=1}^{N}\sum_{m=1}^{M}X(m,n)e^{iq(\xi,m,n)}\right| \tag{S1.5}$$

As $q(\xi, m, n)$ is a quadratic in $m,\ n$ while simplifying (S1.5) we need similar expectation calculations, Result 2 for linear $q(\phi, m, n)$ and Result 1 with zeroth degree polynomial in $m,\ n$. Now in our original case we need similar expectation calculation for rth degree polynomial of m and n. If we would take $q(\alpha, m, n) = \sum_{p=1}^{r}\sum_{j=0}^{p} \alpha^0(j, p-j) m^j n^{p-j}$, i.e. $q(\alpha, m, n)$ is rth degree polynomial of m and n, where $\alpha = (\alpha^0(j, p-j), j = 0, \cdots, p, p = 1, \cdots, r)$ then our object of interest will be $E \sup_{\alpha} \left| \frac{1}{MN} \sum_{n=1}^{N} \sum_{m=1}^{M} X(m, n) e^{iq(\alpha,m,n)} \right|$, for which we would need r many results, like Result 1 and 2, for $r-1, r-2 \cdots, 1, 0$ th degree polynomials. Also note that for quadratic case we need the existence of fourth moment whereas for rth degree polynomial case we need existence of $2r$th moment. Now we can proceed for the proof of Lemma 1. Along same line as before

$$
\begin{aligned}
& E \sup_{\alpha} \left| \frac{1}{MN} \sum_{n=1}^{N} \sum_{m=1}^{M} X(m, n) e^{iq(\alpha,m,n)} \right| \\
& = O((MN)^{-\frac{1}{2^{r+1}}})
\end{aligned}
$$

In next step, similar way as before, we get

$$
\sup_{\alpha} \left| \frac{1}{(MN)^{2^{r+1}+1}} \sum_{n=1}^{N^{2^{r+1}+1}} \sum_{m=1}^{M^{2^{r+1}+1}} X(m, n) e^{iq(\alpha,m,n)} \right| \rightarrow 0 \quad a.s.
$$

For rest of the proof replacing subsequences M^9, N^9 by $M^{2^{r+1}+1}, N^{2^{r+1}+1}$, and ξ by α we arrive the final conclusion. $\square$.

Proof of Lemma 2: It can be obtained along the same line.

To prove the Theorem-1 we need the following Lemma.

Lemma 1. Let $\widehat{\theta}$ be the least squares estimator of θ^0, and consider the set $S_c = \{\theta : \theta \in \Theta; |A - A^0| \geq c, |B - B^0| \geq c, |\alpha(j, p-j) - \alpha^0(j, p-j)| \geq c,\ j = 0, \cdots, p,\ p = 1, \cdots, r\}$. If for any $c > 0$, $\liminf \inf_{\theta \in S_c} \frac{1}{MN}(Q(\theta) - Q(\theta^0)) > 0 \quad a.s.$ then $\widehat{\theta} \to \theta^0 \quad a.s.$. Here the function $Q(\theta)$ is same as defined in Section 2.3.

Proof of Lemma 4: The proof can be obtained by contradiction, along the lines of lemma 1 of Wu(1981). If $\widehat{\theta^{(N)}}$ does not converges to θ^0 then there exists a subsequence $\{N_k\}_{k=1}^{\infty}$ along which it fails to converge. Let the collection of ω's, on which it fails to converge is Ω_0. Now $\widehat{\theta^{(N_k)}}$ is LSE for θ^0 and so minimize the quantity $Q_1(\theta)$. That implies on Ω_0 (which is subset of whole set of ω) $\frac{1}{N_k}(Q(\widehat{\theta^{(N_k)}}) - Q_1(\theta^0)) < 0$. Then on whole set of Ω $\liminf \inf_{\theta \in S_c} \frac{1}{N}(Q(\theta) - Q(\theta^0)) \geq \quad 0 \quad a.s.$ which is a contradiction.

