# ROBUST TIME SERIES ANALYSIS VIA MEASUREMENT ERROR MODELING

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## Supplementary Material

This note presents outlines of the proofs of the main results in Section 2.2 and Section 3.1 on the qualitative robustness properties of score functions of the new robust estimators. When the kernel of the sporadic, gross-error model has polynomial or exponential tails, the proofs use arguments similar to those used by Hwang & Stefanski (1994) to establish properties of regression curves in the presence of measurement error. When the kernel is Gaussian, the proofs use standard calculus arguments.

## A. Independent-Data Model

Under model (2.1) and (2.2), the score function (2.4)

$$\begin{aligned}\boldsymbol{\psi}(w;\boldsymbol{\theta},\epsilon,\tau) &= \frac{\partial}{\partial\boldsymbol{\theta}}\ln\{f_{W_i}(w;\boldsymbol{\theta},\boldsymbol{\eta})\}\\ &= \frac{(1-\epsilon)\dot{f}_{X_i}(w;\boldsymbol{\theta})+\epsilon(\partial/\partial\boldsymbol{\theta})\int f_{X_i}(t;\boldsymbol{\theta})g(k(w))/(\tau\sigma)\ dt}{(1-\epsilon)f_{X_i}(w;\boldsymbol{\theta})+\epsilon\int f_{X_i}(t;\boldsymbol{\theta})g(k(w))/(\tau\sigma)\ dt}, \quad \text{(A-1)}\end{aligned}$$

where $\dot{f}_{X_i}(w;\boldsymbol{\theta}) = (\partial/\partial\boldsymbol{\theta})f_{X_i}(w;\boldsymbol{\theta})$ and $k(w) = (w-t)/(\tau\sigma)$. The robustness properties (redescending, bounded, unbounded) of $\boldsymbol{\psi}(w;\boldsymbol{\theta},\epsilon,\tau)$ can be deduced from the expression above under certain regularity conditions and assumptions on the tail behavior of $g(\cdot)$. First re-express $\boldsymbol{\psi}(w;\boldsymbol{\theta},\epsilon,\tau)$ as

$$\boldsymbol{\psi}(w;\boldsymbol{\theta},\epsilon,\tau) = \frac{(1-\epsilon)\tau\sigma\dot{f}_{X_i}(w;\boldsymbol{\theta})/g(w/(\tau\sigma))+\epsilon(\partial/\partial\boldsymbol{\theta})\int f_{X_i}(t;\boldsymbol{\theta})g(k(w))/g(w/(\tau\sigma))dt}{(1-\epsilon)\tau\sigma f_{X_i}(w;\boldsymbol{\theta})/g(w/(\tau\sigma))+\epsilon\int f_{X_i}(t;\boldsymbol{\theta})g(k(w))/g(w/(\tau\sigma))\ dt}.$$

It follows from Assumptions (T1) and (T2) that

$$\lim_{|w|\to\infty}\frac{f_{X_i}(w;\boldsymbol{\theta})}{g(w/(\tau\sigma))} = 0, \quad \text{and} \quad \lim_{|w|\to\infty}\frac{\dot{f}_{X_i}(w;\boldsymbol{\theta})}{g(w/(\tau\sigma))} = 0.$$

Thus the behavior of $\boldsymbol{\psi}(w;\boldsymbol{\theta},\epsilon,\tau)$ for large $w$ is determined by the tail behavior of $g(h(w))/g(w/(\tau\sigma))$, which in turn is dictated by the tail behavior of $g(\cdot)$.

As $|w| \to \infty$, the $\sigma$-component of the score function is

$$\lim_{|w|\to\infty} \boldsymbol{\psi}_\sigma(w;\boldsymbol{\theta},\epsilon,\tau) = \lim_{|w|\to\infty} \frac{(\partial/\partial\sigma)\{\int f_{X_i}(t;\boldsymbol{\theta})g(k(w))/(\tau\sigma)dt\}/g(w/(\tau\sigma))/(\tau\sigma)}{\int f_{X_i}(t;\boldsymbol{\theta})g(k(w))/g(w/(\tau\sigma))dt}.$$

**Polynomial-Like Tails.** When $g(\cdot)$ has polynomial like tails, (T3) holds, i.e.,

$$\lim_{|w|\to\infty} g(w+b)\Big/g(w) = 1.$$

Thus when the interchange of limits, integration and differentiation is justified,

$$\lim_{|w|\to\infty} \int f_{X_i}(t;\boldsymbol{\theta})\frac{g(k(w))}{g(w/(\tau\sigma))}\,dt = 1 \quad \text{and} \quad \lim_{|w|\to\infty} \frac{\partial}{\partial\mu}\int f_{X_i}(t;\boldsymbol{\theta})\frac{g(k(w))}{g(w/(\tau\sigma))}\,dt = 0.$$

It follows that when $g(\cdot)$ has polynomial-like tails, $\lim_{|w|\to\infty} \boldsymbol{\psi}_\mu(w;\boldsymbol{\theta},\epsilon,\tau) = 0$, i.e., $\boldsymbol{\psi}_\mu(w;\boldsymbol{\theta},\epsilon,\tau)$ redescends to zero.

