Statistica Sinica **10**(2000), 1369-1383

GENERALIZED CONFIDENCE INTERVALS FOR THE LARGEST VALUE OF SOME FUNCTIONS OF PARAMETERS UNDER NORMALITY

Yi-Ping Chang and Wen-Tao Huang

Soochow University and Tamkang University

Abstract: This paper deals with generalized confidence intervals (GCIs) for the maximum value of functions of parameters of interest in the presence of nuisance parameters. For $k(\geq 2)$ normal populations, we propose GCIs for, respectively, the largest mean, the largest quantile and the largest signal-to-noise ratio.

For the case of the largest mean, it is shown that the proposed GCIs are better than those of Chen and Dudewicz (1973a, b). A new measure of efficiency is proposed and some Monte Carlo comparisons between the proposed method and the known method are performed. We also show that in several situations the GCIs are equivalent to Bayesian confidence intervals by employing improper prior distributions. Illustration is made to some real data.

Key words and phrases: Bayesian confidence interval, generalized confidence interval, quantile, signal-to-noise ratio.

1. Introduction

To illustrate the underlying problem and to formulate the generalized confidence interval (GCI), consider an observable random vector $\boldsymbol{X}_i$ from population π_i, $i = 1, \ldots, k$, with cumulative distribution function $F_{\boldsymbol{\zeta}_i}$, where $\boldsymbol{\zeta}_i = (\theta_i, \boldsymbol{\delta}_i)$ is a vector of unknown parameters, θ_i the parameter of interest and $\boldsymbol{\delta}_i$ a vector of nuisance parameters. For convenience, let $\boldsymbol{x}_i$ be an observation of $\boldsymbol{X}_i$, $i = 1, \ldots, k$, $\boldsymbol{X} = (\boldsymbol{X}_1, \ldots, \boldsymbol{X}_k)$, $\boldsymbol{x} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k)$, and $\boldsymbol{\zeta} = (\boldsymbol{\zeta}_1, \ldots, \boldsymbol{\zeta}_k)$. Based on $\boldsymbol{x}$, our goal is to derive a $100(1-\alpha)\%$ GCI for $\theta = \max_{1\leq j\leq k} \theta_j$.

It is well known that confidence intervals (CIs) in statistical problems involving nuisance parameters are available only in special cases. To remedy this, Weerahandi (1993, 1995) proposed a generalized pivotal quantity and derived the GCI as an extension of the classical CI. Based on that work, we make

Definition 1.1. Let $R = R(\boldsymbol{X}, \boldsymbol{x}, \boldsymbol{\zeta})$ be a function of $\boldsymbol{X}$ and possibly $\boldsymbol{x}$ and $\boldsymbol{\zeta}$ as well. Then R is said to be a generalized pivotal quantity if it satisfies the following conditions :
(1) the distribution of R is free from unknown parameters;
(2) r_{obs} defined as $r_{\text{obs}} = R(\boldsymbol{x}, \boldsymbol{x}, \boldsymbol{\zeta})$ does not depend on nuisance parameters.

Definition 1.2. Let Θ be the parameter space of θ. If a subset $C_{1-\alpha}$ of the sample space of R satisfies $P(R \in C_{1-\alpha}) = 1 - \alpha$, then the subset Θ_c of the parameter space given by $\Theta_c(1-\alpha) = \{\theta \in \Theta | r_{\text{obs}} \in C_{1-\alpha}\}$ is said to be a $100(1-\alpha)\%$ GCI for θ.

As in a Bayesian treatment, the idea is to do the best with the observed data rather than treat all possible samples. The GCIs are not based on conventional repeated sampling considerations, but rather on exact probability statements. For further discussions and details in this direction, see Weerahandi (1995). In fact, by Meng (1994), the distribution of R sometimes can be obtained as a limit of the posterior distributions of $r_{\text{obs}}|\boldsymbol{X} = \boldsymbol{x}$ in a Bayesian formulation, and the GCI can be obtained as a limit of Bayesian CIs. However, this is not always the case.

Let $\pi_1, \ldots, \pi_k$ be $k(\geq 2)$ populations where observations X_{ij} from π_i are independently distributed as $\mathcal{N}(\mu_i, \sigma_i^2), i = 1, \ldots, k, j = 1, \ldots, n_i$. All means μ_i and variances σ_i^2 are unknown. Let $g(\mu, \sigma^2)$ be some function of the mean and variance. We consider π_i to be better than π_j if $g(\mu_i, \sigma_i^2) > g(\mu_j, \sigma_j^2)$. Our goal is to construct an interval estimator of $\max_{1\leq i\leq k} g(\mu_i, \sigma_i^2)$.

When $g(\mu_i, \sigma_i^2) = \mu_i$ and the variances are equal, the problem has been considered by several authors (e.g., Dudewicz (1972), Chen and Dudewicz (1973a, b), Alam, Saxena and Tong (1973), Alam and Saxena (1974), among others). A discussion of these approaches and various related methods can be found in Gupta and Panchapakesan (1979). When the variances are unequal, we consider $100(1-\alpha)\%$ generalized upper confidence intervals (GUCIs) for the largest mean by using the generalized pivotal quantity derived in Section 2.

In many practical situations, an experimenter is not only interested in selecting the population in terms of the means, but also in considering other quantities such as the signal-to-noise ratio (Box (1988)). The latter is an important measure in industrial statistics.

In this paper, we consider two specific cases: $g(\mu, \sigma^2) = \mu + \sigma\Phi^{-1}(p)$ and $g(\mu, \sigma^2) = \mu/\sigma$, i.e. the pth quantile and the signal-to-noise ratio. They are studied in Section 3 and Section 4, respectively. Proofs are presented in the Appendix.