Proof of Theorem 1:

To prove Theorem 1, it is enough to prove that

$$\liminf \inf_{\theta \in S_c} \frac{1}{MN}(Q(\theta) - Q(\theta^0)) > \quad 0 \quad a.s.$$

Observe that

$$\frac{1}{MN}[Q(\theta) - Q(\theta^0)] = f(\theta) + g(\theta),$$

where

$$f(\theta) = \frac{1}{MN}\sum_{n=1}^{N}\sum_{m=1}^{M}\Bigg[A\cos(\sum_{p=1}^{r}\sum_{j=0}^{p}\alpha(j,p-j)m^j n^{p-j}) + B\sin(\sum_{p=1}^{r}\sum_{j=0}^{p}\alpha(j,p-j)m^j n^{p-j})$$
$$-A^0\cos(\sum_{p=1}^{r}\sum_{j=0}^{p}\alpha^0(j,p-j)m^j n^{p-j}) - B^0\sin(\sum_{p=1}^{r}\sum_{j=0}^{p}\alpha^0(j,p-j)m^j n^{p-j})\Bigg]^2$$

$$g(\theta) = \frac{2}{MN}\sum_{n=1}^{N}\sum_{m=1}^{M}\Bigg[A^0\cos(\sum_{p=1}^{r}\sum_{j=0}^{p}\alpha^0(j,p-j)m^j n^{p-j}) + B^0\sin(\sum_{p=1}^{r}\sum_{j=0}^{p}\alpha^0(j,p-j)m^j n^{p-j})$$
$$-A\cos(\sum_{p=1}^{r}\sum_{j=0}^{p}\alpha(j,p-j)m^j n^{p-j}) - B\sin(\sum_{p=1}^{r}\sum_{j=0}^{p}\alpha(j,p-j)m^j n^{p-j})\Bigg]X(m,n)$$

Now $g(\theta)$ is going to zero $a.s.$ because of Lemma-1 . We now observe that

$$S_c \subset S_c^A \cup S_c^B \cup_{p=1}^{r} \cup_{j=0}^{p} S_c^{\alpha(j,p-j)}$$

where $S_c^A = \{\theta; \theta \in \Theta, |A - A^0| \geq c\}$, and the other sets are also similarly defined. It can be shown along the same line as in Kundu (1997) that

$$\liminf \inf_{S_c^{\mathfrak{t}}} f(\theta) > 0,\ a.s. \quad \text{for } \mathfrak{t} \text{ is any one of } A, B, \alpha(j,p-j),\ j = 0,\cdots,p,\ p = 1,\cdots,r$$

and hence the result is proved. $\square$

Proof of Lemma 3:

Let us denote $Q'(\theta)$ as the $(2 + \frac{r(r+3)}{2}) \times 1$ first derivative matrix and $Q''(\theta)$ as the $(2 + \frac{r(r+3)}{2}) \times (2 + \frac{r(r+3)}{2})$ second derivative matrix. Now using multivariate Taylor series expansion of $Q'(\widehat{\theta})$ around θ^0 and we get

$$Q'(\widehat{\theta}) - Q'(\theta^0) = (\widehat{\theta} - \theta^0)Q''(\bar{\theta}) \tag{S1.6}$$

where $\bar{\theta}$ is a point on line joining $\widehat{\theta}$ and θ^0. Since, $Q'(\widehat{\theta}) = 0$, then for the diagonal matrix D, same as defined in Theorem 2, we obtain

$$-Q'(\theta^0)D = (\widehat{\theta} - \theta^0)D^{-1}[DQ''(\bar{\theta})D] \tag{S1.7}$$

which gives,

$$(\widehat{\theta} - \theta^0)D^{-1} = [-Q'(\theta^0)D][DQ''(\bar{\theta})D]^{-1} \tag{S1.8}$$

Dividing by $\sqrt{MN}$ the expression becomes

$$(\widehat{\theta} - \theta^0)(\sqrt{MN}D)^{-1} = [-\frac{1}{\sqrt{MN}}Q'(\theta^0)D][DQ''(\bar{\theta})D]^{-1} \tag{S1.9}$$

Since, $\widehat{\theta} \to \theta^0 \;\; a.s.$, $\bar{\theta} \to \theta^0 \;\; a.s.$. Therefore,

$$\left[DQ''(\bar{\theta})D\right]^{-1} \to \left[DQ''(\theta^0)D\right]^{-1}$$

Moreover, using Lemma 1 and Proposition 2,

$$\frac{1}{\sqrt{MN}}Q'(\theta^0)D \to 0 \; a.s. \tag{S1.10}$$

So,

$$(\widehat{\theta} - \theta^0)(\sqrt{MN}D)^{-1} \to 0 \; a.s. \tag{S1.11}$$

Hence using (S1.11) we get for $j = 0, \cdots, p, \; p = 1, \cdots, r$

$$M^j N^{p-j}(\widehat{\alpha}(j, p-j) - \alpha^0(j, p-j)) \to 0 \; a.s.$$

□

Proof of Theorem 2: We recall $q(\alpha, m, n) = \sum_{p=1}^{r}\sum_{j=0}^{p} \alpha(j, p-j) m^j n^{p-j}$.

