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FORM OF THE CONDITIONAL VARIANCE FOR STABLE RANDOM VARIABLES

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Abstract. Non-Gaussian stable random variables always have infinite variance, but the conditional second moment $E[X_2^2|X_1]$ for a jointly α-stable vector (X_1, X_2) with index $1/2 < \alpha < 2$ may exist when some conditions on the spectral measure are met. Wu and Cambanis (1991) obtained a functional form of the conditional variance Var $[X_2|X_1 = x]$ for symmetric α-stable vectors with $1 < \alpha < 2$. This paper extends their result to the whole range $1/2 < \alpha < 2$ and also provides a formula for the conditional variance in the case where (X_1, X_2) are skewed and $\alpha \neq 1$.

Key words and phrases: Stable distributions, bivariate stable distributions, domain of attraction, conditional moments, regression, nonlinear regression.

1. Introduction and the Main Result

Let (X_1, X_2) be a symmetric α-stable ($S\alpha S$) random vector with index $0 < \alpha < 2$. Assume that X_1 and X_2 are non-degenerate. Then $E|X_i|^p < \infty, i = 1, 2$, if and only if $p < \alpha$. However, conditional moments $E[|X_2|^p|X_1]$ can exist with higher values of p. If the components of (X_1, X_2) are linearly dependent, conditional moments of any order exist. If we exclude the trivial case of linear dependence, then as shown in Cioczek-Georges and Taqqu (1994a), conditional moments of order up to $p < 2\alpha + 1$ may exist. One can therefore investigate the regression $E[X_2|X_1 = x]$ if $0 < \alpha < 2$ and the conditional variance $E[X_2^2|X_1 = x] - E^2[X_2|X_1 = x]$ if $1/2 < \alpha < 2$.

Recall that the joint characteristic function ϕ of an $S\alpha S$ vector (X_1, X_2) is given by

$$\phi(t, r) = E\exp(i(tX_1 + rX_2)) = \exp(-\int_{S_2} |ts_1 + rs_2|^\alpha \Gamma(d\mathbf{s})),$$

where Γ, called the spectral measure, is a finite symmetric measure on the Borel sets of the unit circle S_2 in $\mathbb{R}^2$ (see for example Samorodnitsky and Taqqu (1994)). There is a one-to-one correspondence between the measure Γ and the distribution of (X_1, X_2). The marginal characteristic function ϕ_1 of X_1 is given by

$$\phi_1(t) = E\exp(itX_1) = \exp\{-\sigma_1^\alpha |t|^\alpha\},$$

where $\sigma_1 = (\int_{S_2} |s_1|^\alpha \Gamma(d\mathbf{s}))^{1/\alpha}$ is the scale parameter of X_1.

For the case $1 < \alpha < 2$ the regression always exists, and Kanter (1972) proved that for all $x \in \mathbb{R}$,

$$E[X_2|X_1 = x] = \frac{\int_{S_2} s_1^{<\alpha-1>} s_2 \Gamma(d\mathbf{s})}{\int_{S_2} |s_1|^\alpha \Gamma(d\mathbf{s})} x, \tag{1.1}$$

where $a^{<\beta>} = |a|^\beta \operatorname{sign}(a)$ for $a, \beta \in \mathbb{R}$. This result can be extended in some situations to $\alpha \leq 1$. We have shown (Cioczek-Georges and Taqqu (1994a,b)) that $E[|X_2|\,|X_1 = x] < \infty$ for all $x \in \mathbb{R}$ if and only if

$$\int_{S_2} |s_1|^{-(1-\alpha)} \Gamma(d\mathbf{s}) < \infty \text{ for the case } 0 < \alpha < 1, \tag{1.2}$$

and if and only if

$$-\int_{S_2} \ln|s_1| \Gamma(d\mathbf{s}) < \infty \text{ for the case } \alpha = 1. \tag{1.3}$$

The equality (1.1) also holds for $0 < \alpha \leq 1$ as shown by Samorodnitsky and Taqqu (1991). They obtained this result under a condition slightly stronger than (1.2) or (1.3), namely, assuming that $\int_{S_2} |s_1|^{-\nu} \Gamma(d\mathbf{s}) < \infty$ for some $\nu > 1 - \alpha$, $0 < \alpha \leq 1$. (The form of the first derivative of the characteristic function $\phi_{X_2|x}$ of X_2 given $X_1 = x$ used by Samorodnitsky and Taqqu to establish (1.1) follows from (1.2) and (1.3) as well, and to establish its existence one does not need the stronger condition stated above.[1])

In the Gaussian case ($\alpha = 2$) the regression is always linear. In view of (1.1), linearity extends to symmetric α-stable, $0 < \alpha < 2$, vectors whenever the regression exists. However, the behavior of the conditional variance is completely different in the two cases, as will be seen below.