2. GCIs for the Largest Mean

Let $\pi_1, \ldots, \pi_k$ be $k(\geq 2)$ populations where observations X_{ij} from π_i are independently distributed as $\mathcal{N}(\mu_i, \sigma_i^2), i = 1, \ldots, k, j = 1, \ldots, n_i$. Let $n^* = \sum_{i=1}^k (n_i - 1)$, $\bar{X}_i = \sum_{j=1}^{n_i} X_{ij}/n_i$, $S_p^2 = \sum_{i=1}^k \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2/n^*$, $S_i^2 = \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2/n_i$, and $\bar{x}_i$, s_p^2, and s_i^2 the observed values of $\bar{X}_i$, S_p^2, and S_i^2, respectively. We want to construct $100(1-\alpha)\%$ GUCIs for $\max_{1\leq i\leq k} \mu_i$.

2.1. When $\sigma_1^2 = \cdots = \sigma_k^2 = \sigma^2$ are known

Consider the identity $\mu_i = \bar{X}_i - n_i^{-1/2}\sigma Z_i$, where $Z_i = n_i^{1/2}(\bar{X}_i - \mu_i)/\sigma \sim \mathcal{N}(0,1)$, and $Z_1,\ldots,Z_k$ are independent. Let $Y_i = \bar{x}_i - n_i^{-1/2}\sigma Z_i$ and $R = \max_{1\le i\le k} Y_i$. Clearly, the observed value of R is $\max_{1\le i\le k}\mu_i$ and the distribution of R is free from the unknown parameters. Therefore, R is a generalized pivotal quantity. In fact, the components of the random vector $(n_1^{1/2}(Y_1 - \bar{x}_1)/\sigma,\ldots,n_k^{1/2}(Y_k - \bar{x}_k)/\sigma) = (-Z_1,\ldots,-Z_k)$ have independent normal distributions. The following theorem shows that the distribution of $(Y_1,\ldots,Y_k)$ is equivalent to a posterior distribution of $(\mu_1,\ldots,\mu_k)|\,\boldsymbol{x}$ based on a linear transformation of independent normal random variables.

Theorem 2.1. *Suppose the joint prior distribution of* $(\mu_1,\ldots,\mu_k)$ *is the improper prior* $p(\mu_1,\ldots,\mu_k)\propto 1$. *Then the joint posterior distribution of* $(n_1^{1/2}(\mu_1 - \bar{x}_1)/\sigma,\ldots,\ n_k^{1/2}(\mu_k - \bar{x}_k)/\sigma)|\,\boldsymbol{x}$ *is equivalent to* k *independent normal distributions.*

The proof is staightforward and thus is omitted. Now, for given $1-\alpha$, we need to find a constant c such that

$$\begin{aligned}
1-\alpha =& P\left(\max_{1\le i\le k} Y_i \le c\right) = P\left(Y_i \le c \ \text{ for all } i = 1,\ldots,k\right)\\
=& P\left\{Z_i \ge n_i^{1/2}(\bar{x}_i - c)/\sigma \ \text{ for all } i = 1,\ldots,k\right\}\\
=& \prod_{i=1}^{k}\left\{1-\Phi\left[n_i^{1/2}(\bar{x}_i - c)/\sigma\right]\right\},
\end{aligned} \tag{2.1}$$

where $\Phi(\cdot)$ denotes the standard normal distribution function. It is evident that $(-\infty,\ c_{1-\alpha}(\bar{x}_1,\ldots,\bar{x}_k))$ is a $100(1-\alpha)\%$ GUCI for $\max_{1\le i\le k}\mu_i$, where $c_{1-\alpha}(\bar{x}_1,\ldots,\bar{x}_k)$ is the value of c that satisfies (2.1). By Theorem 2.1, it is equivalent to a $100(1-\alpha)\%$ Bayesian upper CI.

Chen and Dudewicz (1973a) (see also Gupta and Panchapakesan (1979)) also proposed a $100(1-\alpha)\%$ upper CI given by $(-\infty, \max_{1\le i\le k}(\bar{x}_i + n_i^{-1/2}\sigma\Phi^{-1}((1-\alpha)^{1/k})))$. The following theorem shows that the proposed $100(1-\alpha)\%$ GUCI is always contained in this upper CI, henceforth abbreviated as UCI1.

Theorem 2.2. *Let* $c_{1-\alpha}(\bar{x}_1,\ldots,\bar{x}_k)$ *denote the value* c *satisfying* (2.1). *Then* $c_{1-\alpha}(\bar{x}_1,\ldots,\bar{x}_k) \le \max_{1\le i\le k}(\bar{x}_i + n_i^{-1/2}\sigma\Phi^{-1}((1-\alpha)^{1/k}))$, *and equality holds if and only if* $\bar{x}_i + n_i^{-1/2}\sigma\Phi^{-1}((1-\alpha)^{1/k})$, $i = 1,\ldots,k$, *are all equal.*

To compare the empirical coverage probabilities of the proposed GUCI and UCI1, suppose that $k=5$, $n_1=\cdots=n_5=10$, $\mu_i = i$, and $\sigma^2 = 1$. Figure 1 is based on 10000 simulations.

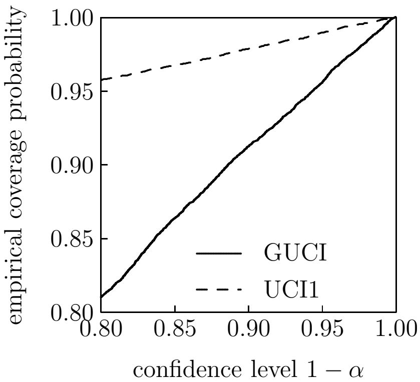

Figure 1. Empirical coverage probabilities of GUCI and UCI1 based on 10000 simulations, $k = 5$, $n_1 = \cdots = n_5 = 10$, $\mu_i = i$, $\sigma^2 = 1$.

To compare the two CIs, we propose a new measure of efficiency between two intervals. For any observations $\bar{x}_1, \ldots, \bar{x}_k$ and confidence level $1-\alpha$, a UCI1 for $\max_{1\le i\le k} \mu_i$ can be constructed and, based on this, we can find $1-\alpha'$ such that

$$1-\alpha' = \prod_{i=1}^{k} \left(1 - \Phi\left\{\frac{n_i^{1/2}}{\sigma}\left[\bar{x}_i - \max_{1\le i\le k}\left(\bar{x}_i + n_i^{-1/2}\sigma\Phi^{-1}\left((1-\alpha)^{1/k}\right)\right)\right]\right\}\right).$$

We define the efficiency of the proposed GUCI to UCI1 by

$$\text{eff} = \frac{1-\alpha'}{1-\alpha}. \tag{2.2}$$

Based on 10000 simulations with $k = 5$, $n_1 = \cdots = n_5 = 10$, $\mu_i = i$, $\sigma^2 = 1$, efficiencies are shown in Figure 2.