Note that

$$\left[Q'(\theta^0)D\right]^T = \begin{bmatrix} -\frac{2}{\sqrt{MN}} \sum_{n=1}^{N}\sum_{m=1}^{M} \cos(q(\alpha^0, m, n))X(m,n) \\ -\frac{2}{\sqrt{MN}} \sum_{n=1}^{N}\sum_{m=1}^{M} \sin(q(\alpha^0, m, n))X(m,n) \\ \frac{2}{M\sqrt{MN}} \sum_{n=1}^{N}\sum_{m=1}^{M} m[A^0 \sin(q(\alpha^0, m, n)) - B^0 \cos(q(\alpha^0, m, n))]\ X(m,n) \\ \frac{2}{M^2\sqrt{MN}} \sum_{n=1}^{N}\sum_{m=1}^{M} m^2[A^0 \sin(q(\alpha^0, m, n)) - B^0 \cos(q(\alpha^0, m, n))]\ X(m,n) \\ \frac{2}{N\sqrt{MN}} \sum_{n=1}^{N}\sum_{m=1}^{M} n[A^0 \sin(q(\alpha^0, m, n)) - B^0 \cos(q(\alpha^0, m, n))]\ X(m,n) \\ \frac{2}{N^2\sqrt{MN}} \sum_{n=1}^{N}\sum_{m=1}^{M} n^2[A^0 \sin(q(\alpha^0, m, n)) - B^0 \cos(q(\alpha^0, m, n))]\ X(m,n). \end{bmatrix}$$

Now using Central Limit Theorem of linear processes, see Fuller (1996,), page 329, it follows that

$$\left[Q'(\theta^0)D\right]^T \rightarrow N_6(0, 2c\sigma^2\Sigma). \tag{S1.12}$$

Also,

$$\left.\frac{\partial^2 Q(\theta)}{\partial A^2}\right|_{\theta^0} = 2\sum_{n=1}^{N} \cos^2(\sum_{p=1}^{r}\sum_{j=0}^{p} \alpha^0(j, p-j) m^j n^{p-j}),$$
$$\left.\frac{\partial^2 Q(\theta)}{\partial A \partial B}\right|_{\theta^0} = 2\sum_{n=1}^{N} \sin(\sum_{p=1}^{r}\sum_{j=0}^{p} \alpha^0(j, p-j) m^j n^{p-j}) \cos(\sum_{p=1}^{r}\sum_{j=0}^{p} \alpha^0(j, p-j) m^j n^{p-j}),$$
$$\left.\frac{\partial^2 Q(\theta)}{\partial B^2}\right|_{\theta^0} = 2\sum_{n=1}^{N} \sin^2(\sum_{p=1}^{r}\sum_{j=0}^{p} \alpha^0(j, p-j) m^j n^{p-j}),$$

$$\frac{\partial^2 Q(\theta)}{\partial A \partial \alpha(j,p-j)}\Big|_{\theta^0} = 2\sum_{n=1}^{N} m^j n^{p-j} \sin(\sum_{p=1}^{r}\sum_{j=0}^{p} \alpha^0(j,p-j)m^j n^{p-j}) \times X(n)$$
$$- 2\sum_{n=1}^{N} m^j n^{p-j} \cos(\sum_{p=1}^{r}\sum_{j=0}^{p} \alpha^0(j,p-j)m^j n^{p-j})$$
$$\times [A^0 \sin(\sum_{p=1}^{r}\sum_{j=0}^{p} \alpha^0(j,p-j)m^j n^{p-j}) - B^0 \cos(\sum_{p=1}^{r}\sum_{j=0}^{p} \alpha^0(j,p-j)m^j n^{p-j})],$$

$$\frac{\partial^2 Q(\theta)}{\partial B \partial \alpha(j,p-j)}\Big|_{\theta^0} = -2\sum_{n=1}^{N} m^j n^{p-j} \cos(\sum_{p=1}^{r}\sum_{j=0}^{p} \alpha^0(j,p-j)m^j n^{p-j}) \times X(n)$$
$$- 2\sum_{n=1}^{N} m^j n^{p-j} \sin(\sum_{p=1}^{r}\sum_{j=0}^{p} \alpha^0(j,p-j)m^j n^{p-j})$$
$$\times [A^0 \sin(\sum_{p=1}^{r}\sum_{j=0}^{p} \alpha^0(j,p-j)m^j n^{p-j}) - B^0 \cos(\sum_{p=1}^{r}\sum_{j=0}^{p} \alpha^0(j,p-j)m^j n^{p-j})],$$

