OPTIMAL BOUNDS FOR INVERSE PROBLEMS WITH JACOBI-TYPE EIGENFUNCTIONS

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Supplementary Material

This note contains in section S1 the proof for the main result given in Theorem 2, and in section S2 the proof for two preliminary results on needlets, i.e. Lemma 1 and Theorem 7 in the main paper.

S1. Proof of the main result

We recall the three conditions required on the families of functions $f_0, \dots, f_m$:

- ***Condition (i):*** for all $i \in \{0, 1, \dots, m\}$, $f_i \in B^s_{\pi,r}(M)$,
- ***Condition (ii):*** for all $i \neq j$, $\|f_i - f_j\|_p^p \geq 2\delta$ for some $\delta > 0$,
- ***Condition (iii'):*** for all $i \in \{1, \dots, m\}$, $P_{f_i} \ll P_{f_0}$ and $\frac{1}{m}\sum_{i\geq 1}\mathcal{K}(P_{f_i}, P_{f_0}) \leq \theta \log(M+1)$, where $0 < \theta < \frac{1}{8}$ and P_f denotes the probability distribution of the process Y under the hypothesis f.

Consider Condition (iii'). Let $I = [a, b]$ (the case $I = [a, b[$ is similar). If we define the variables $\widetilde{Y}(w) = Y(w(. - a)/\sqrt{\lambda(.)})$ and $\widetilde{\xi}(w) = \xi(w(. - a)/\sqrt{\lambda(.)})$ for all $w \in \widetilde{V} = \mathbb{L}^2([0, b - a], dx)$ then Model (2.1) in the main text is equivalent to: $\widetilde{Y}(w) = (Kf(. + a)\sqrt{\lambda(. + a)}, w)_{\widetilde{V}} + \epsilon\widetilde{\xi}(w)$, which is equivalent to the stochastic equation: $\forall t \in [0, b - a]$, $d\widetilde{Y}_t = Kf(t + a)\sqrt{\lambda(t + a)}dt + \epsilon dW_t$ where $(W_t)_{t\geq 0}$ denotes the standard Wiener process. Then using Girsanov's formula, for all $f, g \in U$, P_f is absolutely continuous with respect to P_g, and under the hypothesis g the likelihood ratio $\Lambda_\epsilon(f, g) := \frac{dP_f}{dP_g}(Y)$ is distributed as: $\log \Lambda_\epsilon(f, g) \sim \mathcal{N}(-\frac{1}{2}\|\frac{K(f-g)}{\epsilon}\|_V^2, \|\frac{K(f-g)}{\epsilon}\|_V)$. Thus

$$\mathcal{K}(P_f, P_g) = E_f \ln(\Lambda_\epsilon(f, g)) = -E_f \log(\Lambda_\epsilon(g, f)) = \frac{1}{2}\|\frac{K(f - g)}{\epsilon}\|_V^2.$$

Then Condition (iii') can be replaced by

Condition (iii): $f_0 = 0$ and for all $i \in \{1, \ldots, m\}$, $\|Kf_i\|_V^2 \leq \theta \log(M+1)\epsilon^2$ where $0 < \theta < \frac{1}{4}$.

Sparse cases

Condition (i) is satisfied if $u_j := 2^{js}(\sum_{\eta \in \mathbb{Z}_j} |\langle f_1, \psi_{j,\eta}\rangle|^\pi \|\psi_{j,\eta}\|_\pi^\pi)^{1/\pi}$ belongs to $l^r(M)$, where $f_1 = \gamma\psi_{j_0,\eta_1}$. Using the first part of Lemma 1, $u_j = 0$ whenever $|j - j_0| \geq 2$. So in the sequel we assume that $j \in \{j_0 - 1, j_0, j_0 + 1\}$, and the l^r norm of (u_j) is bounded by a constant M (independent of $\gamma > 0$ and j_0), if for instance $u_j \leq 3^{-\frac{1}{r}}M$. We have $u_j^\pi = 2^{j\pi s}\gamma^\pi \sum_{\eta \in \mathbb{Z}_j} |\langle \psi_{j_0,\eta_1}, \psi_{j,\eta}\rangle|^\pi \|\psi_{j,\eta}\|_\pi^\pi \leq c(I_1 + I_2)$, with, using the bound of Theorem 6