2.2. When $\sigma_1^2 = \cdots = \sigma_k^2 = \sigma^2$ are unknown

Consider the identity $\mu_i = \bar{X}_i - (n^* S_p^2/n_i V)^{1/2} Z_i$, where

$$Z_i = n_i^{1/2}(\bar{X}_i - \mu_i)/\sigma \sim \mathcal{N}(0,1), \qquad V = n^* S_p^2/\sigma^2 \sim \chi^2_{n^*}, \tag{2.3}$$

and $Z_1, \ldots, Z_k$ and V are independent. Let $Y_i = \bar{x}_i - (n^* s_p^2/n_i V)^{1/2} Z_i$ and $R = \max_{1\le i\le k} Y_i$. Clearly R is a generalized pivotal quantity. In fact, the distribution of the random vector $(n_1^{1/2}(Y_1 - \bar{x}_1)/s_p, \ldots, n_k^{1/2}(Y_k - \bar{x}_k)/s_p) = (-Z_1(V/n^*)^{-1/2}, \ldots, -Z_k(V/n^*)^{-1/2})$ has a k-variate t distribution with n^* degrees of freedom. The following theorem shows that the distribution of $(Y_1, \ldots, Y_k)$ is equivalent to a posterior distribution of $(\mu_1, \ldots, \mu_k)|\,\boldsymbol{x}$ which is a linear transformation of a k-variate t distribution.

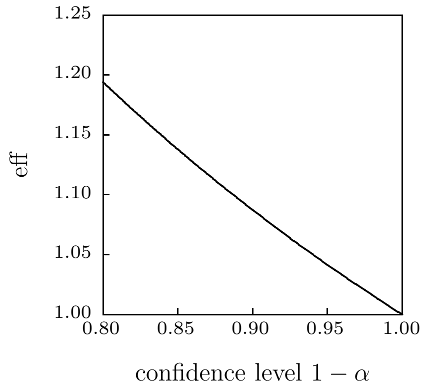

Figure 2. Efficiency of GUCI to UCI1 defined by (2.2) based on 10000 simulations, $k = 5$, $n_1 = \cdots = n_5 = 10$, $\mu_i = i$, $\sigma^2 = 1$.

Theorem 2.3. *Consider the improper prior $p(\mu_1, \ldots, \mu_k, \sigma^2) \propto \sigma^{-2}$. The joint posterior distribution of $(n_1^{1/2}(\mu_1 - \bar{x}_1)/s_p, \ldots, n_k^{1/2}(\mu_k - \bar{x}_k)/s_p)|\,\boldsymbol{x}$ is a k-variate t distribution with n^* degrees of freedom.*

Now, for given $1-\alpha$, we need to find a constant c such that

$$1-\alpha = P\left(\max_{1\le i\le k} Y_i \le c\right) = P\left\{Z_i \ge \left(\frac{n_i V}{n^*}\right)^{1/2}\left(\frac{\bar{x}_i - c}{s_p}\right) \quad \text{for all } i = 1, \ldots, k\right\}$$
$$= \int_0^\infty \prod_{i=1}^k \left\{1 - \Phi\left[\left(\frac{n_i v}{n^*}\right)^{1/2}\left(\frac{\bar{x}_i - c}{s_p}\right)\right]\right\} p_{\chi^2_{n^*}}(v)dv, \tag{2.4}$$

where $p_{\chi^2_{n^*}}(v)$ is the probability density function of a χ^2 distribution with n^* degrees of freedom. It is evident that

$$(-\infty, c_{1-\alpha}(\bar{x}_1, \ldots, \bar{x}_k, s_p)) \tag{2.5}$$

is a $100(1-\alpha)\%$ GUCI for $\max_{1\le i\le k}\mu_i$, where $c_{1-\alpha}(\bar{x}_1, \ldots, \bar{x}_k, s_p)$ is the value of c satisfying (2.4) and, by Theorem 2.3, is equivalent to a $100(1-\alpha)\%$ Bayesian upper CI.

Chen and Dudewicz (1973b) proposed a $100(1-\alpha)\%$ upper CI for the case $n_1 = \cdots = n_k = n$ which is given by $(-\infty, \max_{1\le i\le k}(\bar{x}_i + n^{-1/2}s_p F^{-1}_{k,k(n-1)}(1-\alpha)))$, where $F^{-1}_{k,k(n-1)}(1-\alpha)$ satisfies $F_{k,k(n-1)}(F^{-1}_{k,k(n-1)}(1-\alpha), \ldots, F^{-1}_{k,k(n-1)}(1-\alpha)) = 1-\alpha$, and $F_{k,k(n-1)}(\cdot, \ldots, \cdot)$ denotes the distribution function of a k-variate t distribution with $k(n-1)$ degrees of freedom and identity correlation matrix. The following theorem shows that the proposed $100(1-\alpha)\%$ GUCI is a non-trivial subset of the $100(1-\alpha)\%$ upper CI given by Chen and Dudewicz (1973b), henceforth abbreviated as UCI2.

Theorem 2.4. *Let* $n_1 = \cdots = n_k = n$ *and* $c_{1-\alpha}(\bar{x}_1, \ldots, \bar{x}_k, s_p)$ *be the value of* c *satisfying* (2.4). *Then* $c_{1-\alpha}(\bar{x}_1, \ldots, \bar{x}_k, s_p) \leq \max_{1\leq i\leq k}(\bar{x}_i + n^{-1/2} s_p F^{-1}_{k,k(n-1)}(1-\alpha))$, *and equality holds if and only if* $\bar{x}_1 = \cdots = \bar{x}_k$.