$$\frac{\partial^2 Q(\theta)}{\partial \alpha(j,p-j) \partial \alpha(k,q-k)}\Big|_{\theta^0} = 2\sum_{n=1}^{N} m^{j+k} n^{p+q-j-k+1} [A^0 \cos(\sum_{p=1}^{r}\sum_{j=0}^{p} \alpha^0(j,p-j)m^j n^{p-j})$$
$$+ B^0 \sin(\sum_{p=1}^{r}\sum_{j=0}^{p} \alpha^0(j,p-j)m^j n^{p-j})] \times X(n)$$
$$+ 2\sum_{n=1}^{N} m^{j+k} n^{p+q-j-k+1} [A^0 \sin(\sum_{p=1}^{r}\sum_{j=0}^{p} \alpha^0(j,p-j)m^j n^{p-j}) - B^0 \cos(\sum_{p=1}^{r}\sum_{j=0}^{p} \alpha^0(j,p-j)m^j n^{p-j})]^2,$$

for $k = 0, \cdots, q, \ q = 1, \cdots, r$ Since

$$\left[DQ''(\theta^0)D\right] \to \Sigma$$

we immediately get

$$(\widehat{\theta} - \theta^0)D^{-1} \to \ N_6(0, 2c\sigma^2\Sigma^{-1}).$$

□

S2 Simulation Results

In case of Model 1, for Gaussian errors the results are presented in Tables 1 - 4, and in case of Laplace errors the results are reported in Tables 5 - 8. For Model 2, the results are reported in Table 9.

Table 1: The MEAN, MSE, VAR and ASYV of the least squares estimators when $\sigma^2 = 0.05$, Error-I and $\varepsilon(m,n)$'s are Gaussian random variables (Model 1)

M=N=50						
PARA	5.00	5.00	1.00	0.05	1.50	0.50
MEAN	4.998749	5.000799	0.999997	0.050000	1.500020	0.500000
MSE	(0.12381E-02)	(0.10095E-02)	(0.11243E-06)	(0.39859E-10)	(0.38522E-06)	(0.11305E-09)
VAR	(0.12365E-02)	(0.10089E-02)	(0.11242E-06)	(0.39853E-10)	(0.38490E-06)	(0.11297E-09)
ASYV	(0.72000E-03)	(0.72000E-03)	(0.61440E-07)	(0.23040E-10)	(0.61440E-07)	(0.23040E-10)
M=N=75						
PARA	5.00	5.00	1.00	0.05	1.50	0.50
MEAN	5.001370	4.998416	1.000027	0.050000	1.499981	0.500000
MSE	(0.43412E-03)	(0.43844E-03)	(0.37391E-07)	(0.63734E-11)	(0.37549E-07)	(0.63756E-11)
VAR	(0.43225E-03)	(0.43594E-03)	(0.36688E-07)	(0.62422E-11)	(0.37172E-07)	(0.63756E-11)
ASYV	(0.32000E-03)	(0.32000E-03)	(0.12136E-07)	(0.20227E-11)	(0.12136E-07)	(0.20221E-11)
M=N=100						
PARA	5.00	5.00	1.00	0.05	1.50	0.50
MEAN	4.983871	5.015692	1.000094	0.049999	1.500076	0.500000
MSE	(0.35075E-03)	(0.32362E-03)	(0.15354E-07)	(0.14174E-11)	(0.75770E-08)	(0.75615E-12)
VAR	(0.90565E-04)	(0.77356E-04)	(0.64723E-08)	(0.62758E-12)	(0.17133E-08)	(0.32169E-12)
ASYV	(0.24000E-03)	(0.24000E-03)	(0.68285E-08)	(0.85333E-12)	(0.68285E-08)	(0.85333E-12)

Table 2: The MEAN, MSE, VAR and ASYV of the least squares estimators when $\sigma^2 = 0.5$, Error-I and $\varepsilon(m,n)$'s are Gaussian random variables (Model 1)