$$I_1 = 2^{j[\pi s + (\pi-2)(\alpha+1)]}\gamma^\pi \sum_{k=1}^{2^{j-1}} |\langle \psi_{j_0,\eta_1}, \psi_{j,\eta}\rangle|^\pi k^{-(\pi-2)(\alpha+1/2)},$$

$$I_2 = 2^{j[\pi s + (\pi-2)(\beta+1)]}\gamma^\pi \sum_{k=2^{j-1}+1}^{2^j} |\langle \psi_{j_0,\eta_1}, \psi_{j,\eta}\rangle|^\pi (2^j - k + 1)^{-(\pi-2)(\beta+1/2)}.$$

Using the second part of Lemma 1, we have for any ζ, $|\langle \psi_{j_0,\eta_1}, \psi_{j,\eta_k}\rangle| \leq c\frac{1}{k^\zeta}$. Thus choosing any $\zeta > \frac{-(\pi-2)(\alpha+1/2)+1}{\pi}$, we obtain $I_1 \leq c2^{j[\pi s + (\pi-2)(\alpha+1)]}\gamma^\pi$. Moreover $\sum_{k=1}^{2^{j-1}} \frac{(2^j-k+1)^{-(\pi-2)(\beta+1/2)}}{k^{\zeta\pi}} \leq c2^{-\zeta\pi j}2^{j[1-(\pi-2)(\beta+1/2)]_+}$, so for a large enough ζ, $I_2 \leq c2^{j(\pi s + (\pi-2)(\beta+1) - \zeta\pi + [1-(\pi-2)(\beta+1/2)]_+)}\gamma^\pi \leq cI_1$. Thus, we have, for all $j \in \{j_0 - 1, j_0, j_0 + 1\}$, $u_j^\pi \leq c2^{j_0[\pi s + (\pi-2)(\alpha+1)]}\gamma^\pi$, and condition (i) is satisfied if, for a small enough c depending on M,

$$\gamma \leq c2^{-j_0[s + (1-\frac{2}{\pi})(\alpha+1)]}.$$

Condition (ii), using Theorem 6, is fulfilled with $\delta \asymp \gamma^p 2^{j_0(p-2)(\alpha+1)}$.

Condition (iii) is satisfied if $\int_I (\frac{K(\gamma\psi_{j_0,\eta_1})(t)}{\epsilon})^2 d\lambda(t) \leq C$. We have $\psi_{j_0,\eta}(x) = \sum_{l=2^{j-2}+1}^{2^j-1} c_{j,\eta,l}P_l(x)$ and $K^*KP_l = b_l^2 P_l$, thus

$$\|K(\psi_{j_0,\eta_1})\|_V^2 = \sum_l [b_l c_{j,\eta,l}]^2 \asymp 2^{-2\nu j_0} \sum_l [c_{j,\eta,l}]^2 = 2^{-2\nu j_0}\|\psi_{j_0,\eta_1}\|_U^2 \leq C2^{-2\nu j_0}.$$

Condition (iii) is then satisfied if $\frac{\gamma 2^{-\nu j_0}}{\epsilon} \leq c$.

Regular case

Condition (i): for $\varepsilon \in E_{j_0}$, let $u_j := 2^{js}(\sum_{\eta\in\mathbb{Z}_j} |\langle f_\varepsilon, \psi_{j,\eta}\rangle|^\pi \|\psi_{j,\eta}\|_\pi^\pi)^{1/\pi}$. Once again $u_j = 0$ whenever $|j - j_0| \geq 2$. Now let $j \in \{j_0 - 1, j_0, j_0 + 1\}$. Then we have $u_j^\pi \leq c(I_1 + I_2)$, with

$$I_1 = \quad 2^{j[\pi s+(\pi-2)(\alpha+1)]}\gamma^\pi \sum_{k=1}^{2^{j-1}} k^{-(\pi-2)(\alpha+1/2)}(\sum_{l=1}^{2^{j_0-1}} l^\delta |\langle \psi_{j_0,\eta_l}, \psi_{j,\eta_k}\rangle|)^\pi,$$

$$I_2 = \quad 2^{j[\pi s+(\pi-2)(\beta+1)]}\gamma^\pi \sum_{k=2^{j-1}+1}^{2^j} (2^j - k + 1)^{-(\pi-2)(\beta+1/2)}(\sum_{l=1}^{2^{j_0-1}} l^\delta |\langle \psi_{j_0,\eta_l}, \psi_{j,\eta_k}\rangle|)^\pi.$$