Based on 10000 simulations with $k = 5$, $n_1 = \cdots = n_k = 10$, $\mu_i = i$, $\sigma^2 = 1$, Figure 3 shows that UCI2 is rather conservative and that the empirical coverage probability of GUCI is very close to (never below) the confidence level.

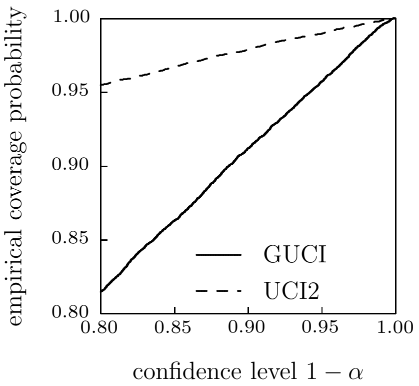

Figure 3. Empirical coverage probabilities for the case of equal unknown variances.

In Figure 4 the efficiencies of GUCI relative to UCI2 is based on 10000 simulations with $k = 5$, $n_1 = \cdots = n_5 = 10$, $\mu_i = i$, $\sigma^2 = 1$.

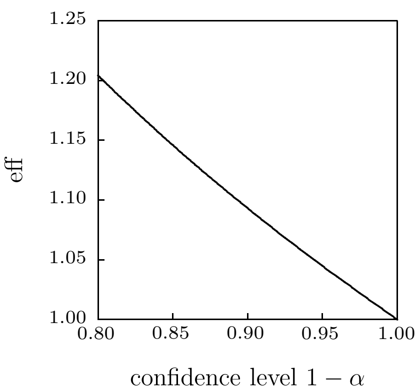

Figure 4. Efficiency of GUCI to UCI2 when the variances are equal and unknown.

2.3. When σ_i^2 are unequal and unknown

Following an identity of Weerahandi (1995), $\mu_i = \bar{X}_i - n_i^{-1/2}\sigma_i Z_i = \bar{X}_i - S_i Z_i V_i^{-1/2}$, where

$$Z_i = n_i^{1/2}(\bar{X}_i - \mu_i)/\sigma_i \sim \mathcal{N}(0,1), \qquad V_i = n_i S_i^2/\sigma_i^2 \sim \chi^2_{n_i-1}, \tag{2.6}$$

and Z_i and V_i, $i = 1,\ldots,k$ are independent. Let $Y_i = \bar{x}_i - s_i Z_i V_i^{-1/2}$ and $R = \max_{1\le i\le k} Y_i$. Clearly R is a generalized pivotal quantity. It can be shown that the random variables $(n_i - 1)^{1/2}(Y_i - \bar{x}_i)/s_i = -Z_i\{V_i/(n_i - 1)\}^{-1/2}$, $i = 1,\ldots,k$, follow independent t distributions with $n_i - 1$ degrees of freedom, respectively, and the distribution of the random vector $(Y_1,\ldots,Y_k)$ is equivalent to a posterior distribution of $(\mu_1,\ldots,\mu_k)|\,\boldsymbol{x}$ which is a linear transformation of independent t distributions.

Theorem 2.5. *Consider the improper prior* $p(\mu_1,\ldots,\mu_k,\sigma_1^2,\ldots,\sigma_k^2) \propto \prod_{i=1}^k \sigma_i^{-2}$. *The joint posterior distribution of* $((n_1 - 1)^{1/2}(\mu_1 - \bar{x}_1)/s_1,\ldots,(n_k - 1)^{1/2}(\mu_k - \bar{x}_k)/s_k)|\,\boldsymbol{x}$ *is equivalent to* k *independent* t *distributions with* $n_i - 1$ *degrees of freedom, respectively.*

The proof is similar to that of Theorem 2.3 and is omitted. Now, for given $1 - \alpha$, there is some constant c satisfying

$$\begin{aligned}
1 - \alpha &= P\left(\max_{1\le i\le k} Y_i \le c\right)\\
&= P\left\{\frac{Z_i}{\sqrt{V_i/(n_i - 1)}} \ge (n_i - 1)^{1/2}(\bar{x}_i - c)/s_i \text{ for all } i = 1,\ldots,k\right\}\\
&= \prod_{i=1}^k P\left\{t_{n_i-1} \ge (n_i - 1)^{1/2}(\bar{x}_i - c)/s_i\right\},
\end{aligned} \tag{2.7}$$

where t_{n_i-1} denotes a random variable with t distribution with $n_i - 1$ degrees of freedom. Thus

$$(-\infty, c_{1-\alpha}(\bar{x}_1,\ldots,\bar{x}_k,s_1,\ldots,s_k)) \tag{2.8}$$

is a $100(1 - \alpha)\%$ GUCI for $\max_{1\le i\le k}\mu_i$, where $c_{1-\alpha}(\bar{x}_1,\ldots,\bar{x}_k,s_1,\ldots,s_k)$ is the value of c that satisfies (2.7) and, by Theorem 2.5, is equivalent to a $100(1-\alpha)\%$ Bayesian upper CI.

Figure 5 shows the empirical coverage probabilities of the proposed GUCI, based on 10000 simulations with $k = 5$, $n_1 = \cdots = n_5 = 10$, and $\mu_i = \sigma_i^2 = i, i = 1,\ldots,5$. It is seen that the empirical coverage probability is larger than the associated confidence level.

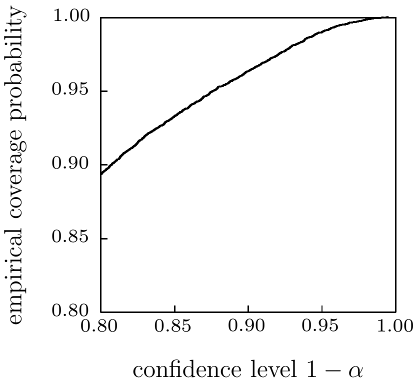

Figure 5. Empirical coverage probabilities when the variances are unequal and unknown.

What is the efficiency loss if GUCI for the unequal variances case is applied when the variances are in fact all equal ? Under the same setting as in Section 2.1, we computed the efficiency of (2.5) relative to (2.8). Figure 6 shows that, over the range $1-\alpha \in [0.8, 1)$, (2.8) is almost as efficient as (2.5). Therefore, unless there is strong evidence for equality of variances, it is recommended to use the GUCI at (2.8) derived under the unequal variances assumption.