M=N=50						
PARA	5.00	5.00	1.00	0.05	1.50	0.50
MEAN	4.993152	5.003687	1.000072	0.049999	1.500018	0.499999
MSE	(0.14862E-01)	(0.89375E-02)	(0.11333E-05)	(0.42206E-09)	(0.56846E-05)	(0.21873E-08)
VAR	(0.14815E-01)	(0.89239E-02)	(0.11280E-05)	(0.42062E-09)	(0.56842E-05)	(0.21871E-08)
ASYV	(0.72000E-02)	(0.72000E-02)	(0.61440E-06)	(0.23040E-09)	(0.61440E-06)	(0.23040E-09)
M=N=75						
PARA	5.00	5.00	1.00	0.05	1.50	0.50
MEAN	4.998179	5.000843	1.000020	0.050000	1.500002	0.500000
MSE	(0.30544E-02)	(0.30886E-02)	(0.27571E-06)	(0.46163E-10)	(0.27355E-06)	(0.47246E-10)
VAR	(0.30511E-02)	(0.30879E-02)	(0.27533E-06)	(0.46128E-10)	(0.27354E-06)	(0.47246E-10)
ASYV	(0.32000E-02)	(0.32000E-02)	(0.12136E-06)	(0.20227E-10)	(0.12136E-06)	(0.20227E-10)
M=N=100						
PARA	5.00	5.00	1.00	0.05	1.50	0.50
MEAN	4.984965	5.013754	1.000111	0.049999	1.500057	0.500000
MSE	(0.69261E-03)	(0.71473E-03)	(0.52219E-07)	(0.55155E-11)	(0.56498E-08)	(0.96203E-12)
VAR	(0.46656E-03)	(0.52554E-03)	(0.39950E-07)	(0.42982E-11)	(0.24600E-08)	(0.65270E-12)
ASYV	(0.24000E-02)	(0.24000E-02)	(0.68285E-07)	(0.85333E-11)	(0.68285E-07)	(0.85333E-11)

Table 3: The MEAN, MSE, VAR and ASYV of the least squares estimators when $\sigma^2 = 0.05$, Error-II and $\varepsilon(m,n)$'s are Gaussian random variables (Model 1)

M=N=50						
PARA	5.00	5.00	1.00	0.05	1.50	0.50
MEAN	4.998425	5.001181	1.000016	0.050000	1.500002	0.500000
MSE	(0.68604E-03)	(0.68611E-03)	(0.11592E-06)	(0.44304E-10)	(0.11251E-06)	(0.40000E-10)
VAR	(0.68357E-03)	(0.68472E-03)	(0.11565E-06)	(0.44250E-10)	(0.11250E-06)	(0.40000E-10)
ASYV	(0.97840E-03)	(0.97840E-03)	(0.83490E-07)	(0.31309E-10)	(0.83490E-07)	(0.31309E-10)
M=N=75						
PARA	5.00	5.00	1.00	0.05	1.50	0.50
MEAN	5.000978	4.998699	1.000024	0.050000	1.499987	0.500000
MSE	(0.48896E-03)	(0.47950E-03)	(0.44114E-07)	(0.76008E-11)	(0.45481E-07)	(0.78724E-11)
VAR	(0.48799E-03)	(0.47780E-03)	(0.43524E-07)	(0.74974E-11)	(0.45366E-07)	(0.78669E-11)
ASYV	(0.43484E-03)	(0.43484E-03)	(0.16492E-07)	(0.27486E-11)	(0.16492E-07)	(0.27486E-11)
M=N=100						
PARA	5.00	5.00	1.00	0.05	1.50	0.50
MEAN	4.983525	5.016331	1.000103	0.049999	1.500071	0.500000
MSE	(0.43866E-03)	(0.42898E-03)	(0.24280E-07)	(0.21339E-11)	(0.67342E-08)	(0.68071E-12)
VAR	(0.16725E-03)	(0.16233E-03)	(0.13803E-07)	(0.12060E-11)	(0.16579E-08)	(0.31423E-12)
ASYV	(0.32613E-03)	(0.32613E-03)	(0.92768E-08)	(0.11596E-11)	(0.92768E-08)	(0.11596E-11)

Table 4: The MEAN, MSE, VAR and ASYV of the least squares estimators when $\sigma^2 = 0.5$, Error-II and $\varepsilon(m, n)$'s are Gaussian random variables (Model 1)