Let us focus first on the second conditional moment. Again, results of Cioczek-Georges and Taqqu (1994a,b) (c.f. also Wu and Cambanis (1991) for

[1]In fact, even Relation (1.3) is stronger than what is needed for the existence of the first derivative in the case $\alpha = 1$. The first derivative of $\phi_{X_2|x}$ for $\alpha = 1$ exists without any additional assumption, and evaluating $-i\phi'_{X_2|x}(0)$ yields the right hand side of (1.1). But, as it is known, the finiteness of an odd derivative of a characteristic function does not guarantee the existence of the corresponding odd moment.

$1 < \alpha < 2$) imply that, in the case $1/2 < \alpha < 2, E[X_2^2|X_1 = x] < \infty$ for all $x \in \mathbb{R}$ if and only if

$$\int_{S_2} |s_1|^{-(2-\alpha)}\Gamma(d\mathbf{s}) < \infty. \tag{1.4}$$

Note that (1.4) implies that the scale parameter $\sigma_1 = (\int_{S_2} |s_1|^\alpha \Gamma(d\mathbf{s}))^{1/\alpha}$ of X_1 is non-zero and hence $X_1 \not\equiv 0$. A functional form of the conditional variance for $1 < \alpha < 2$, under the above assumption, was obtained by Wu and Cambanis (1991). The next theorem extends their result to the case $1/2 < \alpha \le 1$ and shows that the conditional variance is of the form $C(\Gamma, \alpha)R^2(x/\sigma_1; \alpha)$.

Theorem 1.1. *Let (X_1, X_2) be a $S\alpha S$, $1/2 < \alpha < 2$, random vector. Assume that* (1.4) *holds in the case $1 \le \alpha < 2$ and assume*

$$\int_{S_2} |s_1|^{-\nu}\Gamma(d\mathbf{s}) < \infty \tag{1.5}$$

for some $\nu > 2 - \alpha$ for the case $1/2 < \alpha < 1$. Then the conditional variance $\mathrm{Var}[X_2|X_1 = x]$, *for $x \in \mathbb{R}$, has the form*

$$\mathrm{Var}[X_2|X_1 = x] = \sigma_1^{2-2\alpha}[\sigma_1^\alpha \int_{S_2} |s_1|^{\alpha-2} s_2^2 \Gamma(d\mathbf{s}) - (\int_{S_2} s_1^{<\alpha-1>} s_2 \Gamma(d\mathbf{s}))^2] R^2(x/\sigma_1; \alpha), \tag{1.6}$$

where

$$R^2(x;\alpha) = \frac{\alpha^2 \int_0^\infty \cos tx\, e^{-t^\alpha} t^{2\alpha-2} dt}{\int_0^\infty \cos tx\, e^{-t^\alpha}\, dt} + x^2.$$

The function $R(x;\alpha)$ is symmetric and

$$R^2(x;\alpha) - x^2 \begin{cases} \to \infty, & \text{if } 1/2 < \alpha < 1, \\ = 1, & \text{if } \alpha = 1, \\ \to -\infty, & \text{if } 1 < \alpha < 2. \end{cases}$$

Moreover, if $\alpha \ne 1$,

$$R(x;\alpha) = x + o(x), \quad \text{as } x \to \infty,$$
$$R(x;\alpha) = R(0;\alpha) + B(\alpha)x^2 + o(x^2), \quad \text{as } x \to 0,$$

where

$$R(0;\alpha) = [\alpha(\alpha-1)\Gamma(1-1/\alpha)/\Gamma(1/\alpha)]^{1/2},$$
$$B(\alpha)R(0;\alpha) = ((\alpha-1)/4)[-1 + \alpha\Gamma(3/\alpha)\Gamma(1-1/\alpha)/\Gamma^2(1/\alpha)] \ge 0.$$

Remarks

- For the case $1 < \alpha < 2$ the statement of Theorem 1.1 is equivalent to Theorem 2 of Wu and Cambanis (1991). To verify this, integrate by parts (twice) the top integral of $R^2(x;\alpha)$:

$$\begin{aligned} R^2(x;\alpha) &= \frac{-\alpha x \int_0^\infty \sin tx\, e^{-t^\alpha} t^{\alpha-1} dt}{\int_0^\infty \cos tx\, e^{-t^\alpha} dt} + \frac{\alpha(\alpha-1)\int_0^\infty \cos tx\, e^{-t^\alpha} t^{\alpha-2} dt}{\int_0^\infty \cos\ tx\, e^{-t^\alpha} dt} + x^2 \\ &= \alpha(\alpha-1)\frac{\int_0^\infty \cos tx\, e^{-t^\alpha} t^{\alpha-2} dt}{\int_0^\infty \cos tx\, e^{-t^\alpha} dt} \equiv \alpha(\alpha-1) S^2(x;\alpha), \end{aligned} \tag{1.7}$$