Using Lemma 1 with some ζ given later, we have $|\langle \psi_{j_0,\eta_l}, \psi_{j,\eta_k}\rangle| \leq c\frac{1}{(1+|l-2^{j_0-j}k|)^\zeta}$. Then, for $x \in \mathbb{R}$, let $\lfloor x\rfloor$ denote the largest integer smaller than x. We have

$$\sum_{l\leq\lfloor 2^{j_0-j}k\rfloor} \frac{l^\delta}{(1+|l-2^{j_0-j}k|)^\zeta} \leq ck^\delta \sum_{l\leq\lfloor 2^{j_0-j}k\rfloor} \frac{1}{(1+\lfloor 2^{j_0-j}k\rfloor - l)^\zeta} \leq ck^\delta \sum_{l\geq 1} \frac{1}{l^\zeta} \leq ck^\delta,$$

for ζ large enough. Moreover

$$\begin{aligned}\sum_{l\geq\lfloor 2^{j_0-j}k\rfloor+1} \frac{l^\delta}{(1+|l-2^{j_0-j}k|)^\zeta} \quad &\leq \sum_{l\geq\lfloor 2^{j_0-j}k\rfloor+1} \frac{l^\delta}{(l-\lfloor 2^{j_0-j}k\rfloor)^\zeta} = \sum_{l\geq 1} \frac{(l+\lfloor 2^{j_0-j}K\rfloor)^\delta}{l^\zeta} \\ &\leq c\sum_{l\geq 1} \frac{l^\delta + \lfloor 2^{j_0-j}k\rfloor^\delta}{l^\zeta} \leq Ck^\delta,\end{aligned}$$

for ζ large enough. To obtain the last line, we used the fact that $\delta \geq 1$. Thus $\sum_{l=1}^{2^{j_0-1}} \frac{l^\delta}{(1+|l-2^{j_0-j}k|)^\zeta} \leq ck^\delta$, and

$$I_1 \leq c2^{j[\pi s+(\pi-2)(\alpha+1)]}\gamma^\pi \sum_{k=1}^{2^{j-1}} k^{-(\pi-2)(\alpha+1/2)}k^{\delta\pi} = c2^{j[s+\delta+\frac{1}{2}]}\gamma.$$

For I_2, remark that for any $k \in \{2^{j-1}+1, \dots, 2^j\}$ and any $l \in \{1, \dots, 2^{j_0-1}\}$, we have $|\frac{k}{2^j} - \frac{l}{2^{j_0}}| = \frac{k}{2^j} - \frac{l}{2^{j_0}} \geq |\frac{2^j-k}{2^j} - \frac{l}{2^{j_0}}|$. So for such a k, as previously, $\sum_{l=1}^{2^{j_0-1}} \frac{l^\delta}{(1+|l-2^{j_0-j}k|)^\zeta} \leq \sum_{l=1}^{2^{j_0-1}} \frac{l^\delta}{(1+|l-2^{j_0-j}(2^j-k)|)^\zeta} \leq c(2^j-k)^\delta$, and

$$I_2 \leq c2^{j[\pi s+(\pi-2)(\beta+1)]}\gamma^\pi \sum_{k=2^{j-1}+1}^{2^j} (2^j-k+1)^{-(\pi-2)(\beta+1/2)}(2^j-k+1)^{\delta\pi} = c2^{j[s+\delta+\frac{1}{2}]}\gamma.$$

Finally we have $u_j \leq c2^{j[s+\delta+\frac{1}{2}]}\gamma$ so f_ε belongs to $B^s_{\pi,r}(M)$ if, for a small enough c depending on M,

$$\gamma \leq c2^{-j_0[s+\delta+\frac{1}{2}]}.$$

Condition (ii): for all $\varepsilon^u, \varepsilon^v \in E_{j_0}$ with $u \neq v$, $f_u - f_v = \sum_{k=1}^{2^{j_0-m-1}} \gamma(\varepsilon^u_k - \varepsilon^v_k)k^\delta \psi_{j_0,\eta_{2^m k}}$. So by Theorems 7 and 6, we have

$$\|f_u - f_v\|^2_U \geq c\gamma^2 \sum_{k=1}^{2^{j_0-m-1}} (\varepsilon^u_k - \varepsilon^v_k)^2 k^{2\delta} = c\gamma^2 \sum_{\{k \mid \varepsilon^u_k \neq \varepsilon^v_k\}} k^{2\delta}.$$