M=N=50						
PARA	5.00	5.00	1.00	0.05	1.50	0.50
MEAN	4.995960	5.002028	1.000068	0.049999	1.499953	0.500001
MSE	(0.69557E-02)	(0.69637E-02)	(0.14144E-05)	(0.53330E-09)	(0.11924E-05)	(0.43918E-09)
VAR	(0.69394E-02)	(0.69596E-02)	(0.14098E-05)	(0.53240E-09)	(0.11900E-05)	(0.43844E-09)
ASYV	(0.97840E-02)	(0.97840E-02)	(0.83490E-06)	(0.31309E-09)	(0.83490E-06)	(0.31309E-09)
M=N=75						
PARA	5.00	5.00	1.00	0.05	1.50	0.50
MEAN	4.998765	4.999673	1.000021	0.050000	1.499990	0.500000
MSE	(0.44162E-02)	(0.44612E-02)	(0.39796E-06)	(0.61326E-10)	(0.31483E-06)	(0.53257E-10)
VAR	(0.44147E-02)	(0.44611E-02)	(0.39755E-06)	(0.61311E-10)	(0.31474E-06)	(0.53251E-10)
ASYV	(0.43484E-02)	(0.43484E-02)	(0.16492E-06)	(0.27486E-10)	(0.16492E-06)	(0.27486E-10)
M=N=100						
PARA	5.00	5.00	1.00	0.05	1.50	0.50
MEAN	4.981299	5.018357	1.000159	0.049998	1.500048	0.500000
MSE	(0.86866E-03)	(0.10029E-02)	(0.80940E-07)	(0.82032E-11)	(0.45432E-08)	(0.93503E-12)
VAR	(0.51900E-03)	(0.66586E-03)	(0.55460E-07)	(0.57826E-11)	(0.22498E-08)	(0.76692E-12)
ASYV	(0.32613E-02)	(0.32613E-02)	(0.92768E-07)	(0.11596E-10)	(0.92768E-07)	(0.11596E-10)

Table 5: The MEAN, MSE, VAR and ASYV of the least squares estimators when $\sigma^2 = 0.05$, Error-I and $\varepsilon(m,n)$'s are Laplace random variables (Model 1)

M=N=50						
PARA	5.00	5.00	1.00	0.05	1.50	0.50
MEAN	5.006754	5.001654	1.002316	0.049912	1.500532	0.499913
MSE	(0.13214E-02)	(0.10765E-02)	(0.11765E-06)	(0.40145E-10)	(0.39675E-06)	(0.12267E-09)
VAR	(0.13112E-02)	(0.10498E-02)	(0.11111E-06)	(0.39264E-10)	(0.39541E-06)	(0.12001E-09)
ASYV	(0.72000E-03)	(0.72000E-03)	(0.61440E-07)	(0.23040E-10)	(0.61440E-07)	(0.23040E-10)
M=N=75						
PARA	5.00	5.00	1.00	0.05	1.50	0.50
MEAN	5.003412	5.000132	0.999911	0.049992	1.500016	0.499981
MSE	(0.44563E-03)	(0.44983E-03)	(0.37998E-07)	(0.64876E-11)	(0.37999E-07)	(0.64236E-11)
VAR	(0.44123E-03)	(0.43889E-03)	(0.37076E-07)	(0.63991E-11)	(0.37571E-07)	(0.63998E-11)
ASYV	(0.32000E-03)	(0.32000E-03)	(0.12136E-07)	(0.20227E-11)	(0.12136E-07)	(0.20221E-11)
M=N=100						
PARA	5.00	5.00	1.00	0.05	1.50	0.50
MEAN	4.983871	5.015692	1.000094	0.049999	1.500076	0.500000
MSE	(0.35657E-03)	(0.32463E-03)	(0.15588E-07)	(0.14768E-11)	(0.75991E-08)	(0.75899E-12)
VAR	(0.90786E-04)	(0.77651E-04)	(0.64915E-08)	(0.62887E-12)	(0.17387E-08)	(0.32342E-12)
ASYV	(0.24000E-03)	(0.24000E-03)	(0.68285E-08)	(0.85333E-12)	(0.68285E-08)	(0.85333E-12)

Table 6: The MEAN, MSE, VAR and ASYV of the least squares estimators when $\sigma^2 = 0.5$, Error-I and $\varepsilon(m,n)$'s are Laplace random variables (Model 1)