Assume that $g(w) \sim w^{-p}$ for some $p > 1$ as $|w| \to \infty$. Then, in general,

$$\lim_{w\to\infty} \frac{(\partial/\partial\sigma)\{g((w-t)/(\tau\sigma))/(\tau\sigma)\}}{g(w/(\tau\sigma))/(\tau\sigma)} = \frac{p-1}{\sigma}.$$

It follows that

$$\lim_{w\to\infty} \frac{(\partial/\partial\sigma)\{\int f_{X_i}(t;\boldsymbol{\theta})g((w-t)/(\tau\sigma))/(\tau\sigma)dt\}/g(w/(\tau\sigma))/(\tau\sigma)}{\int f_{X_i}(t;\boldsymbol{\theta})g((w-t)/(\tau\sigma))/g(w/(\tau\sigma))dt} = \frac{p-1}{\sigma}.$$

Similarly, as $w$ goes to $-\infty$, the limit is also $(p-1)/\sigma$.

**Exponential-Like Tails.** When $g(\cdot)$ has polynomial like tails, (T4) holds, i.e.,

$$\lim_{w\to\infty} g(w+b)\Big/g(w) = \exp(c_1 b) \quad \text{and} \quad \lim_{w\to-\infty} g(w+b)\Big/g(w) = \exp(c_2 b)$$

for some constants $c_1 < 0$ and $c_2 > 0$. So for exponential-like tails,

$$\lim_{w\to\infty} \int f_{X_i}(t;\boldsymbol{\theta})\frac{g(k(w))}{g(w/(\tau\sigma))}\,dt = \int f_{X_i}(t;\boldsymbol{\theta})\exp(-c_1 t/(\tau\sigma))\,dt,$$
$$\text{and} \quad \lim_{w\to-\infty} \int f_{X_i}(t;\boldsymbol{\theta})\frac{g(k(w))}{g(w/(\tau\sigma))}\,dt = \int f_{X_i}(t;\boldsymbol{\theta})\exp(-c_2 t/(\tau\sigma))\,dt, \quad \text{(A-2)}$$

assuming that the limits exist and are interchangeable with integration. If in addition, the limits can be interchanged with differentiation, then

$$\lim_{w\to\infty} \boldsymbol{\psi}_\mu(w;\boldsymbol{\theta},\epsilon,\tau) = \frac{\partial}{\partial\mu}\ln\{m_f(-c_1/(\tau\sigma);\boldsymbol{\theta})\}$$

and

$$\lim_{w\to-\infty} \boldsymbol{\psi}_\mu(w;\boldsymbol{\theta},\epsilon,\tau) = \frac{\partial}{\partial\mu}\ln\{m_f(-c_2/(\tau\sigma);\boldsymbol{\theta})\},$$

where $m_f(\cdot;\boldsymbol{\theta})$ is the moment generating function of $f_{x_i}(\cdot;\boldsymbol{\theta})$, assumed to exist for all real values of its argument.

Following (T4)

$$\lim_{h\to0}\lim_{w\to\infty}\frac{g(w+h)-g(w)}{hg(w)} = \lim_{h\to0}\frac{e^{c_1h}-1}{h} = c_1.$$

Assume that the iterative limits are interchangeable, which holds for the Laplace distribution. Then

$$\lim_{w\to\infty}\lim_{h\to0}\frac{g(w+h)-g(w)}{hg(w)} = \lim_{h\to0}\lim_{w\to\infty}\frac{g(w+h)-g(w)}{hg(w)} = \lim_{w\to\infty}\frac{\dot{g}(w)}{g(w)},$$

where $\dot{g}$ denotes the first derivative. It follows that

$$\lim_{w\to\infty}\frac{\dot{g}(w+b)}{g(w)} = \lim_{w\to\infty}\frac{\dot{g}(w+b)}{g(w+b)}\frac{g(w+b)}{g(w)} = c_1\exp(c_1b),$$