Let $N_{u,v}$ denote the cardinality of the set $\{k \in \{1, \ldots, 2^{j_0-m-1}\} \mid \varepsilon^u_k \neq \varepsilon^v_k\}$, then we have $N_{u,v} \geq c2^{j_0}$ and, since $\delta > 0$,

$$\|f_u - f_v\|^2_U \geq c\gamma^2 \sum_{k=1}^{N_{u,v}} k^{2\delta} = \gamma^2 {N_{u,v}}^{1+2\delta} \geq c\gamma^2 2^{j_0(1+2\delta)}. \tag{1.1}$$

Let us distinguish two cases. *Suppose* $2 < p < \infty$ and let $1/p + 1/q = 1$. By (1.1), and Hölder's inequality, we have

$$c2^{j_0(1+2\delta)} \leq \|f_u - f_v\|^2_{\mathbb{L}^2(\mu)} \leq \|f_u - f_v\|_{\mathbb{L}^p(\mu)} \|f_u - f_v\|_{\mathbb{L}^q(\mu)}.$$

Using Theorem 5, and the fact that, under our assumptions, $q\delta - (q-2)(\alpha + 1/2) > -1$, we have

$$\|f_u - f_v\|_{\mathbb{L}^q(\mu)} \leq c\gamma 2^{j\frac{(q-2)}{q}(\alpha+1)} (\sum_{k=1}^{2^{j_0-m-1}} k^{q\delta-(q-2)(\alpha+1/2)})^{1/q} \leq c'\gamma 2^{j_0(\frac{1}{2}+\delta)},$$

therefore $\|f_u - f_v\|^p_{\mathbb{L}^p(\mu)} \geq c\gamma^p 2^{j_0 p(\frac{1}{2}+\delta)}$.

Suppose now $1 < p < 2$, we have, using (1.1),

$$c2^{j_0(1+2\delta)} \leq \|f_u - f_v\|^2_{\mathbb{L}^2(\mu)} \leq \|f_u - f_v\|^p_{\mathbb{L}^p(\mu)} \|f_u - f_v\|^{2-p}_{\mathbb{L}^\infty(\mu)}.$$

From Theorem 4, we infer, for all $0 \leq \theta \leq \pi/2$,

$$|\psi_{j_0,\eta_k}(\cos\theta)| \leq C \frac{2^{j_0(1+\alpha)}}{(1 + 2^{j_0}|\theta - \frac{k\pi}{2^{j_0}}|)^l} \frac{1}{(2^{j_0}\theta + 1)^{\alpha+1/2}}.$$

Thus, for l large enough, $|\psi_{j_0,\eta_k}(\cos\theta)| \leq C \frac{2^{j_0(1+\alpha)}}{k^{\alpha+1/2}} \frac{1}{(1+2^{j_0}|\theta - \frac{k\pi}{2^{j_0}}|)^2}$. Moreover, since $\delta - (\alpha + 1/2) \geq 0$,

$$|f_u(\cos\theta) - f_v(\cos\theta)| \leq c\gamma 2^{j_0(\alpha+1)} \sum_{k=1}^{2^{j_0-m-1}} k^{\delta-(\alpha+1/2)} \frac{1}{(1 + 2^{j_0}|\theta - \frac{k\pi}{2^{j_0}}|)^2} \leq c'\gamma 2^{j_0(\frac{1}{2}+\delta)},$$

where in the last line we used the fact that for any θ, $\sum_{k=1}^{2^{j_0-m-1}} \frac{1}{(1+2^{j_0}|\theta-\frac{k\pi}{2^{j_0}}|)^2} \leq c\sum_{l=1}^{+\infty}\frac{1}{l^2}$. Similarly the same bound holds for any $\pi/2 \leq \theta \leq \pi$, thus, we have $\|f_u - f_v\|_{\mathbb{L}^\infty(\mu)} \leq c2^{j_0(\frac{1}{2}+\delta)}$, and, once again, $\|f_u - f_v\|^p_{\mathbb{L}^p(\mu)} \geq c\gamma^p 2^{j_0 p(\frac{1}{2}+\delta)}$.