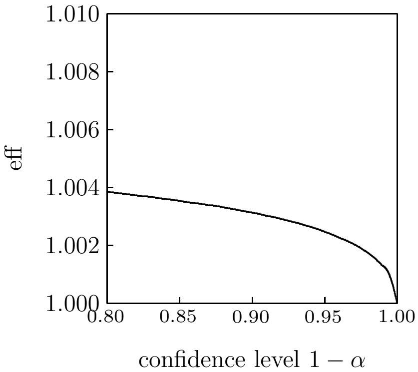

Figure 6. Efficiency of GUCI for the case of equal variances against that of unequal variances based on 10000 simulations, $k=5$, $n_1 = \cdots = n_5 = 10$, $\mu_i = i$, $\sigma_i^2 = 1$.

It is noted that a two-sided GCI can also be obtained by the same method. In fact, a $100(1-\alpha)\%$ two-sided GCI is given by $(c_{1,1-\alpha}(\bar{x}_1, \ldots, \bar{x}_k, s_1, \ldots, s_k),$ $c_{2,1-\alpha}(\bar{x}_1, \ldots, \bar{x}_k, s_1, \ldots, s_k))$ where $c_{1,1-\alpha}(\bar{x}_1, \ldots, \bar{x}_k, s_1, \ldots, s_k)$ and $c_{2,1-\alpha}(\bar{x}_1,$ $\ldots, \bar{x}_k, s_1, \ldots, s_k)$ respectively satisfy

$$\prod_{i=1}^{k} P\left\{t_{n_i-1} \geq (n_i-1)^{1/2}\left(\bar{x}_i - c_{1,1-\alpha}(\bar{x}_1, \ldots, \bar{x}_k, s_1, \ldots, s_k)\right)/s_i\right\} = \alpha_1,$$

$$\prod_{i=1}^{k} P\left\{t_{n_i-1} \geq (n_i-1)^{1/2}\left(\bar{x}_i - c_{2,1-\alpha}(\bar{x}_1,\ldots,\bar{x}_k,s_1,\ldots,s_k)\right)/s_i\right\} = 1-\alpha_2,$$

and $\alpha_1 + \alpha_2 = \alpha$.

3. GCIs for the Largest Quantile

Let the pth quantile corresponding to π_i be denoted by θ_i^p, $\theta_i^p = \mu_i + \sigma_i \Phi^{-1}(p)$. We want to construct a $100(1-\alpha)\%$ GUCIs for $\max_{1\leq i\leq k} \theta_i^p$.

3.1. When $\sigma_1^2 = \cdots = \sigma_k^2 = \sigma^2$ are unknown

Consider the identity $\theta_i^p = \bar{X}_i - (n^*/V)^{1/2} S_p\{n_i^{-1/2} Z_i - \Phi^{-1}(p)\}$, where Z_i and V are defined in (2.3). Let $Y_i = \bar{x}_i - (n^*/V)^{1/2} s_p\{n_i^{-1/2} Z_i - \Phi^{-1}(p)\}$ and $R = \max_{1\leq i\leq k} Y_i$. Clearly R is a generalized pivotal quantity. Also, the distribution of the random vector

$$\begin{aligned}&\left(\frac{n_1^{1/2}(Y_1-\bar{x}_1)}{s_p},\ldots,\frac{n_k^{1/2}(Y_k-\bar{x}_k)}{s_p}\right)\\ &= \left(-\frac{Z_1 - n_1^{1/2}\Phi^{-1}(p)}{(V/n^*)^{1/2}},\ldots,-\frac{Z_k - n_k^{1/2}\Phi^{-1}(p)}{(V/n^*)^{1/2}}\right)\end{aligned}$$

has a k-variate noncentral t distribution with n^* degrees of freedom and noncentral parameter vector $(n_1^{1/2}\Phi^{-1}(p),\ldots,n_k^{1/2}\Phi^{-1}(p))$. Connection of the distribution of the random vector $(Y_1,\ldots,Y_k)$ and a posterior distribution of $(\theta_1^p,\ldots,\theta_k^p)|\boldsymbol{x}$ is shown by the following.

Theorem 3.1. *Consider the improper prior $p(\mu_1,\ldots,\mu_k,\sigma^2) \propto \sigma^{-2}$. The joint posterior distribution of $(n_1^{1/2}(\theta_1^p - \bar{x}_1)/s_p,\ldots,n_k^{1/2}(\theta_k^p - \bar{x}_k)/s_p)|\boldsymbol{x}$ is a k-variate noncentral t distribution with n^* degrees of freedom and noncentral parameter vector $(n_1^{1/2}\Phi^{-1}(p),\ldots,\ n_k^{1/2}\Phi^{-1}(p))$.*

For given $1-\alpha$, find a constant c such that

$$\begin{aligned}1-\alpha &= P\left(\max_{1\leq i\leq k} Y_i \leq c\right)\\ &= P\left\{Z_i \geq \left(\frac{n_i V}{n^* s_p^2}\right)^{1/2}(\bar{x}_i - c) + n_i^{1/2}\Phi^{-1}(p) \ \text{ for all } i=1,\ldots,k\right\}\\ &= \int_0^\infty \prod_{i=1}^{k} P\left\{1-\Phi\left[\left(\frac{n_i v}{n^* s_p^2}\right)^{1/2}(\bar{x}_i - c) + n_i^{1/2}\Phi^{-1}(p)\right]\right\} p_{\chi^2_{n^*}}(v)dv. \quad (3.1)\end{aligned}$$

Then $(-\infty, c_{1-\alpha}(\bar{x}_1,\ldots,\bar{x}_k,s_p))$ is a $100(1-\alpha)\%$ GUCI for $\max_{1\leq i\leq k}\theta_i^p$, where $c_{1-\alpha}(\bar{x}_1,\ldots,\bar{x}_k,s_p)$ is the value c satisfying (3.1) and, by Theorem 3.1, is equivalent to a $100(1-\alpha)\%$ Bayesian upper CI.