M=N=50						
PARA	5.00	5.00	1.00	0.05	1.50	0.50
MEAN	4.996754	4.998767	1.000564	0.050031	1.500784	0.499943
MSE	(0.15123E-01)	(0.94325E-02)	(0.12875E-05)	(0.436574-09)	(0.57865E-05)	(0.22998E-08)
VAR	(0.14993E-01)	(0.91234E-02)	(0.11887E-05)	(0.43018E-09)	(0.57003E-05)	(0.21997E-08)
ASYV	(0.72000E-02)	(0.72000E-02)	(0.61440E-06)	(0.23040E-09)	(0.61440E-06)	(0.23040E-09)
M=N=75						
PARA	5.00	5.00	1.00	0.05	1.50	0.50
MEAN	5.000153	5.001065	1.000776	0.049965	1.500232	0.500116
MSE	(0.32287E-02)	(0.327781-02)	(0.28671E-06)	(0.47861E-10)	(0.28112E-06)	(0.47998E-10)
VAR	(0.31943E-02)	(0.31671E-02)	(0.28001E-06)	(0.46995E-10)	(0.27967E-06)	(0.47848E-10)
ASYV	(0.32000E-02)	(0.32000E-02)	(0.12136E-06)	(0.20227E-10)	(0.12136E-06)	(0.20227E-10)
M=N=100						
PARA	5.00	5.00	1.00	0.05	1.50	0.50
MEAN	5.000119	5.011234	1.000675	0.050014	1.500165	0.500087
MSE	(0.69867E-03)	(0.71675E-03)	(0.52568E-07)	(0.55376E-11)	(0.56621E-08)	(0.96701E-12)
VAR	(0.47212E-03)	(0.53019E-03)	(0.41671E-07)	(0.43789E-11)	(0.25105E-08)	(0.67562E-12)
ASYV	(0.24000E-02)	(0.24000E-02)	(0.68285E-07)	(0.85333E-11)	(0.68285E-07)	(0.85333E-11)

Table 7: The MEAN, MSE, VAR and ASYV of the least squares estimators when $\sigma^2 = 0.05$, Error-II and $\varepsilon(m,n)$'s are Laplace random variables (Model 1)

M=N=50						
PARA	5.00	5.00	1.00	0.05	1.50	0.50
MEAN	4.991154	4.976781	1.004516	0.0499856	1.500176	0.500232
MSE	(0.71657E-03)	(0.71245E-03)	(0.13561E-06)	(0.46678E-10)	(0.13391E-06)	(0.42584E-10)
VAR	(0.70651E-03)	(0.70067E-03)	(0.128971-06)	(0.455423-10)	(0.12891E-06)	(0.41675E-10)
ASYV	(0.97840E-03)	(0.97840E-03)	(0.83490E-07)	(0.31309E-10)	(0.83490E-07)	(0.31309E-10)
M=N=75						
PARA	5.00	5.00	1.00	0.05	1.50	0.50
MEAN	4.999651	4.995543	1.000667	0.050112	1.500143	0.499945
MSE	(0.50187E-03)	(0.500957-03)	(0.45875E-07)	(0.78098E-11)	(0.471541-07)	(0.80009E-11)
VAR	(0.49089E-03)	(0.48761E-03)	(0.43998E-07)	(0.75671E-11)	(0.460091-07)	(0.78998E-11)
ASYV	(0.43484E-03)	(0.43484E-03)	(0.16492E-07)	(0.27486E-11)	(0.16492E-07)	(0.27486E-11)
M=N=100						
PARA	5.00	5.00	1.00	0.05	1.50	0.50
MEAN	5.000561	5.001761	1.000451	0.049965	1.500110	0.499993
MSE	(0.44541E-03)	(0.438716-03)	(0.24678E-07)	(0.21667E-11)	(0.67981E-08)	(0.68121E-12)
VAR	(0.172345-03)	(0.16876E-03)	(0.14365E-07)	(0.14671E-11)	(0.18867E-08)	(0.33451E-12)
ASYV	(0.32613E-03)	(0.32613E-03)	(0.92768E-08)	(0.11596E-11)	(0.92768E-08)	(0.11596E-11)

Table 8: The MEAN, MSE, VAR and ASYV of the least squares estimators when $\sigma^2 = 0.5$, Error-II and $\varepsilon(m,n)$'s are Laplace random variables (Model 1)