and thus, because $c_1<0$, also that

$$\lim_{w\to\infty}\frac{(\partial/\partial\sigma)\{g((w-t)/(\tau\sigma))/(\tau\sigma)\}}{g(w/(\tau\sigma))/(\tau\sigma)} = \lim_{w\to\infty} -\frac{1}{\sigma}k(w)c_1\exp(c_t) - \frac{1}{\sigma}\exp(c_t) = \infty,$$

where $c_t = -c_1t/(\tau\sigma)$. Therefore

$$\lim_{w\to\infty}\frac{(\partial/\partial\sigma)\{\int f_{x_i}(t;\boldsymbol{\theta})g((w-t)/(\tau\sigma))/(\tau\sigma)dt\}/g(w/(\tau\sigma))/(\tau\sigma)}{\int f_{x_i}(t;\boldsymbol{\theta})g((w-t)/(\tau\sigma))/g(w/(\tau\sigma))dt} \quad = \quad \infty.$$

The proof for $w\to-\infty$ is similar.

**Gaussian Contamination.** When $g(\cdot)$ is standard normal, general results like those above are not possible as the tail behavior of the score function depends on the underlying model. When the central model in (A-1) is a Gaussian location-scale model, and $g(\cdot)$ in (A-1) is standard normal, the score function for location is

$$\boldsymbol{\psi}_\mu(w;\boldsymbol{\theta},\epsilon,\tau) = \frac{(w-\mu)\{(1-\epsilon)\phi(a(w))/\sigma^3 + \epsilon\phi(b(w))/(\tau^2\sigma^2+\sigma^2)^{3/2}\}}{(1-\epsilon)\phi(a(w))/\sigma + \epsilon\phi(b(w))/\sqrt{\tau^2\sigma^2+\sigma^2}},$$

where $a(w) = (w-\mu)/\sigma$ and $b(w) = (w-\mu)/\sqrt{\tau^2\sigma^2+\sigma^2}$. Divide the numerator and denominator by $\phi(b(w))$. Because

$$\lim_{|w|\to\infty}\frac{\phi(a(w))}{\phi(b(w))} = 0 \quad \text{and} \quad \lim_{|w|\to\infty}\frac{(w-\mu)\phi(a(w))}{\phi(b(w))} = 0,$$

it follows that $\boldsymbol{\psi}_\mu(w;\boldsymbol{\theta},\epsilon,\tau)$ diverges linearly with asymptotic slope $= 1/(\sigma^2+\tau^2\sigma^2)$.

The score function for scale is

$$\boldsymbol{\psi}_\sigma(w;\boldsymbol{\theta},\epsilon,\tau) \quad = \quad \frac{\{(w-\mu)^2/\sigma^4 - 1/\sigma^2\}(1-\epsilon)\phi(a(w)) + \epsilon d_1 e(w)\phi(c(w))/(\sigma^4\sqrt{\tau^2+1})}{(1-\epsilon)\phi(a(w))/\sigma + \epsilon d_1\phi(c(w))/(\sigma\sqrt{\tau^2+1})},$$

where $a(w) = (w-\mu)/\sigma$, $c(w) = (w-\mu\tau^3)/(\sigma\tau\sqrt{\tau^2+1})$, $d_1 = \exp\{\mu^2(\tau^2-1)/(2\sigma^2)\}$ and $e(w) = c(w)^2\sigma^2 + \mu^2(1-\tau^2) - \sigma^2$. Divide the numerator and denominator by $\phi(c(w))$. Because

$$\lim_{|w|\to\infty}\frac{\phi(a(w))}{\phi(c(w))} = 0 \quad \text{and} \quad \lim_{|w|\to\infty}\frac{(w-\mu)^2\phi(a(w))}{\phi(c(w))} = 0,$$

it follows that $\boldsymbol{\psi}_\sigma(w;\boldsymbol{\theta},\epsilon,\tau)$ diverges quadratically with coefficient $1/(\sigma^3\tau^2(1+\tau^2))$.

## B. Correlated-Data Model

The theoretical robustness results derived for independent and identically distributed data in Section 2.2 can be directly extended to the case where $X_1,\ldots,X_n$ are correlated. Define

$$a_n(\boldsymbol{w};\boldsymbol{\theta}) = \epsilon^n \int\cdots\int f(\boldsymbol{t};\boldsymbol{\theta})\prod_{i=1}^n\left\{\frac{1}{\tau\sigma}g\left(\frac{w_i-t_i}{\tau\sigma}\right)dt_i\right\},$$

and

$$q(\boldsymbol{w};\boldsymbol{\theta}) = h(\boldsymbol{w};\boldsymbol{\theta}) - a_n(\boldsymbol{w};\boldsymbol{\theta}),$$