where $S^2(x;\alpha)$ is the notation used in Wu and Cambanis (1991). Similarly,

$$R(0;\alpha) = (\alpha(\alpha-1))^{1/2} S(0;\alpha),$$

and

$$B(\alpha)R(0;\alpha) = \alpha(\alpha-1)A(\alpha)S(0;\alpha),$$

where again $A(\alpha)$ is defined in Wu and Cambanis (1991). (There is a misprint in that paper. One should have $S(0;\alpha) = (\Gamma(1-1/\alpha)/\Gamma(1/\alpha))^{1/2}$ instead of $S(0;\alpha) = (\alpha\Gamma(1-1/\alpha)/\Gamma(1/\alpha))^{1/2}$.)

- As remarked by Wu and Cambanis (1991), for the Gaussian case $\alpha = 2$, we have $R^2(x;2) \equiv 2$ by (1.7) and hence (1.6) reduces to the usual formula

$$\mathrm{Var}[X_2|X_1 = x] = 2\sigma_2^2(1-\rho^2) = (\mathrm{Var}(X_2))(1-\rho^2),$$

where ρ is the correlation coefficient between X_1 and X_2.

- For the case $1/2 < \alpha < 1$ we make the stronger assumption (1.5). For the existence of the conditional variance it is enough to assume (1.4). But to evaluate it, we need the form of the second derivative $\phi''_{X_2|x}$, which is obtained using Condition (1.5).
- One can draw conclusions similar to those in Wu and Cambanis (1991). The conditional variance is proportional to the function $R^2(\cdot/\sigma_1;\alpha)$ which depends only on the index of stability α and the scale parameter σ_1 of X_1. When $\alpha = 1$, this functional form reduces to the surprisingly simple expression $R^2(x/\sigma_1;1) = 1 + (x/\sigma_1)^2$. The dependence on the joint distribution of (X_1, X_2) expresses itself only through a multiplicative constant (which does not depend on $X_1 = x$). This constant,

$$\sigma_1^{2-2\alpha}[\sigma_1^\alpha \int_{S_2} |s_1|^{\alpha-2} s_2^2 \Gamma(d\mathbf{s}) - (\int_{S_2} s_1^{<\alpha-1>} s_2 \Gamma(d\mathbf{s}))^2],$$

is always nonnegative (by the Cauchy-Schwarz inequality) and finite. It can be zero only if $c_1 s_1^{<\alpha/2>-1} s_2 + c_2 s_1^{<\alpha/2>} = 0$, Γ-a.e., for some $c_1^2 + c_2^2 > 0$, i.e. $s_2 = (c_2/c_1)s_1$, Γ-a.e. ($c_1 \neq 0$ since $s_1 \neq 0$ Γ-a.e.), i.e. $X_2 = (c_2/c_1)X_1$ a.e. Hence, if X_1 and X_2 are linearly independent, the conditional variance $\mathrm{Var}[X_2|X_1 = x]$ can never be constant in contrast to the Gaussian case ($\alpha = 2$). In fact, it tends to infinity as $|x| \to \infty$. The function $R(x;\alpha)$, which is proportional to the conditional standard deviation, is even, approximately quadratic around zero and approximately linear at infinity. Its mean is finite for $\alpha > 1$ but infinite for $\alpha \leq 1$.

Figures 1, 2 and 3 display $R^2(x;\alpha)$ and $R^2(x;\alpha) - x^2$ plotted for various values of parameter α. A graph of $R^2(0;\alpha)$ as a function of α is given in Figure 4. All figures were created with *Mathematica*, v.2.0.

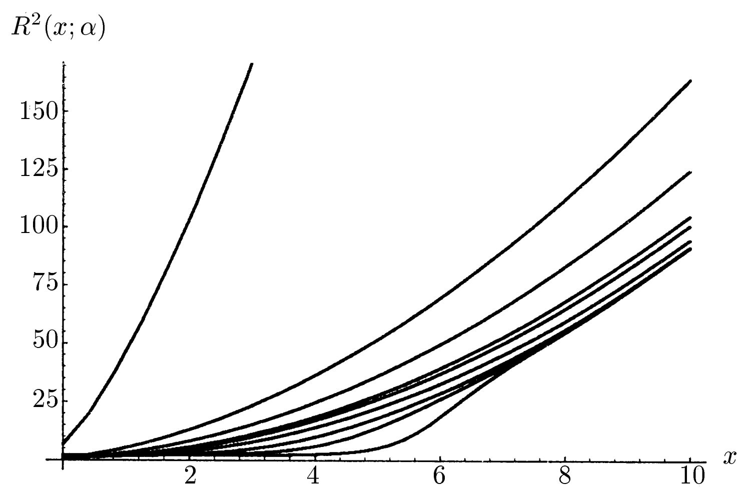

Figure 1. The functions $R^2(x;\alpha)$ for $\alpha = 0.51, 0.6, 0.7, 0.9, 1, 1.3, 1.7, 1.9, 1.99$, starting with the top graph and proceeding down.