Condition (iii): we have $\sqrt{T_{j_0}} \geq \exp(\frac{\rho}{2}2^{j_0})$, so Condition (iii) is satisfied if for all $\varepsilon^u \in E_{j_0}$, $\int_I(\frac{K(f_u)(t)}{\epsilon})^2 d\lambda(t) \leq c2^{j_0}$ for a small enough constant c. We have $f_u = \sum_{k=1}^{2^{j_0-m-1}} \beta_{j_0,k}\psi_{j_0,\eta_{2^m k}} = \sum_{k=1}^{2^{j_0-m-1}} \sum_{l\in\mathbb{N}} \beta_{j_0,k}c_{j_0,\eta_k,l}P_l(x)$, with $\beta_{j_0,k} = \gamma\varepsilon^u_k k^\delta$. Thus,

$$\begin{aligned}\|K(f_u)\|^2_{\mathbb{L}_2(I,\lambda)} &= \sum_l [\sum_{k=1}^{2^{j_0-m-1}} \beta_{j_0,k}b_l c_{j_0,\eta_k,l}]^2 \asymp 2^{-2\nu j_0}\sum_l[\sum_{k=1}^{2^{j_0-m-1}} \beta_{j_0,k}c_{j_0,\eta_k,l}]^2\\ &= 2^{-2\nu j_0}\|\sum_{k=1}^{2^{j_0-m-1}} \beta_{j_0,k}\psi_{j_0,\eta_{2^m k}}\|^2_{\mathbb{L}_2(I,\mu)} \leq c2^{-2\nu j_0}\sum_{k=1}^{2^{j_0-m-1}} \beta^2_{j_0,k}\\ &\leq c2^{-2\nu j_0}\gamma^2 \sum_{k=1}^{2^{j_0-m-1}} k^{2\delta} = c2^{-2\nu j_0}\gamma^2 2^{(2\delta+1)j_0}.\end{aligned}$$

So finally we need $\frac{2^{-\nu j_0}\gamma 2^{(\delta+\frac{1}{2})j_0}}{\epsilon} \leq C2^{j_0/2}$, i.e. $\frac{2^{(\delta-\nu)j_0}\gamma}{\epsilon} \leq C$ with a small enough constant C.

S2. Proof of two new properties of needlets

Lemma 1. *We have*

1. $\forall j, j', k, l$ *such that* $|j'-j| \geq 2$, $\langle\psi_{j,\eta_k}, \psi_{j',\eta_l}\rangle = 0$,
2. $\forall\zeta > 0$, $\exists c_\zeta$ *such that* $\forall j, j', k, l$ *with* $|j'-j| \leq 1$, $|\langle\psi_{j,\eta_k}, \psi_{j',\eta_l}\rangle| \leq \frac{c_\zeta}{(1+|k-2^{j-j'}l|)^\zeta}$.

Proof of Lemma 1. The needlets are $\psi_{j,\eta} = \sum_{l=2^{j-2}+1}^{2^j-1} c_{j,\eta,l}P_l(x)$, with coefficients $c_{j,\eta,l} = a(l/2^{j-1})P_l(\eta)\sqrt{b_{j,\eta}}$. So if $|j'-j| \geq 2$ then $\{2^{j-2}+1,\dots,2^j-1\} \cap \{2^{j'-2}+1,\dots,2^{j'}-1\} = \emptyset$, and $\langle\psi_{j,\eta_k}, \psi_{j',\eta_l}\rangle = 0, \quad \forall(k,l)$.