where $h(\boldsymbol{w};\boldsymbol{\theta})$ is the contaminated density under the contamination model (2.1) and (2.2). With these definitions, the likelihood score function becomes

$$\begin{aligned}\psi(\boldsymbol{w};\boldsymbol{\theta},\epsilon,\tau) &= \frac{\partial}{\partial\boldsymbol{\theta}}\ln h(\boldsymbol{w};\boldsymbol{\theta}) \\ &= \frac{\dot{q}(\boldsymbol{w};\boldsymbol{\theta}) + \dot{a}_n(\boldsymbol{w};\boldsymbol{\theta})}{q(\boldsymbol{w};\boldsymbol{\theta}) + a_n(\boldsymbol{w};\boldsymbol{\theta})},\end{aligned}$$

where $\dot{q}(\boldsymbol{w};\boldsymbol{\theta}) = (\partial/\partial\boldsymbol{\theta})q(\boldsymbol{w};\boldsymbol{\theta})$ and $\dot{a}_n(\boldsymbol{w};\boldsymbol{\theta}) = (\partial/\partial\boldsymbol{\theta})a_n(\boldsymbol{w};\boldsymbol{\theta})$. Define

$$p_n(\boldsymbol{w}) = \frac{1}{\tau\sigma}\prod_{i=1}^n g\left(\frac{w_i}{\tau\sigma}\right),$$

and write

$$\psi(\boldsymbol{w};\boldsymbol{\theta},\epsilon,\tau) = \frac{\dot{q}(\boldsymbol{w};\boldsymbol{\theta})/p_n(\boldsymbol{w}) + \dot{a}_n(\boldsymbol{w};\boldsymbol{\theta})/p_n(\boldsymbol{w})}{q(\boldsymbol{w};\boldsymbol{\theta})/p_n(\boldsymbol{w}) + a_n(\boldsymbol{w};\boldsymbol{\theta})/p_n(\boldsymbol{w})}. \tag{B-1}$$

In Section B.1 we show that if $f$ is multivariate normal and $g$ has polynomial-like tails or exponential-like tails , then

$$\lim_{\min|w_i|\to\infty}\left|\frac{q(\boldsymbol{w};\boldsymbol{\theta})}{p_n(\boldsymbol{w})}\right| + \left|\frac{\dot{q}(\boldsymbol{w};\boldsymbol{\theta})}{p_n(\boldsymbol{w})}\right| = 0. \tag{B-2}$$

Therefore the behavior of the score function is determined by the tail behavior of $a_n(\boldsymbol{w};\boldsymbol{\theta})/p_n(\boldsymbol{w})$ and $\dot{a}_n(\boldsymbol{w};\boldsymbol{\theta})/p_n(\boldsymbol{w})$. Note that

$$\frac{a_n(\boldsymbol{w};\boldsymbol{\theta})}{p_n(\boldsymbol{w})} = \epsilon^n \int\cdots\int f(\boldsymbol{t};\boldsymbol{\theta}) \prod_{i=1}^{n} \left\{ \frac{g((w_i - t_i)/\tau\sigma)}{g(w_i/\tau\sigma)}\ dt_i \right\}.$$

and

$$\frac{\dot{a}_n(\boldsymbol{w};\boldsymbol{\theta})}{p_n(\boldsymbol{w})} = \frac{(\partial/\partial\boldsymbol{\theta})a_n(\boldsymbol{w};\boldsymbol{\theta})}{p_n(\boldsymbol{w})}.$$

Throughout the section we assume that limits, differentiation, and integration are interchangeable, thus the tail behavior of $a_n(\boldsymbol{w};\boldsymbol{\theta})/p_n(\boldsymbol{w})$ and $\dot{a}_n(\boldsymbol{w};\boldsymbol{\theta})/p_n(\boldsymbol{w})$ depends on the tail behavior of $\prod_{i=1}^{n} g((w_i - t_i)/\tau\sigma)/g(w_i/\tau\sigma)$.

**Polynomial-Like Tails.** For the case of polynomial-like tails,

$$\begin{aligned}
&\lim_{\min|w_i|\to\infty} \int\cdots\int f(\boldsymbol{t};\boldsymbol{\theta}) \prod_{i=1}^{n} \left\{ \frac{g((w_i - t_i)/\tau\sigma)}{g(w_i/\tau\sigma)}\ dt_i \right\} \\
&= \int\cdots\int f(\boldsymbol{t};\boldsymbol{\theta}) \prod_{i=1}^{n} \lim_{|w_i|\to\infty} \left\{ \frac{g((w_i - t_i)/\tau\sigma)}{g(w_i/\tau\sigma)}\ dt_i \right\} \\
&= \int\cdots\int f(\boldsymbol{t};\boldsymbol{\theta}) \prod_{i=1}^{n} 1\ dt_i \\
&= 1. 
\end{aligned} \tag{B-3}$$