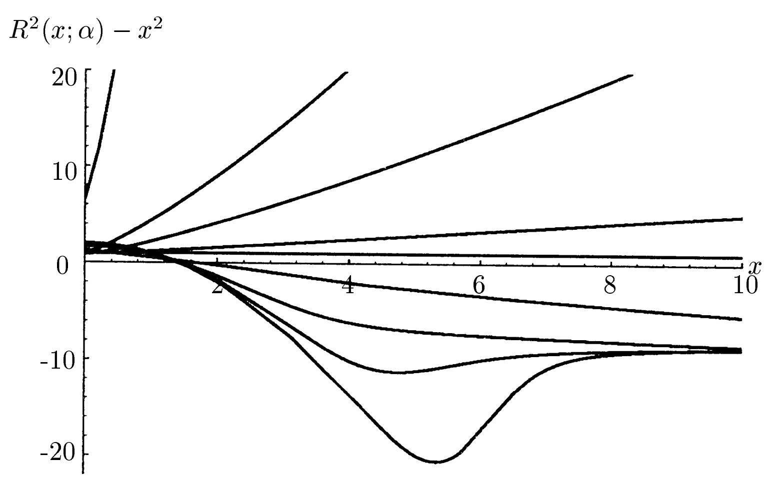

Figure 2. The functions $R^2(x;\alpha) - x^2$ for $\alpha = 0.51, 0.6, 0.7, 0.9, 1, 1.3, 1.7, 1.9, 1.99$, starting with the top graph and proceeding down. Recall that $R^2(x;\alpha) - x^2 \to +\infty$, as $x \to +\infty$, for $\alpha < 1$, and $R^2(x;\alpha) - x^2 \to -\infty$, as $x \to +\infty$, for $\alpha > 1$.

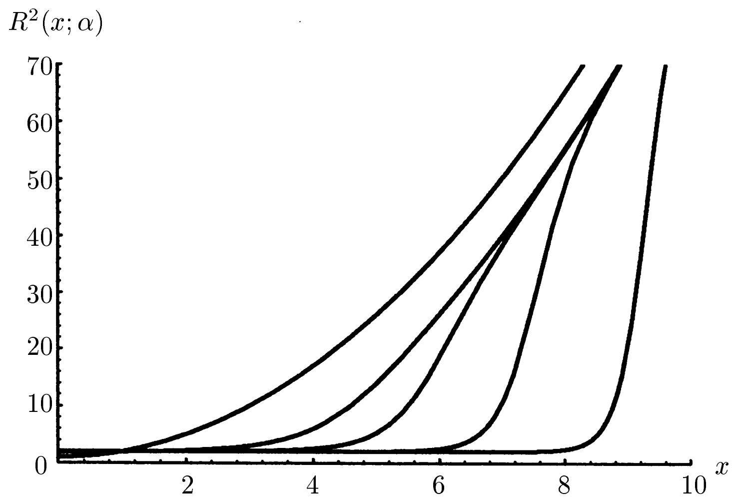

Figure 3. The functions $R^2(x;\alpha)$ for $\alpha = 1$, 1.9, 1.99, 1.9999, 1.9999999, starting with the top graph and proceeding down. Note that the closer α is to 2, the longer $R^2(x;\alpha)$ stays flat and near 2 ($R^2(x;2) \equiv 2$).

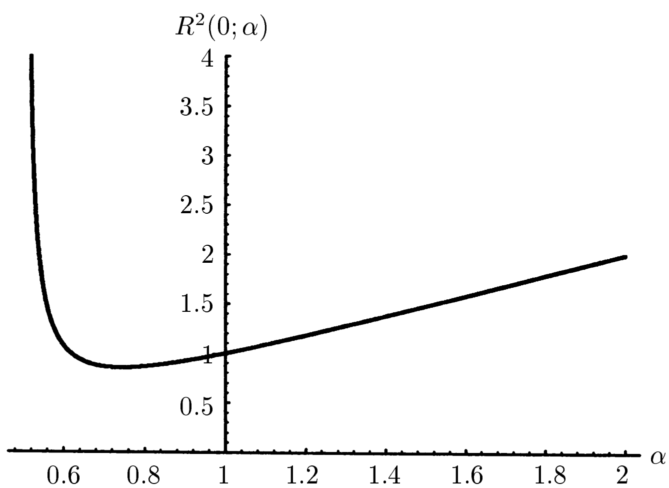

Figure 4. $R^2(0;\alpha)$ as a function of α.

The next section contains the proof of Theorem 1.1. In Section 3 we include a formula for the conditional variance in the skewed α-stable case, $1/2 < \alpha < 2$, $\alpha \neq 1$.