3.2. When $\sigma_1^2, \ldots, \sigma_k^2$ are unequal and unknown

Consider the identity $\theta_i^p = \bar{X}_i - S_i V_i^{-1/2}\{Z_i - n_i^{1/2}\Phi^{-1}(p)\}$, where Z_i and V_i are defined in (2.6). Let $Y_i = \bar{x}_i - s_i V_i^{-1/2}\{Z_i - n_i^{1/2}\Phi^{-1}(p)\}$ and $R = \max_{1\le i\le k} Y_i$. Clearly, R is a generalized pivotal quantity. It can be shown that $(n_i-1)^{1/2}(Y_i - \bar{x}_i)/s_i = -\{Z_i - n_i^{1/2}\Phi^{-1}(p)\}/\{V_i/(n_i-1)\}^{1/2}$, $i = 1, \ldots, k$, are independent noncentral t distributions with $n_i - 1$ degrees of freedom and noncentral parameter $n_i^{1/2}\Phi^{-1}(p)$, respectively, and the distribution of the random vector $(Y_1, \ldots, Y_k)$ is equivalent to a posterior distribution of $(\theta_1^p, \ldots, \theta_k^p)|\,\boldsymbol{x}$ by the following.

Theorem 3.2. *Consider the improper prior $p(\mu_1, \ldots, \mu_k, \sigma_1^2, \ldots, \sigma_k^2) \propto \prod_{i=1}^k \sigma_i^{-2}$. The joint posterior distribution of $((n_1-1)^{1/2}(\theta_1^p - \bar{x}_1)/s_1, \ldots, (n_k-1)^{1/2}(\theta_k^p - \bar{x}_k)/s_k)|\,\boldsymbol{x}$ is equivalent to k independent noncentral t distributions with $n_i - 1$ degrees of freedom and noncentral parameters $n_i^{1/2}\Phi^{-1}(p)$, respectively.*

The proof is similar to that of Theorem 3.1 and is omitted. Now, for given $1-\alpha$, there is some constant c satisfying

$$
\begin{aligned}
1-\alpha &= P\left(\max_{1\le i\le k} Y_i \le c\right)\\
&= P\left\{\frac{Z_i - n_i^{1/2}\Phi^{-1}(p)}{\sqrt{V_i/(n_i-1)}} \ge (n_i-1)^{1/2}(\bar{x}_i - c)/s_i \ \text{ for all } i = 1, \ldots, k\right\}\\
&= \prod_{i=1}^k P\left\{t'_{n_i-1}\left(-n_i^{1/2}\Phi^{-1}(p)\right) \ge (n_i-1)^{1/2}(\bar{x}_i - c)/s_i\right\}, \qquad (3.2)
\end{aligned}
$$

where $t'_{n_i-1}(\lambda_i)$ denotes a random variable of noncentral t distribution with $n_i - 1$ degrees of freedom and the noncentral parameter λ_i. Thus

$$(-\infty, c_{1-\alpha}(\bar{x}_1, \ldots, \bar{x}_k, s_1, \ldots, s_k)), \qquad (3.3)$$

is a $100(1-\alpha)\%$ GUCI for $\max_{1\le i\le k}\theta_i^p$, where $c_{1-\alpha}(\bar{x}_1, \ldots, \bar{x}_k, s_1, \ldots, s_k)$ is the value c satisfying (3.2) and, by Theorem 3.2, is equivalent to a $100(1-\alpha)\%$ Bayesian upper CI.

4. GCIs for the Largest Signal-to-noise Ratio

Let θ_i denote the signal-to-noise ratio corresponding to π_i, $\theta_i = \mu_i/\sigma_i$. We want to construct $100(1-\alpha)\%$ GUCIs for $\max_{1\le i\le k}\theta_i$.

4.1. When $\sigma_1^2 = \cdots = \sigma_k^2 = \sigma^2$ are unknown

Consider the identity $\theta_i = (V/n^*)^{1/2}\bar{X}_i/S_p - n_i^{-1/2}Z_i$, where Z_i and V_i are defined in (2.3). Let $Y_i = (V/n^*)^{1/2}\bar{x}_i/s_p - n_i^{-1/2}Z_i$ and $R = \max_{1\le i\le k} Y_i$.

Clearly R is a generalized pivotal quantity. For given $1-\alpha$, find a constant c such that

$$\begin{aligned} 1-\alpha &= P\left(\max_{1\le i\le k} Y_i \le c\right) \\ &= P\left\{Z_i \ge \left(\frac{n_i V}{n^*}\right)^{1/2}\left(\frac{\bar{x}_i}{s_p}\right) - n_i^{1/2}c \quad \text{for all } i=1,\ldots,k\right\} \\ &= \int_0^\infty \prod_{i=1}^k \left\{1-\Phi\left[\left(\frac{n_i v}{n^*}\right)^{1/2}\left(\frac{\bar{x}_i}{s_p}\right) - n_i^{1/2}c\right]\right\} p_{\chi^2_{n^*}}(v)dv. \end{aligned} \tag{4.1}$$

Then $(-\infty, c_{1-\alpha}(\bar{x}_1,\ldots,\bar{x}_k,s_p))$, is a $100(1-\alpha)\%$ GUCI for $\max_{1\le i\le k}\theta_i$, where $c_{1-\alpha}(\bar{x}_1,\ldots,\bar{x}_k,s_p)$ is the value c satisfying (4.1) and, by the following, it is equivalent to a $100(1-\alpha)\%$ Bayesian upper CI.