Equations (B-1), (B-2) and (B-3) lead to

$$\begin{aligned}
&\lim_{\min|w_i|\to\infty} \psi_{\boldsymbol{\theta}_{(\sigma)}}(\boldsymbol{w};\boldsymbol{\theta},\epsilon,\tau) \\
&= \lim_{\min|w_i|\to\infty} \frac{\partial}{\partial\boldsymbol{\theta}_{(\sigma)}} \int\cdots\int f(\boldsymbol{t};\boldsymbol{\theta}) \prod_{i=1}^{n} \left\{ \frac{g((w_i - t_i)/\tau\sigma)}{g(w_i/\tau\sigma)}\ dt_i \right\},
\end{aligned}$$

where $\boldsymbol{\theta}_{(\sigma)}$ denotes all parameters except $\sigma$. In light of (B-3) and the assumption that limits, integration and differentiation are interchangeable it follows that

$$\begin{aligned}
&\lim_{\min|w_i|\to\infty} \frac{\partial}{\partial\boldsymbol{\theta}_{(\sigma)}} \int\cdots\int f(\boldsymbol{t};\boldsymbol{\theta}) \prod_{i=1}^{n} \left\{ \frac{g((w_i - t_i)/\tau\sigma)}{g(w_i/\tau\sigma)}\ dt_i \right\} \\
&= \frac{\partial}{\partial\boldsymbol{\theta}_{(\sigma)}} \lim_{\min|w_i|\to\infty} \int\cdots\int f(\boldsymbol{t};\boldsymbol{\theta}) \prod_{i=1}^{n} \left\{ \frac{g((w_i - t_i)/\tau\sigma)}{g(w_i/\tau\sigma)}\ dt_i \right\} \\
&= 0.
\end{aligned}$$

Therefore

$$\lim_{\min|w_i|\to\infty} \psi_{\boldsymbol{\theta}_{(\sigma)}}(\boldsymbol{w};\boldsymbol{\theta},\epsilon,\tau) = 0.$$

Since the gross error model (2.1) includes $\sigma$, we need to treat the $\sigma$ component of the score function differently, which is written as

$$\begin{aligned}
&\lim_{\min|w_i|\to\infty} \psi_\sigma(\boldsymbol{w};\boldsymbol{\theta},\epsilon,\tau) \\
&= \frac{\int\cdots\int(\partial/\partial\sigma)f(\boldsymbol{t};\boldsymbol{\theta})\prod_{i=1}^n\{g((w_i-t_i)/\tau\sigma)/\tau\sigma\}\ dt_i}{\prod_{i=1}^n g(w_i/\tau\sigma)/\tau\sigma}. \qquad \text{(B-4)}
\end{aligned}$$

where

$$\begin{aligned}
&\frac{\partial}{\partial\sigma}\prod_{i=1}^n\left\{\frac{g((w_i-t_i)/\tau\sigma)}{\tau\sigma}\right\} \\
&= \sum_{i=1}^n\left[\prod_{j\neq i}\left\{\frac{g((w_j-t_j)/\tau\sigma)}{\tau\sigma}\right\}\frac{\partial}{\partial\sigma}\frac{g((w_i-t_i)/\tau\sigma)}{\tau\sigma}\right].
\end{aligned}$$

In Section A we have shown that

$$\lim_{|w|\to\infty}\frac{\partial\{g((w-t)/(\tau\sigma))/(\tau\sigma)\}/\partial\sigma}{g(w/\tau\sigma)/\tau\sigma} = \frac{p-1}{\sigma},$$

with $g(w)\sim w^{-p}$. It follows that

$$\begin{aligned}
&\lim_{\min|w_i|\to\infty}\frac{(\partial/\partial\sigma)\prod_{i=1}^n\{g(w_i-t_i/\tau\sigma)/\tau\sigma\}}{\prod_{i=1}^n g(w_i/\tau\sigma)/\tau\sigma} \\
&= \sum_{i=1}^n \lim_{\min|w_i|\to\infty}\left[\prod_{j\neq i}\left\{\frac{g((w_j-t_j)/\tau\sigma)}{g(w_j/\tau\sigma)}\right\}\frac{(\partial/\partial\sigma)\prod_{i=1}^n\{g(w_i-t_i/\tau\sigma)/\tau\sigma\}}{g(w_i/\tau\sigma)/\tau\sigma}\right] \\
&= \sum_{i=1}^n\frac{p-1}{\sigma} \\
&= \frac{n(p-1)}{\sigma}.
\end{aligned}$$