2. Proof of Theorem 1.1

The statement of Theorem 1.1 for $1 < \alpha < 2$ follows from Theorem 2 of Wu and Cambanis (1991) as noted earlier. Hence, we focus only on the case $1/2 < \alpha \leq 1$.

In Proposition 2.2 of Cioczek-Georges and Taqqu (1994a) we showed that, for $1/2 < \alpha \leq 1$,

$$
\begin{aligned}
&\mathrm{Re}\phi''_{X_2|x}(r) \\
&= \frac{\alpha^2}{2\pi f(x)} \int_{-\infty}^{\infty} \cos tx \exp\Big(-\int_{S_2} |ts_1+rs_2|^{\alpha}\Gamma(d\mathbf{s})\Big)\Big(\int_{S_2} (ts_1+rs_2)^{<\alpha-1>} s_2\Gamma(d\mathbf{s})\Big)^2 dt \\
&\quad - \frac{\alpha x}{2\pi f(x)} \int_{-\infty}^{\infty} \sin tx \exp\Big(-\int_{S_2} |ts_1+rs_2|^{\alpha}\Gamma(d\mathbf{s})\Big)\Big(\int_{S_2} (ts_1+rs_2)^{<\alpha-1>} s_2^2 s_1^{-1}\Gamma(d\mathbf{s})\Big) dt \\
&\quad - \frac{\alpha^2}{2\pi f(x)} \int_{-\infty}^{\infty} \cos tx \exp\Big(-\int_{Z_2} |ts_1+rs_2|^{\alpha}\Gamma(d\mathbf{s})\Big)\Big(\int_{S_2} (ts_1+rs_2)^{<\alpha-1>} s_1\Gamma(d\mathbf{s})\Big) \\
&\qquad\qquad \times \Big(\int_{S_2} (ts_1+rs_2)^{<\alpha-1>} s_2^2 s_1^{-1}\Gamma(d\mathbf{s})\Big) dt,
\end{aligned}
\tag{2.1}
$$

where $\phi_{X_2|x}$ is the conditional characteristic function of X_2 given $X_1 = x$ and $f(x)$ is the density of X_1. Relation (2.1) was proved using (1.5) in the case $1/2 < \alpha < 1$, but only (1.4) in the case $\alpha = 1$. The second moment $E[X_2^2|X_1 = x]$ is then finite for all $x \in \mathbb{R}$ and

$$
\begin{aligned}
&E[X_2^2|X_1 = x] = -\mathrm{Re}\phi''_{X_2|x}(0) \\
&= \Big[\int_{S_2} |s_1|^{<\alpha>}\Gamma(d\mathbf{s}) \int_{S_2} |s_1|^{\alpha-2} s_2^2\Gamma(d\mathbf{s}) \\
&\quad - \Big(\int_{S_2} s_1^{<\alpha-1>} s_2\Gamma(d\mathbf{s})\Big)^2\Big] \frac{\alpha^2}{\pi f(x)} \int_0^{\infty} \cos tx \exp(-\sigma_1^{\alpha} t^{\alpha}) t^{2\alpha-2} dt \\
&\quad + \Big(\int_{S_2} |s_1|^{\alpha-2} s_2^2\Gamma(d\mathbf{s})\Big) \frac{\alpha x}{\pi f(x)} \int_0^{\infty} \sin tx \exp(-\sigma_1^{\alpha} t^{\alpha}) t^{\alpha-1} dt.
\end{aligned}
\tag{2.2}
$$

Relation (1.4) (or (1.5)) implies $\sigma_1 > 0$, i.e. $X_1 \not\equiv 0$, and thus the characteristic function of X_1 is absolutely integrable. The following equality holds for $x \in \mathbb{R}$:

$$
f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} e^{-\sigma_1^{\alpha}|t|^{\alpha}} dt = \frac{1}{\pi} \int_0^{\infty} \cos tx\, e^{-\sigma_1^{\alpha} t^{\alpha}} dt. \tag{2.3}
$$

Using (2.3) and integrating by parts, we get

$$
\frac{\alpha x}{\pi f(x)} \int_0^{\infty} \sin tx \exp(-\sigma_1^{\alpha} t^{\alpha}) t^{\alpha-1} dt = \frac{x^2}{\sigma_1^{\alpha}\pi f(x)} \int_0^{\infty} \cos tx \exp(-\sigma_1^{\alpha} t^{\alpha}) dt = \frac{x^2}{\sigma_1^{\alpha}}.
$$