M=N=50						
PARA	5.00	5.00	1.00	0.05	1.50	0.50
MEAN	5.000541	5.006571	1.000675	0.050071	1.499786	0.500112
MSE	(0.71453E-02)	(0.71981E-02)	(0.166547-05)	(0.55543E-09)	(0.13675E-05)	(0.45871E-09)
VAR	(0.69878E-02)	(0.700876-02)	(0.157643-05)	(0.54018E-09)	(0.12287E-05)	(0.45001E-09)
ASYV	(0.97840E-02)	(0.97840E-02)	(0.83490E-06)	(0.31309E-09)	(0.83490E-06)	(0.31309E-09)
M=N=75						
PARA	5.00	5.00	1.00	0.05	1.50	0.50
MEAN	4.994311	5.000113	1.000254	0.050176	1.500087	0.500267
MSE	(0.46654E-02)	(0.46098E-02)	(0.41076E-06)	(0.63245E-10)	(0.33528E-06)	(0.55267E-10)
VAR	(0.45186E-02)	(0.459981-02)	(0.40185E-06)	(0.62376E-10)	(0.32675E-06)	(0.54987E-10)
ASYV	(0.43484E-02)	(0.43484E-02)	(0.16492E-06)	(0.27486E-10)	(0.16492E-06)	(0.27486E-10)
M=N=100						
PARA	5.00	5.00	1.00	0.05	1.50	0.50
MEAN	4.982453	5.012546	1.000441	0.050014	1.500067	0.500132
MSE	(0.88876E-03)	(0.11768E-02)	(0.82001E-07)	(0.83176E-11)	(0.46651E-08)	(0.94093E-12)
VAR	(0.52176E-03)	(0.66987E-03)	(0.55860E-07)	(0.58019E-11)	(0.22998E-08)	(0.77016E-12)
ASYV	(0.32613E-02)	(0.32613E-02)	(0.92768E-07)	(0.11596E-10)	(0.92768E-07)	(0.11596E-10)

Table 9: The MEAN, MSE, VAR and ASYV of the least squares estimators when $\sigma^2 = 0.5$, Error-I and $\varepsilon(m, n)$'s are Laplace random variables (Model 2)

M=N=50								
PARA	2.00	2.00	1.00	0.05	0.01	1.0	0.05	0.01
MEAN	1.987911	2.012565	1.000834	0.050032	0.010001	1.000878	0.050028	0.010003
MSE	(0.37671E-01)	(0.35645E-01)	(0.25645E-05)	(0.75678E-09)	(0.41178E-12)	(0.25538E-05)	(0.76017E-09)	(0.43176E-12)
VAR	(0.29879E-01)	(0.276756-01)	(0.21786E-05)	(0.74564E-09)	(0.39876E-12)	(0.22176E-05)	(0.74018E-09)	(0.40173E-12)
ASYV	(0.21543E-01)	(0.21543E-01)	(0.18232E-05)	(0.68124E-09)	(0.13672E-12)	(0.18232E-05)	(0.68124E-09)	(0.13672E-12)
M=N=75								
PARA	2.00	2.00	1.00	0.05	0.01	1.0	0.05	0.01
MEAN	1.995671	1.994532	1.000675	0.050010	0.010000	1.000112	0.500112	0.010000
MSE	(0.16523E-01)	(0.166587-01)	(0.62367E-06)	(0.82567E-09)	(0.12765E-13)	(0.61786E-06)	(0.81987E-09)	(0.13451E-13)
VAR	(0.13668E-01)	(0.140171-01)	(0.58764E-06)	(0.81198E-09)	(0.111453-13)	(0.599765-06)	(0.81076E-09)	(0.12017E-13)
ASYV	(0.94789E-02)	(0.94789E-02)	(0.34578E-06)	(0.61235E-10)	(0.90123E-14)	(0.34587E-06)	(0.61235E-10)	(0.90123E-14)
M=N=100								
PARA	2.00	2.00	1.00	0.05	0.01	1.0	0.05	0.01
MEAN	2.000897	2.000765	1.000675	0.050006	0.010000	1.499786	0.500112	0.010000
MSE	(0.99865E-02)	(0.98769E-02)	(0.29876E-05)	(0.35672E-11)	(0.15467E-16)	(0.30156E-06)	(0.34561E-11)	(0.15569E-16)
VAR	(0.90167E-02)	(0.914536-02)	(0.26778E-05)	(0.31987E-11)	(0.14221E-16)	(0.27801E-06)	(0.31675E-11)	(0.14451E-16)
ASYV	(0.70167E-02)	(0.70167E-02)	(0.22167E-06)	(0.23156E-11)	(0.10176E-16)	(0.22167E-06)	(0.23156E-11)	(0.10176E-16)