For the second part of the lemma we use Theorem 4. For any δ, there exists c_δ such that, for all j, k,

$$|\psi_{j,\eta_k}(\cos\theta)| \leq c_\delta \frac{1}{\sqrt{\omega_{\alpha,\beta}(2^j,\cos\theta)}}\frac{2^{j/2}}{(1+2^j|\theta-\frac{\pi k}{2^j}|)^\delta}, \quad 0 \leq \theta \leq \pi.$$

We recall that $\omega_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$, and $\omega_{\alpha,\beta}(2^j;x) = (1-x+2^{-2j})^{\alpha+1/2}(1+x+2^{-2j})^{\beta+1/2}$. For a given $\zeta > 0$ and j, j', k, l such that $|j'-j| \leq 1$, we use this inequality for $|\psi_{j,\eta_k}|$ with $\delta = \zeta + 2$, and for $|\psi_{j',\eta_l}|$ with $\delta = \zeta$. Noticing that $\omega_{\alpha,\beta}(2^j, \cos\theta) \asymp \omega_{\alpha,\beta}(2^{j'}, \cos\theta)$ we obtain

$$|\langle \psi_{j,\eta_k}, \psi_{j',\eta_l}\rangle| \leq c2^j \int_0^\pi \frac{\omega_{\alpha,\beta}(\cos\theta)}{\omega_{\alpha,\beta}(2^j, \cos\theta)} \frac{\sin\theta d\theta}{(1+2^j|\theta - \frac{\pi k}{2^j}|)^{\zeta+2}(1+2^{j'}|\theta - \frac{\pi l}{2^{j'}}|)^{\zeta}}$$
$$\leq c \frac{I_{j,k,\alpha,\beta}}{(\min_{0\leq\theta\leq\pi} f_{j,j',k,l}(\theta))^{\zeta}},$$

with $f_{j,j',k,l}(\theta) = (1+2^j|\theta - \frac{\pi k}{2^j}|)(1+2^{j'}|\theta - \frac{\pi l}{2^{j'}}|)$, $0 \leq \theta \leq \pi$, and $I_{j,k,\alpha,\beta} = 2^j \int_0^\pi \frac{\omega_{\alpha,\beta}(\cos\theta)}{\omega_{\alpha,\beta}(2^j,\cos\theta)} \frac{\sin\theta d\theta}{(1+2^j|\theta - \frac{\pi k}{2^j}|)^2}$.

First, we have $\min_{0\leq\theta\leq\pi} f_{j,j',k,l}(\theta) = \min\{f_{j,j',k,l}(\frac{\pi k}{2^j}), f_{j,j',k,l}(\frac{\pi l}{2^{j'}})\} \geq 1 + \frac{\pi}{2^{|j-j'|}}|k - 2^{j-j'}l| \geq c(1+|k-2^{j-j'}l|)$. Second, let us divide $I_{j,k,\alpha,\beta}$ into two terms: $I_{j,k,\alpha,\beta} = I^1_{j,k,\alpha,\beta} + I^2_{j,k,\alpha,\beta}$, with

$$I^1_{j,k,\alpha,\beta} = 2^j \int_0^{\frac{\pi}{2}} \frac{\omega_{\alpha,\beta}(\cos\theta)}{\omega_{\alpha,\beta}(2^j, \cos\theta)} \frac{\sin\theta d\theta}{(1+2^j|\theta - \frac{\pi k}{2^j}|)^2},$$
$$I^2_{j,k,\alpha,\beta} = 2^j \int_{\frac{\pi}{2}}^{\pi} \frac{\omega_{\alpha,\beta}(\cos\theta)}{\omega_{\alpha,\beta}(2^j, \cos\theta)} \frac{\sin\theta d\theta}{(1+2^j|\theta - \frac{\pi k}{2^j}|)^2}$$
$$= 2^j \int_0^{\frac{\pi}{2}} \frac{\omega_{\alpha,\beta}(-\cos\theta)}{\omega_{\alpha,\beta}(2^j, -\cos\theta)} \frac{\sin\theta d\theta}{(1+2^j|\pi - \theta - \frac{\pi k}{2^j}|)^2}$$
$$= 2^j \int_0^{\frac{\pi}{2}} \frac{\omega_{\beta,\alpha}(\cos\theta)}{\omega_{\beta,\alpha}(2^j, \cos\theta)} \frac{\sin\theta d\theta}{(1+2^j|\theta - \frac{\pi(2^j-k)}{2^j}|)^2}$$
$$= I^1_{j,2^j-k,\beta,\alpha}.$$

We have $\sin\theta\omega_{\alpha,\beta}(\cos\theta) = \sin\theta(2\sin^2(\theta/2))^{\alpha}(2\cos^2(\theta/2))^{\beta} \leq c_1\theta^{2\alpha+1}$, for all $0 \leq \theta \leq \frac{\pi}{2}$, and

$$\omega_{\alpha,\beta}(2^j; \cos\theta) = (2\sin^2(\theta/2) + 2^{-2j})^{\alpha+1/2}(2\cos^2(\theta/2) + 2^{-2j})^{\beta+1/2} \geq c_2\theta^{2\alpha+1}.$$