Now, we are able to calculate the conditional variance $\mathrm{Var}[X_2|X_1 = x]$. Relations (2.2), (1.1), (2.3) and the last one imply

$$\begin{aligned}\mathrm{Var}[X_2|X_1 = x] &= E[X_2^2|X_1 = x] - E^2[X_2|X_1 = x] \\ &= \Big[\sigma_1^\alpha \int_{S_2} |s_1|^{\alpha-2} s_2^2 \Gamma(d\mathbf{s}) - (\int_{S_2} s_1^{<\alpha-1>} s_2 \Gamma(d\mathbf{s}))^2\Big] \\ &\quad \cdot \Big\{ \frac{\alpha^2}{\pi f(x)} \int_0^\infty \cos tx \exp(-\sigma_1^\alpha t^\alpha) t^{2\alpha-2} dt + \frac{x^2}{\sigma_1^{2\alpha}} \Big\} \\ &= \sigma_1^{2-2\alpha} \Big[\sigma_1^\alpha \int_{S_2} |s_1|^{\alpha-2} s_2^2 \Gamma(d\mathbf{s}) - (\int_{S_2} s^{<\alpha-1>} s_2 \Gamma(d\mathbf{s}))^2\Big] \\ &\quad \cdot \Big\{ \frac{\alpha^2 \int_0^\infty \cos(t\frac{x}{\sigma_1}) e^{-t^\alpha} t^{2\alpha-2} dt}{\int_0^\infty \cos(t\frac{x}{\sigma_1}) e^{-t^\alpha} dt} + \frac{x^2}{\sigma_1^2} \Big\}.\end{aligned}$$

Thus, (1.6) is established.

We now turn to the asymptotic behavior of $R(x;\alpha), \alpha \neq 1$, as $x \to \infty$ and $x \to 0$. First note that

$$\lim_{x\to\infty} \frac{R^2(x;\alpha)}{x^2} = 1 + \lim_{x\to\infty} \frac{\alpha^2 \int_0^\infty \cos tx\, e^{-t^\alpha} t^{2\alpha-2} dt}{x^2 \int_0^\infty \cos tx\, e^{-t^\alpha} dt} = 1.$$

The second limit is zero because $\int_0^\infty \cos tx\, e^{-t^\alpha} dt \sim const\ x^{-\alpha-1}$, as $x \to \infty$ (the integral $\int_0^\infty \cos tx\ e^{-t^\alpha} dt$ is proportional to the density of the normalized one-dimensional stable distribution), and $\int_0^\infty \cos tx\, e^{-t^\alpha} t^{2\alpha-2} dt \sim \Gamma(2\alpha-1) \sin(\pi(1-\alpha)) x^{1-2\alpha}$, as $x \to \infty$, by Theorems 126 and 127 of Titchmarsh (1986). Note, however, that when $x \to \infty, R^2(x;\alpha) - x^2$ becomes infinitely large $(+\infty)$ if $\alpha < 1$ and infinitely small $(-\infty)$ if $\alpha > 1$ (see Figure 2).

We also have

$$\begin{aligned}&\lim_{x\to 0} \frac{R^2(x;\alpha) - R^2(0;\alpha)}{x^2} = 1 + \lim_{x\to 0} \frac{\Big(\frac{\alpha^2 \int_0^\infty \cos tx\, e^{-t^\alpha} t^{2\alpha-2} dt}{\int_0^\infty \cos tx\, e^{-t^\alpha} dt}\Big)'}{2x} \\ &= 1 - \lim_{x\to 0} \frac{\alpha^2 \int_0^\infty \sin tx\, e^{-t^\alpha} t^{2\alpha-1} dt}{2x \int_0^\infty \cos tx\, e^{-t^\alpha} dt} \\ &\quad + \lim_{x\to 0} \frac{\alpha^2 (\int_0^\infty \cos tx\, e^{-t^\alpha} t^{2\alpha-2} dt)(\int_0^\infty \sin tx\, e^{-t^\alpha} t dt)}{2x (\int_0^\infty \cos\ tx\, e^{-t^\alpha} dt)^2} \\ &= 1 - \frac{\alpha^2}{2\int_0^\infty e^{-t^\alpha} dt} \lim_{x\to 0} (\int_0^\infty \sin tx\, e^{-t^\alpha} t^{2\alpha-1} dt)' \\ &\quad + \frac{\alpha^2 \int_0^\infty e^{-t^\alpha} t^{2\alpha-2} dt}{2(\int_0^\infty e^{-t^\alpha} dt)^2} \lim_{x\to 0} (\int_0^\infty \sin tx\, e^{-t^\alpha} t dt)' \\ &= 1 - \frac{\alpha^2 \int_0^\infty e^{-t^\alpha} t^{2\alpha} dt}{2\int_0^\infty e^{-t^\alpha} dt} + \frac{\alpha^2 (\int_0^\infty e^{-t^\alpha} t^{2\alpha-2} dt)(\int_0^\infty e^{-t^\alpha} t^2 dt)}{2(\int_0^\infty e^{-t^\alpha} dt)^2}.\end{aligned}$$