Theorem 4.1. *Consider the improper prior $p(\mu_1,\ldots,\mu_k,\sigma^2)\propto\sigma^{-2}$. The joint posterior distribution of $(\mu_1/\sigma,\ldots,\mu_k/\sigma)|\,\boldsymbol{x}$ is equivalent to the joint distribution of $(Y_1,\ldots,Y_k)$.*

4.2. When σ_i^2 are unequal and unknown

Following an identity of Weerahandi (1995), $\theta_i = (V_i/n_i)^{1/2}\bar{X}_i/S_i - n_i^{-1/2}Z_i$, where Z_i and V_i are defined in (2.6). Let $Y_i = (V_i/n_i)^{1/2}\bar{x}_i/s_i - n_i^{-1/2}Z_i$ and $R=\max_{1\le i\le k}Y_i$. Clearly R is a generalized pivotal quantity. Now, for given $1-\alpha$, there is some constant c satisfying

$$\begin{aligned} 1-\alpha &= P\left(\max_{1\le i\le k} Y_i \le c\right) \\ &= P\left\{\frac{Z_i + n_i^{1/2}c}{\sqrt{V_i/(n_i-1)}} \ge (n_i-1)^{1/2}\bar{x}_i/s_i \;\text{ for all } i=1,\ldots,k\right\} \\ &= \prod_{i=1}^k P\left\{t'_{n_i-1}(n_i^{1/2}c) \ge (n_i-1)^{1/2}\bar{x}_i/s_i\right\}. \end{aligned} \tag{4.2}$$

Then

$$(-\infty, c_{1-\alpha}(\bar{x}_1,\ldots,\bar{x}_k,s_1,\ldots,s_k)) \tag{4.3}$$

is a $100(1-\alpha)\%$ GUCI for $\max_{1\le i\le k}\theta_i$, where $c_{1-\alpha}(\bar{x}_1,\ldots,\bar{x}_k,s_1,\ldots,s_k)$ is the value c satisfying (4.2) and, by the following, is equivalent to a $100(1-\alpha)\%$ Bayesian upper CI.

Theorem 4.2. *Consider the improper prior $p(\mu_1,\ldots,\mu_k,\sigma_1^2,\ldots,\sigma_k^2)\propto\prod_{i=1}^k\sigma_i^{-2}$. The joint posterior distribution of $(\mu_1/\sigma_1,\ldots,\mu_k/\sigma_k)|\,\boldsymbol{x}$ is equivalent to the joint distribution of $(Y_1,\ldots,Y_k)$.*

The proof is similar to that of Theorem 4.1 and is omitted.

5. Real Data

Table 1 presents repeated determinations of bilirubin, a red bile pigment at intervals of a week or more in the serum of healthy young men. Data are from Table 10.1 of Bliss (1967) and originated from Drill (1947).

Table 1. Concentration of bilirubin in serum samples from each of 8 young men, in units of milligrams per milliliter, listed in order of size. (Drill, 1947.)

Term	A	B	C	D	E	F	G	H
	0.14	0.20	0.32	0.41	0.61	0.53	0.61	0.48
	0.20	0.27	0.41	0.68	0.61	0.55	0.83	0.68
	0.23	0.32	0.41	0.68	0.68	0.68	0.83	0.75
	0.27	0.34	0.55	0.68	0.68	0.75	0.89	0.96
	0.27	0.34	0.55	0.68	0.74	0.79	0.96	1.03
$\boldsymbol{x}$	0.34	0.38	0.62	0.75	0.75	0.82	0.96	1.23
	0.41	0.41	0.71	0.75	0.75	0.82	1.10	1.30
	0.41	0.41	0.91	0.98	0.82	1.16	1.10	1.30
	0.55	0.48		1.00	0.83	1.23	1.44	1.30
	0.61	0.55		1.03	1.03		1.51	1.51
	0.66				1.16			
n_i	11	10	8	10	11	9	10	10
$\bar{x}_i$	0.372	0.370	0.560	0.764	0.787	0.814	1.023	1.054
s_i	0.165	0.095	0.177	0.181	0.163	0.228	0.263	0.315
$\bar{x}_i + s_i\Phi^{-1}(.9)$	0.583	0.493	0.787	0.996	0.996	1.106	1.361	1.458
$\bar{x}_i/s_i$	2.253	3.879	3.159	4.225	4.835	3.580	3.884	3.347

From Table 1, it can be seen that the sample standard deviation is large when the sample mean is large. To test homogeneity of variances, apply Bartlett's test and find a p-value of 0.0441. Thus, significant differences in the variances at the 0.05 significance level. Applying the proposed methods at (2.8), (3.3), and (4.3), Table 2 gives the associated generalized upper confidence bounds for the largest mean, the largest 0.9-quantile, and the largest signal-to-noise ratio at various confidence levels.

Table 2. Generalized upper confidence bounds for the largest mean, 0.9-quantile, and signal-to-noise ratio.

	confidence level		
	0.9	0.95	0.99
mean	1.2190	1.2632	1.3639
0.9-quantile	1.7752	1.8702	2.1016
signal-to-noise ratio	6.0856	6.4458	7.1805

Acknowledgements

The authors would like to thank a referee for helpful comments and suggestions. This research was partially supported by NSC 88-2118-M-001-004 and NSC 89-2118-M-031-001 from the National Science Council, R.O.C.

Appendix

Proof of Theorem 2.2. Since $\prod_{i=1}^{k}\{1-\Phi(n_i^{1/2}(\bar{x}_i-c)/\sigma)\}$ is increasing in c,

$$\begin{aligned}&\prod_{i=1}^{k}\left(1-\Phi\left\{\frac{n_i^{1/2}}{\sigma}\left[\bar{x}_i-\max_{1\le i\le k}\left(\bar{x}_i+n_i^{-1/2}\sigma\Phi^{-1}\left((1-\alpha)^{1/k}\right)\right)\right]\right\}\right)\\ &\ge\prod_{i=1}^{k}\left\{1-\Phi\left[-\Phi^{-1}\left((1-\alpha)^{1/k}\right)\right]\right\}\\ &=1-\alpha.\end{aligned}$$

It is noted that the equality holds if and only if $\bar{x}_i+n_i^{-1/2}\sigma\Phi^{-1}((1-\alpha)^{1/k})$ are all equal.