Thus, $I^1_{j,k,\alpha,\beta} \leq c2^j \int_0^{\frac{\pi}{2}} \frac{d\theta}{(1+2^j|\theta - \frac{\pi k}{2^j}|)^2} \leq c \int_0^{\frac{\pi 2^j}{2}} \frac{d\theta}{(1+|\theta - \pi k|)^2} \leq C$, since $\int_{-\infty}^{+\infty} \frac{d\theta}{(1+\theta)^2}$ is finite, and the same goes for $I^2_{j,k,\alpha,\beta}$.

Thus, there exists $C(\alpha,\beta) > 0$ such that, for all (j,k), $I_{j,k,\alpha,\beta} \leq C(\alpha,\beta)$, which completes the proof of the lemma. □

Theorem 7. *If* $p \in 2\mathbb{N}^*$*, there exist a constant* $c_p > 0$ *and an integer* n_p *such that, for any collection of numbers* $\{\lambda_k : k \in I_j\}$*,* $j \geq 0$*, where* $I_j \subset \{1, 2, \ldots, 2^j\}$ *and* $k, l \in I_j, k \neq l \Longrightarrow |k - l| \geq n_p$*,*

$$\| \sum_{k \in I_j} \lambda_k \psi_{j,\eta_k} \|^p_{\mathbb{L}^p(\mu)} \geq c_p \sum_{k \in I_j} |\lambda_k|^p \|\psi_{j,\eta_k}\|^p_{\mathbb{L}^p(\mu)}.$$

Proof of Theorem 7. Let $p \in 2\mathbb{N}^*$ and $I_j \subset \{1, 2, \ldots, 2^j\}$. We have the decomposition $\|(\sum_{k \in I_j} \lambda_k \psi_{j,\eta_k})\|^p_{\mathbb{L}^p(\mu)} = A + B$, where

$$A = \sum_{k \in I_j} \lambda_k^p \|\psi_{j,\eta_k}\|^p_{\mathbb{L}^p(\mu)},$$

$$B = \sum_{(p_k)_{k \in I_j} \in \Lambda} \frac{p! \prod_{k \in I_j} \lambda_k^{p_k}}{\prod_{k \in I_j} p_k!} \int_{-1}^{1} (\prod_{k \in I_j} \psi^{p_k}_{j,\eta_k}(x)) \mu(x) dx,$$

and $\Lambda = \{(p_k)_{k \in I_j} \;\mid\; p_k \in \mathbb{N}, \; \sum_{k \in I_j} p_k = p$ and $\exists u \neq v$ such that $p_u > 0$ and $p_v > 0\}$.

Let $\varphi_{j,k}(x) = \frac{1}{\sqrt{\omega_{\alpha,\beta}(2^j, x)}} \frac{2^{j/2}}{(1 + 2^j|\arccos x - \frac{\pi k}{2^j}|)^{\frac{2}{s}}}$ for some $0 < s < \min\{1, \frac{p}{\alpha \vee \beta + 1}\}$. For $(p_k)_{k \in I_j} \in \Lambda$, we use Theorem 4 with $l = \frac{2}{s} + 1$ for every ψ_{j,η_k}, $k \in I_j$. There exists C such that

$$\prod_{k \in I_j} |\psi_{j,\eta_k}(\cos\theta)|^{p_k} \leq C \prod_{k \in I_j} \varphi_{j,k}(\cos\theta)^{p_k} \prod_{k \in I_j} \frac{1}{(1 + 2^j|\theta - \frac{\pi k}{2^j}|)^{p_k}}.$$

Let $u, v \in I_j, u \neq v$ such that $p_u > 0$ and $p_v > 0$, and let $n_{inf} = \min_{k,l \in I_j, k \neq l} |k - l|$. We have

$$\prod_{k \in I_j} (1 + 2^j|\theta - \frac{\pi k}{2^j}|)^{p_k} \geq (1 + 2^j|\theta - \frac{\pi u}{2^j}|)(1 + 2^j|\theta - \frac{\pi v}{2^j}|) \geq c|u - v| \geq c n_{inf}.$$