Using the equalities $\int_0^\infty e^{-t^\alpha} t^\beta dt = \Gamma((\beta+1)/\alpha)/\alpha$ for $\beta > -1$, and $\Gamma(y+1) = y\Gamma(y)$ for $y > -1, y \neq 0$, we get

$$\lim_{x\to 0} \frac{R^2(x;\alpha) - R^2(0;\alpha)}{x^2} = 1 - \frac{\alpha^2\Gamma(2+\frac{1}{\alpha})}{2\Gamma(\frac{1}{\alpha})} + \frac{\alpha^2\Gamma(2-\frac{1}{\alpha})\Gamma(\frac{3}{\alpha})}{2\Gamma^2(\frac{1}{\alpha})}$$
$$= \frac{(\alpha-1)}{2}\Big[-1 + \alpha\Gamma\Big(\frac{3}{\alpha}\Big)\Gamma\Big(1-\frac{1}{\alpha}\Big)/\Gamma^2\Big(\frac{1}{\alpha}\Big)\Big] \equiv 2B(\alpha)R(0;\alpha).$$

Hence

$$\lim_{x\to 0} \frac{R(x;\alpha) - R(0;\alpha)}{x^2} = B(\alpha),$$

which completes the proof.

3. Conditional Variance for Skewed Stable Random Variables

Condition (1.4) is also sufficient for the existence a.e. of the second conditional moment $E[X_2^2|X_1 = x]$ in the case when (X_1, X_2) is a skewed α-stable, $1/2 < \alpha < 2$, random vector (c.f. Cioczek-Georges and Taqqu (1994a)), i.e. when the joint characteristic function of (X_1, X_2) is of the form

$$\phi(t,r) = \begin{cases} \exp\{-\int_{S_2} |ts_1 + rs_2|^\alpha (1 - i\tan\frac{\pi\alpha}{2}\mathrm{sign}(ts_1+rs_2))\Gamma(d\mathbf{s}) \\ \qquad + i(t\mu_1 + r\mu_2)\}, & \text{if } \alpha \neq 1, \\ \exp\{-\int_{S_2} |ts_1 + rs_2|(1 + i\frac{2}{\pi}\mathrm{sign}(ts_1+rs_2)\ln|ts_1+rs_2|)\Gamma(d\mathbf{s}) \\ \qquad + i(t\mu_1 + r\mu_2)\}, & \text{if } \alpha = 1, \end{cases}$$

where $(\mu_1, \mu_2) \in \mathbb{R}^2$ and the spectral measure Γ on S_2 is not assumed symmetric anymore. In this case, "a.e." means "for all x such that $f(x) \neq 0$," where f is the density of X_1.

The regression $E[X_2|X_1 = x]$ in the skewed α-stable case is given in Theorem 3.1 of Hardin Jr., Samorodnitsky and Taqqu (1991a) and techniques for its numerical computation are described in Hardin Jr., Samorodnitsky and Taqqu (1991b). The regression equals

$$E[X_2|X_1 = x] = \kappa x + \frac{a^2(\lambda - \beta_1\kappa)\beta_1}{1 + a^2\beta_1^2}x + \frac{a(\lambda - \beta_1\kappa)}{1 + a^2\beta_1^2}\frac{1 - xH(x)}{\pi f(x)}$$
$$= \frac{(\kappa + a^2\beta_1\lambda)}{1 + a^2\beta_1^2}x + \frac{a(\lambda - \beta_1\kappa)}{1 + a^2\beta_1^2}\frac{1 - xH(x)}{\pi f(x)}, \tag{3.1}$$

where

$$\sigma_1 = \Big(\int_{S_2} |s_1|^\alpha \Gamma(d\mathbf{s})\Big)^{1/\alpha}, \qquad \beta_1 = \frac{1}{\sigma_1^\alpha}\int_{S_2} s_1^{<\alpha>}\Gamma(d\mathbf{s})$$

are the scale and skewness parameters of X_1, and

$$a = \tan(\frac{\pi\alpha}{2}), \quad \kappa = \frac{1}{\sigma_1^\alpha}\int_{S_2} s_1^{<\alpha-1>} s_2 \Gamma(d\mathbf{s}), \quad \lambda = \frac{1}{\sigma_1^\alpha}\int_{S_2} |s_1|^{\alpha-1} s_2 \Gamma(d\mathbf{s}),$$
$$H(x) = \int_0^\infty \sin(tx - a\beta_1\sigma_1^\alpha t^\alpha)\exp(-\sigma_1^\alpha t^\alpha)dt,$$
$$\pi f(x) = \int_0^\infty \cos(tx - a\beta_1\sigma_1^\alpha t^\alpha)\exp(-\sigma_1^\alpha t^\alpha)dt.$$

Set $H(x) = (1/\sigma_1^\alpha)H_s(x/\sigma_1)$ and $f(x) = (1/\sigma_1^\alpha)f_s(x/\sigma_1)$, where

$$H_s(x) = \int_0^\infty \sin(xt - a\beta_1 t^\alpha)\exp(-t^\alpha)dt,$$
$$\pi f_s(x) = \int_0^\infty \cos(xt - a\beta_1 t^\alpha)\exp(-t^\alpha)dt,$$

and f_s is the probability density function of the standard α-stable random variable with index $\alpha \neq 1$, scale parameter $\sigma_1 = 1$ and skewness parameter β_1.