After suitable interchange of limits, differentiation and integration,

$$\begin{aligned}
&\lim_{\min|u_i|\to\infty}\psi_\sigma(\boldsymbol{w};\boldsymbol{\theta},\epsilon,\tau) \\
&= \int\cdots\int f(\boldsymbol{s};\boldsymbol{\theta})\lim_{\min|w_i|\to\infty}\frac{(\partial/\partial\sigma)\prod_{i=1}^n\{g(w_i-t_i/\tau\sigma)/\tau\sigma\}}{\prod_{i=1}^n g(u_i/\tau\sigma)/\tau\sigma}\ d\boldsymbol{t} \\
&= \int\cdots\int f(\boldsymbol{t};\boldsymbol{\theta})\frac{n(p-1)}{\sigma}\ d\boldsymbol{t} \\
&= \frac{n(p-1)}{\sigma}.
\end{aligned}$$

So the Cauchy error contamination model ($p=2$) yields a redescending score function except for scale estimation, which is bounded by $n/\sigma$.

**Exponential-Like Tails.** For the case of exponential-like tails, we allow $g(w)$ to have different tails as $w \to \infty$ and $w \to -\infty$:

$$\lim_{w\to\infty} \frac{g(w+b)}{g(w)} = \exp(c_1 b), \qquad \lim_{w\to-\infty} \frac{g(w+b)}{g(w)} = \exp(c_2 b),$$

for some constants $c_1$ and $c_2$. Set $N_1$ to be any subset of $N = \{1, \ldots, n\}$, and let $N_2$ be the set difference $N - N_1$. Then

$$\begin{aligned}
&\lim_{\substack{w_k \to \infty,\, k \in N_1 \\ u_j \to -\infty,\, j \in N_2}} \int \cdots \int f(\boldsymbol{t};\boldsymbol{\theta}) \prod_{i=1}^{n} \left\{ \frac{g((w_i - t_i)/\tau\sigma)}{g(w_i/\tau\sigma)}\ dt_i \right\} \\
&= \int \cdots \int f(\boldsymbol{t};\boldsymbol{\theta}) \prod_{k\in N_1} \lim_{w_k\to\infty} \frac{g((w_k - t_k)/\tau\sigma)}{g(w_k/\tau\sigma)} \prod_{j\in N_2} \lim_{w_j\to-\infty} \frac{g((w_j - t_j)/\tau\sigma)}{g(w_j/\tau\sigma)}\ d\boldsymbol{t} \\
&= \int f(\boldsymbol{t};\boldsymbol{\theta}) \exp\left( -\frac{c_1}{\tau\sigma} \sum_{k\in N_1} t_k - \frac{c_2}{\tau\sigma} \sum_{j\in N_2} t_j \right)\ d\boldsymbol{t}. \qquad \text{(B-5)}
\end{aligned}$$

It follows that

$$\lim_{\substack{w_k \to \infty,\, k \in N_1 \\ w_j \to -\infty,\, j \in N_2}} \psi_{\boldsymbol{\theta}_{(\sigma)}}(\boldsymbol{w};\boldsymbol{\theta},\epsilon,\tau) = \frac{\partial}{\partial \boldsymbol{\theta}_{(\sigma)}} \ln\{m_f(\boldsymbol{v};\boldsymbol{\theta})\},$$

where $v_k = -c_1/\tau\sigma$ for $k \in N_1$, $v_j = -c_2/\tau\sigma$ for $j \in N_2$ and $m_f(\cdot,\boldsymbol{\theta})$ is the moment generating function of $f(\cdot,\boldsymbol{\theta})$, assumed to exist. So the error contamination model with the exponential-like tails results in a bounded score function except for $\sigma$.

In light of (B-5), showing the score function of $\sigma$ is unbounded is equivalent to showing that

$$\lim_{\substack{w_k \to \infty,\, k \in N_1 \\ w_j \to -\infty,\, j \in N_2}} \frac{\int \cdots \int (\partial/\partial\boldsymbol{\theta}) f(\boldsymbol{t};\boldsymbol{\theta}) \prod_{i=1}^{n} \{g(w_i - t_i/\tau\sigma)/\tau\sigma\}\ dt_i}{\prod_{i=1}^{n} g(w_i/\tau\sigma)/\tau\sigma},$$