Proof of Theorem 2.3. By the technique of Lee (1997), the joint posterior distribution is

$$\begin{aligned}&p(\mu_1,\ldots,\mu_k,\sigma^2|\,\boldsymbol{x})\\ &\propto(\sigma^2)^{-1-n^*/2}\exp\left\{-\frac{1}{2\sigma^2}n^*s_p^2\right\}(\sigma^2)^{-k/2}\exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{k}n_i(\mu_i-\bar{x}_i)^2\right\}.\end{aligned}$$

Consider the transformation $\mu_i'=n_i^{1/2}(\mu_i-\bar{x}_i)/\sigma$, $i=1,\ldots,k$, and $\tau=\sigma^{-2}$. Then, conditioning on $\boldsymbol{x}$,

$$\mu_i'\sim\mathcal{N}(0,1),\qquad n^*s_p^2\tau\sim\chi^2_{n^*},\tag{A.1}$$

and $\mu_1',\ldots,\mu_k'$ and $n^*s_p^2\tau$ are independent. Therefore, by Johnson and Kotz (1972), the posterior distribution of $(n_1^{1/2}(\mu_1-\bar{x}_1)/s_p,\ldots,n_k^{1/2}(\mu_k-\bar{x}_k)/s_p)|\,\boldsymbol{x}$ is a k-variate t-distribution with n^* degrees of freedom and identity correlation matrix.

Proof of Theorem 2.4. When $n_1=\cdots=n_k=n$, (2.4) becomes

$$1-\alpha=P\left(\frac{Z_i}{\sqrt{V/(k(n-1))}}\ge n^{1/2}\left(\frac{\bar{x}_i-c}{s_p}\right)\quad\text{for all } i=1,\ldots,k\right).$$

Since the distribution of the random vector $(Z_1\{V/(k(n-1))\}^{-1/2},\ldots,Z_k\{V/(k(n-1))\}^{-1/2})$ has a k-variate t distribution with $k(n-1)$ degrees of freedom and $P(Z_i\{V/(k(n-1))\}^{-1/2} \geq n^{1/2}(\bar{x}_i - c)/s_p$ for all $i = 1, \ldots, k)$ is increasing in c,

$$
\begin{aligned}
&P\left(\frac{Z_i}{\sqrt{V/(k(n-1))}} \geq \frac{n^{1/2}}{s_p}\left\{\bar{x}_i - \max_{1\leq i\leq k}\left(\bar{x}_i + n^{-1/2}s_p F^{-1}_{k,k(n-1)}(1-\alpha)\right)\right\}\right. \\
&\qquad\qquad \left. \text{for all } i = 1, \ldots, k\right) \\
&\geq P\left(\frac{Z_i}{\sqrt{V/(k(n-1)}} \geq -F^{-1}_{k,k(n-1)}(1-\alpha) \;\; \text{for all } i = 1, \ldots, k\right) \\
&= 1-\alpha,
\end{aligned}
$$

and the equality holds if and only if $\bar{x}_1 = \cdots = \bar{x}_k$.

Proof of Theorem 3.1. Similar to the proof of Theorem 2.1, by (A.1), the posterior distribution of

$$
\left(\frac{n_1^{1/2}(\theta_1^p - \bar{x}_1)}{s_p}, \ldots, \frac{n_k^{1/2}(\theta_k^p - \bar{x}_k)}{s_p}\right)\Bigg|\boldsymbol{x} = \left(\frac{\mu_1' + n_1^{1/2}\Phi^{-1}(p)}{(s_p^2\tau)^{1/2}}, \ldots, \frac{\mu_k' + n_k^{1/2}\Phi^{-1}(p)}{(s_p^2\tau)^{1/2}}\right)\Bigg|\boldsymbol{x}
$$

is the k-variate noncentral t-distribution with n^* degrees of freedom and noncentral parameter vector $(n_1^{1/2}\Phi^{-1}(p), \ldots, n_k^{1/2}\Phi^{-1}(p))$.

Proof of Theorem 4.1. It is similar to the proof of Theorem 3.1. Note that $\mu_i/\sigma = (n^* s_p^2\tau/n^*)^{1/2}\bar{x}_i/s_p - (n_i\tau)^{1/2}(\bar{x}_i - \mu_i)/n_i^{1/2}$. By (A.1), the proof is complete.

References

Alam, K., Saxena, K. M. L. and Tong, Y. L. (1973). Optimal confidence interval for a ranked parameter. *J. Amer. Statist. Assoc.* **68**, 720-725.

Alam, K. and Saxena, K. M. L. (1974). On interval estimation of a ranked parameter. *J. Roy. Statist. Soc. Ser. B* **36**, 277-283.

Bliss, C. I. (1967). *Statistics in Biology, Volume* 1. McGraw-Hill, New York.

Box, G. (1988). Signal-to-noise ratios, performance criteria and transformations. *Technometrics* **30**, 1-17.

Chen, H. J. and Dudewicz, E. J. (1973a). Interval estimation of a ranked mean of k normal populations with unequal sample sizes. Technical Report No. 73-21, Dept. of Math. Sciences, Memphis State University, Memphis, Tennessee.

Chen, H. J. and Dudewicz, E. J. (1973b). Estimation of ordered parameters and subset selection. Technical Report, Division of Statistics, Ohio State University, Columbus, Ohio.

Drill, V. A. (1947). Original data.

Dudewicz, E. J. (1972). Two-sided confidence intervals for a ranked means. *J. Amer. Statist. Assoc.* **67**, 462-464.

Gupta, S. S. and Panchapakesan, S. (1979). *Multiple Decision Procedures.* Wiley, New York.

Johnson, N. L. and Kotz, S. (1972). *Distribution in Statistics: Continuous Multivariate Distributions.* Wiley, New York.

Lee, P. M. (1997). *Bayesian Statistics: An Introduction.* Arnold, London.

Meng, X. L. (1994). Posterior predictive p-values. *Ann. Statist.* **22**, 1142-1160.

Weerahandi, S. (1993). Generalized confidence intervals. *J. Amer. Statist. Assoc.* **88**, 899-905, (correction in 89, 726).

Weerahandi, S. (1995). *Exact Statistical Methods for Data Analysis.* Springer-Verlag, New York.

Department of Business Mathematics, Soochow University, Taipei, Taiwan.

E-mail: ypchang@bmath.scu.edu.tw

Department of Management Sciences, Tamkang University, Tamsui, Taiwan.

E-mail : wentao@mail.tku.edu.tw

(Received September 1998; accepted March 2000)