The following theorem gives the form of the conditional variance for $1/2 < \alpha < 2$, $\alpha \neq 1$. Its proof, which is omitted, uses the techniques developed in Cioczek-Georges and Taqqu (1994a) to establish a form of $\mathrm{Re}\phi''_{X_2|x}(r)$ for $\alpha \neq 1$ under the same conditions as in the symmetric case. The case $\alpha = 1$ appears more complicated.

We suppose here that the shift vector (μ_1, μ_2) is zero. This can be done without loss of generality because $\mathrm{Var}[X_2|X_1 = x] = \mathrm{Var}[X_2 + \mu_2|X_1 + \mu_1 = x + \mu_1]$.

Theorem 3.1. *Under* (1.4) *if* $1 < \alpha < 2$ *and* (1.5) *if* $1/2 < \alpha < 1$, *we get for* (X_1, X_2) *with* $(\mu_1, \mu_2) = (0, 0)$,

$$\begin{aligned}
&\mathrm{Var}[X_2|X_1 = x] \\
&= (\frac{x}{\sigma_1})^2\left[\frac{a^2\beta_1\sigma_1^{2-\alpha}}{1+a^2\beta_1^2}\int_{S_2} s_1^{<\alpha-2>}s_2^2\Gamma(d\mathbf{s}) + \frac{\sigma_1^{2-\alpha}}{1+a^2\beta_1^2}\int_{S_2}|s_1|^{\alpha-2}s_2^2\Gamma(d\mathbf{s}) - \frac{(\kappa+a^2\beta_1\lambda)^2\sigma_1^2}{(1+a^2\beta_1^2)^2}\right] \\
&\quad - \left(\frac{1-(\frac{x}{\sigma_1})H_s(\frac{x}{\sigma_1})}{\pi f_s(\frac{x}{\sigma_1})}\right)^2 \frac{a^2(\lambda-\beta_1\kappa)^2\sigma_1^2}{(1+a^2\beta_1^2)^2} + (\frac{x}{\sigma_1})\frac{1-(\frac{x}{\sigma_1})H_s(\frac{x}{\sigma_1})}{\pi f_s(\frac{x}{\sigma_1})} \\
&\quad \times \left[\frac{a\sigma_1^{2-\alpha}}{1+a^2\beta_1^2}\int_{S_2} s_1^{<\alpha-2>}s_2^2\Gamma(d\mathbf{s}) - \frac{a\beta_1\sigma_1^{2-\alpha}}{1+a^2\beta_1^2}\int_{S_2}|s_1|^{\alpha-2}s_2^2\Gamma(d\mathbf{s}) - 2\frac{a(\lambda-\beta_1\kappa)(\kappa+a^2\beta_1\lambda)\sigma_1^2}{(1+a^2\beta_1^2)^2}\right] \\
&\quad + \frac{1}{\pi f_s(\frac{x}{\sigma_1})}\int_0^\infty \cos(\frac{x}{\sigma_1}t - a\beta_1 t^\alpha)\exp(-t^\alpha)t^{2\alpha-2}dt \\
&\quad \times \left[\alpha^2(a^2\lambda^2-\kappa^2)\sigma_1^2 - a^2\alpha^2\beta_1\sigma_1^{2-\alpha}\int_{S_2} s_1^{<\alpha-2>}s_2^2\Gamma(d\mathbf{s}) + \alpha^2\sigma_1^{2-\alpha}\int_{S_2}|s_1|^{\alpha-2}s_2^2\Gamma(d\mathbf{s})\right]
\end{aligned}$$

$$
+ \frac{1}{\pi f_s(\frac{x}{\sigma_1})} \int_0^\infty \sin(\frac{x}{\sigma_1} t - a\beta_1 t^\alpha) \exp(-t^\alpha) t^{2\alpha-2} dt
\times \Big[2a\alpha^2\lambda\kappa\sigma_1^2 - a\alpha^2\sigma_1^{2-\alpha} \int_{S_2} s_1^{<\alpha-2>} s_2^2 \Gamma(d\mathbf{s}) - a\alpha^2\beta_1\sigma_1^{2-\alpha} \int_{S_2} |s_1|^{\alpha-2} s_2^2 \Gamma(d\mathbf{s}) \Big].
$$

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