Thus we obtain

$$\sum_{(p_k)_{k \in I_j} \in \Lambda} \frac{p! \prod_{k \in I_j} |\lambda_k^{p_k}|}{\prod_{k \in I_j} p_k!} \prod_{k \in I_j} |\psi_{j,\eta_k}|^{p_k} \leq \frac{C}{n_{inf}} \sum_{(p_k)_{k \in I_j} \in \Lambda} \frac{p! \prod_{k \in I_j} |\lambda_k|^{p_k}}{\prod_{k \in I_j} p_k!} \prod_{k \in I_j} \varphi^{p_k}_{j,\eta_k}$$

$$\leq C \frac{(\sum_{k \in I_j} |\lambda_k| \varphi_{j,\eta_k})^p}{n_{inf}}.$$

Now let us proceed similarly to the sketch of the proof of Theorem 6 in the main text, given in Kerkyacharian, Picard, Petrushev, and Willer (2007). Let us recall the two main tools.

First, consider the maximal operator $(M_s f)(x) = \sup_{J \ni x} \left(\frac{1}{|J|} \int_J |f(u)|^s du \right)^{1/s}$, where the supremum is taken over all intervals $J \subset [-1, 1]$ which contain x, $s > 0$, and $|J|$ denotes the length of J. Then one can infer the following bound from the Fefferman-Stein maximal inequality (see Fefferman and Stein (1971)). If $0 < p, r < \infty$ and $0 < s < \min\{p, r, \frac{p}{\alpha \vee \beta + 1}\}$, then for any sequence of functions (f_k) on $[-1, 1]$

$$\Big\| \Big(\sum_k (M_s f_k)^r \Big)^{1/r} \Big\|_{\mathbb{L}^p(\mu)} \leq C \Big\| \Big(\sum_k |f_k|^r \Big)^{1/r} \Big\|_{\mathbb{L}^p(\mu)}.$$

Second, set $\eta_0 = 1$ and $\eta_{2^j+1} = -1$, denote $I_k = [\frac{\eta_k + \eta_{k+1}}{2}, \frac{\eta_k + \eta_{k-1}}{2}]$ and put $H_k = h_k 1_{I_k}$ with $h_k = \left(\frac{2^j}{\omega_{\alpha,\beta}(2^j; \eta_k)} \right)^{1/2}$, where 1_{I_k} is the indicator function of I_k. Then $\|H_k\|_{\mathbb{L}^p(\mu)} \asymp \|\psi_{j,\eta_k}\|_{\mathbb{L}^p(\mu)}$, and one shows in Kerkyacharian, Picard, Petrushev, and Willer (2007) that, for any $s > 0$,

$$\varphi_{j,\eta_k}(x) \leq c (M_s H_k)(x), \quad x \in [-1, 1], \quad \forall k = 1, 2, \ldots, 2^j, j \geq 0.$$

We use these two results, with $f_k = H_k$ and $r = 1$. Noticing that the (H_k) have disjoint supports, we obtain

$$\begin{aligned} \| \sum_{k=1}^{2^j} |\lambda_k| \varphi_{j,\eta_k} \|^p_{\mathbb{L}^p(\mu)} &\leq C \| \sum_{k=1}^{2^j} |\lambda_k| H_k \|^p_{\mathbb{L}^p(\mu)} = C \sum_{k=1}^{2^j} |\lambda_k|^p \|H_k\|^p_{\mathbb{L}^p(\mu)} \\ &\leq C' \sum_{k=1}^{2^j} |\lambda_k|^p \|\psi_{j,\eta_k}\|^p_{\mathbb{L}^p(\mu)}. \end{aligned}$$

So there exists $C > 0$ such that $|B| \leq C \frac{A}{n_{inf}}$, and if we impose the condition on I_j that $n_{inf} \geq 2C$, then we obtain $|B| \leq \frac{1}{2} A$, and thus

$$\| (\sum_{k \in I_j} \lambda_k \psi_{j,\eta_k}) \|^p_{\mathbb{L}^p(\mu)} \geq \frac{1}{2} \sum_{k \in I_j} \lambda_k^p \|\psi_{j,\eta_k}\|^p_{\mathbb{L}^p(\mu)}.$$

□