is unbounded. It is direct to show that,

$$\begin{aligned}
&\lim_{\substack{w_k \to \infty,\, k \in N_1 \\ u_j \to -\infty,\, j \in N_2}} \frac{\int \cdots \int (\partial/\partial\boldsymbol{\theta}) f(\boldsymbol{t};\boldsymbol{\theta}) \prod_{i=1}^{n} \{g(w_i - t_i/\tau\sigma)/\tau\sigma\}\ dt_i}{\prod_{i=1}^{n} g(w_i/\tau\sigma)/\tau\sigma} \\
&= \lim_{\substack{w_k \to \infty,\, k \in N_1 \\ w_j \to -\infty,\, j \in N_2}} \int \cdots \int f(\boldsymbol{t};\boldsymbol{\theta}) \sum_{i=1}^{n} b_i(\boldsymbol{w};\boldsymbol{\theta})\ dt_i
\end{aligned}$$

where

$$b_i(\boldsymbol{w};\boldsymbol{\theta}) = \prod_{j\neq i} \left\{ \frac{g((w_j - t_j)/\tau\sigma)}{g(w_j/\tau\sigma)} \right\} \frac{(\partial/\partial\sigma)\{g((w_i - t_i)/\tau\sigma)/\tau\sigma\}}{g(w_i/\tau\sigma)/\tau\sigma}.$$

In Section B we have shown that

$$\lim_{w\to\pm\infty} \frac{\partial\{g((w-t)/(\tau\sigma))/(\tau\sigma)\}/\partial\sigma}{g(w/(\tau\sigma))/(\tau\sigma)} = \infty.$$

It follows that

$$\lim_{\substack{w_k \to \infty,\, k \in N_1 \\ w_j \to -\infty,\, j \in N_2}} \frac{\int\cdots\int(\partial/\partial\boldsymbol{\theta})f(\boldsymbol{t};\boldsymbol{\theta})\prod_{i=1}^{n}\{g(w_i-t_i/\tau\sigma)/\tau\sigma\}\ dt_i}{\prod_{i=1}^{n} g(w_i/\tau\sigma)/\tau\sigma} = \infty.$$

### B1. Proof for Equation (B-2)

The contaminated density $h(\boldsymbol{w};\boldsymbol{\theta})$ can be written as

$$\begin{aligned} h(\boldsymbol{w};\boldsymbol{\theta}) &= \int\cdots\int f(w_1-\tau\sigma z_1,\ldots,w_n-\tau\sigma z_n;\boldsymbol{\theta})\prod_{s=1}^{n} G_\epsilon(dz_s) \\ &= \sum_{j=0}^{n}(1-\epsilon)^{n-j}\epsilon^j c_j(\boldsymbol{w};\boldsymbol{\theta}), \end{aligned}$$

where

$$\begin{aligned} c_0(\boldsymbol{w};\boldsymbol{\theta}) &= f(\boldsymbol{w};\boldsymbol{\theta}), \\ c_1(\boldsymbol{w};\boldsymbol{\theta}) &= \sum_{t=1}^{n}\int f(w_1,\ldots,w_{s-1},u_s-\tau\sigma z_s,u_{s+1},\ldots,u_n;\boldsymbol{\theta})g(z_s)\ dz_s, \end{aligned}$$

and so on up to

$$c_n(\boldsymbol{w};\boldsymbol{\theta}) = \sum_{t=1}^{n}\int\cdots\int f(w_1-\tau\sigma z_1,\ldots,w_s,\ldots,w_n-\tau\sigma z_n;\boldsymbol{\theta})\prod_{j\neq s}\{g(z_j)\ dz_j\}.$$

Then

$$\begin{aligned} q(\boldsymbol{w};\boldsymbol{\theta}) &= h(\boldsymbol{w};\boldsymbol{\theta}) - a_n(\boldsymbol{w};\boldsymbol{\theta}) \\ &= \sum_{j=0}^{n-1}(1-\epsilon)^{n-j}\epsilon^j c_j(\boldsymbol{w};\boldsymbol{\theta}). \end{aligned}$$

It follows that showing (B-2) is equivalent to showing that

$$\lim_{\min|w_t|\to\infty}\left|\frac{c_j(\boldsymbol{w};\boldsymbol{\theta})}{p_n(\boldsymbol{w})}\right| + \left|\frac{\dot{c}_j(\boldsymbol{w};\boldsymbol{\theta})}{p_n(\boldsymbol{w})}\right| = 0, \quad j=0,1,\ldots,n-1.$$

Assume that $f \sim \mathrm{MVN}(\boldsymbol{\mu},\Omega)$, where $\boldsymbol{\mu} = (\mu,\ldots,\mu)^T$ , then

$$f(\boldsymbol{w};\boldsymbol{\theta}) = \frac{1}{(2\pi)^{n/2}}\frac{1}{|\Omega|^{1/2}}\exp\left\{-\frac{(\boldsymbol{w}-\boldsymbol{\mu})^T\Omega^{-1}(\boldsymbol{w}-\boldsymbol{\mu})}{2}\right\}.$$

Since $\Omega^{-1}$ is positive definite,

$$(\boldsymbol{w}-\boldsymbol{\mu})^T\Omega^{-1}(\boldsymbol{w}-\boldsymbol{\mu}) \geq \lambda_{(1)}||\boldsymbol{w}-\boldsymbol{\mu}||^2,$$

where $\lambda_{(1)}$ is the smallest eigenvalue of $\Omega^{-1}$, and $||\boldsymbol{w}-\boldsymbol{\mu}||^2 = \sum_{s=1}^n (w_s-\mu)^2$. In the case that $g$ has polynomial-like tails or exponential-like tails,

$$\lim_{|w_s|\to\infty} \left|\frac{\exp\{-\lambda_{(1)}(w_s-\mu)^2/2\}}{g(w_s/\tau\sigma)}\right| + \left|\frac{(\partial/\partial\boldsymbol{\theta})\exp\{-\lambda_{(1)}(w_s-\mu)^2/2\}}{g(w_s/\tau\sigma)}\right| = 0, \quad s=1,\dots,n.$$

Therefore

$$\lim_{\min|w_s|\to\infty} \left|\frac{c_0(\boldsymbol{w};\boldsymbol{\theta})}{p_n(\boldsymbol{w})}\right| + \left|\frac{\dot{c}_0(\boldsymbol{w};\boldsymbol{\theta})}{p_n(\boldsymbol{w})}\right| = 0.$$

Ignoring the constant,

$$\begin{aligned} c_1(\boldsymbol{w};\boldsymbol{\theta}) \;\; &\leq \;\; \sum_{s=1}^n \prod_{j\neq t} \exp\left\{-\frac{1}{2}\lambda_{(1)}(w_j-\mu)^2\right\} \\ &\times \;\; \int \exp\left\{-\frac{1}{2}\lambda_{(1)}(w_s-\tau\sigma z_s-\mu)^2\right\} g(z_s)\; dz_s. \end{aligned} \tag{B-6}$$

Let $v_s = w_s-\mu-\tau\sigma z_s$, then

$$\begin{aligned} &\lim_{|w_s|\to\infty} \frac{\int \exp\left\{-\lambda_{(1)}(w_s-\tau\sigma z_s-\mu)^2/2\right\} g(z_s)\; dz_s}{g(z_s)/\tau\sigma)/\tau\sigma} \\ &= \lim_{|w_s|\to\infty} \frac{\int \exp\left\{-\lambda_{(1)}v_s^2/2\right\} g((w_s-\mu-v_s)/\tau\sigma)/\tau\sigma\; dz_s}{g(z_s)/\tau\sigma)/\tau\sigma} \\ &= \lim_{|w_s|\to\infty} \int \exp\left\{-\lambda_{(1)}v_s^2/2\right\} \frac{g((w_s-\mu-v_s)/\tau\sigma)}{g(w_s)/\tau\sigma)}\; dv_s \\ &= \int \exp\left\{-\lambda_{(1)}v_s^2/2\right\} \lim_{|w_s|\to\infty} \frac{g((w_s-\mu-v_s)/\tau\sigma)}{g(w_s)/\tau\sigma)}\; dv_s, \end{aligned}$$

which is a constant when $g$ has polynomial-like tails or exponential-like tails. It follows that

$$\lim_{\min|w_s|\to\infty} \left|\frac{c_1(\boldsymbol{w};\boldsymbol{\theta})}{p_n(\boldsymbol{w})}\right| + \left|\frac{\dot{c}_1(\boldsymbol{w};\boldsymbol{\theta})}{p_n(\boldsymbol{w})}\right| = 0.$$

For $j = 2,\dots,n-1$, a similar proof works. The only difference is that the term in the right hand side of (B-6) is more complicated. For example, when $j=2$, (B-6) is replaced by

$$
\begin{aligned}
&c_2(\boldsymbol{w};\boldsymbol{\theta})\\
&\leq \sum_{t=1}^{n}\sum_{s=1}^{n}\prod_{j\neq s,t}\exp\left\{-\frac{1}{2}\lambda_{(1)}(w_j-\mu)^2\right\}\\
&\times \int\int\exp\left\{-\frac{1}{2}\lambda_{(1)}(w_s-\tau\sigma z_s-\mu)^2\right\}\left\{-\frac{1}{2}\lambda_{(1)}(w_t-\tau\sigma z_t-\mu)^2\right\}\prod_{j=t,s}\{g(z_j)\ dz_j\}.
\end{aligned}
$$