Statistica Sinica: Supplement

CENTRAL LIMIT THEOREMS FOR DIRECTIONAL AND LINEAR RANDOM VARIABLES WITH APPLICATIONS

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Supplementary Material

This supplement is organized as follows. Section S1 contains the detailed proofs of the required technical lemmas used to prove the main results in the paper. The section is divided into four subsections to classify the lemmas used in the CLT of the ISE, the independence test and the goodness-of-fit test, with an extra subsection for general purpose lemmas. Section S2 presents closed expressions that can be used in the independence test, the extension of the results to the directional-directional situation and some numerical experiments to illustrate the convergence to the asymptotic distribution. Section S3 describes in detail the simulation study of the goodness-of-fit test to allow its reproducibility: parametric models employed, estimation and simulation methods, the construction of the alternatives, the bandwidth choice and further results omitted in the paper. Finally, Section S4 shows deeper insights on the real data application.

S1 Technical lemmas

S1.1 CLT for the ISE

Lemma 1 presents a generalization of Theorem 1 in Hall (1984) for degenerate U-statistics that, up to the authors' knowledge, was first stated by Zhao and Wu (2001) under different conditions, but without providing a formal proof. This lemma, written under a general notation, is used to prove asymptotic convergence of the ISE when the variance is large relative to the bias ($n\phi(h,g)h^q g \to 0$) and when the bias is balanced with the variance ($n\phi(h,g)h^q g \to \delta$).

Lemma 1. *Let $\{X_i\}_{i=1}^n$ be a sequence of independent and identically distributed random* *variables. Assume that $H_n(x,y)$ is symmetric in x and y,*

$$\mathbb{E}\left[H_n\left(X_1,X_2\right)|X_1\right]=0 \text{ almost surely and } \mathbb{E}\left[H_n^4(X_1,X_2)\right]<\infty, \forall n. \tag{S1.1}$$

Define $G_n\left(x,y\right)=\mathbb{E}\left[H_n\left(x,X_1\right)H_n\left(y,X_1\right)\right]$ and φ_n, satisfying $\mathbb{E}\left[\varphi_n(X_1)\right]=0$ and $\mathbb{E}\left[\varphi_n^4(X_1)\right]<\infty$. Define also:

$$\begin{aligned}
M_n(X_1) &= \mathbb{E}\left[\varphi_n(X_2)H_n(X_1,X_2)|X_1\right],\\
A_n &= n\mathbb{E}\left[\varphi_n^4(X_1)\right]+n^2\mathbb{E}\left[M_n^2(X_1)\right]+n^3\mathbb{E}\left[H_n^4(X_1,X_2)\right]+n^4\mathbb{E}\left[G_n^2(X_1,X_2)\right],\\
B_n &= n\mathbb{E}\left[\varphi_n^2(X_1)\right]+\frac{1}{2}n^2\mathbb{E}\left[H_n^2(X_1,X_2)\right].
\end{aligned}$$

If $A_nB_n^{-2}\to 0$ as $n\to\infty$ and $U_n=\sum_{i=1}^n\varphi_n(X_i)+\sum_{1\leq i<j\leq n}H_n\left(X_i,X_j\right)$,

$$B_n^{-\frac{1}{2}}U_n\stackrel{d}{\longrightarrow}\mathcal{N}(0,1).$$

Note that when $\varphi_n\equiv 0$, U_n is an U-statistic and Theorem 1 in Hall (1984) is a particular case of Lemma 1.

Proof of Lemma 1. To begin with, let consider the sequence of random variables $\{Y_{n_i}\}_{i=1}^n$, defined by

$$Y_{n_i}=\begin{cases}\varphi_n(X_1), & i=1,\\ \varphi_n(X_i)+\sum_{j=1}^{i-1}H_n(X_i,X_j), & 2\leq i\leq n.\end{cases}$$

This sequence generates a martingale $S_i=\sum_{j=1}^iY_{n_j}$, $1\leq i\leq n$ with respect to the sequence of random variables $\{X_i\}_{i=1}^n$, with differences Y_{n_i} and with $S_n=U_n$. To see that $S_i=\sum_{j=1}^iY_{n_j}$, $1\leq i\leq n$ is indeed a martingale with respect to $\{X_i\}_{i=1}^n$, recall that

$$\begin{aligned}
\mathbb{E}\left[S_{i+1}|X_1,\ldots,X_i\right] &= \sum_{j=1}^{i+1}\mathbb{E}\left[\varphi_n(X_j)|X_1,\ldots,X_i\right]+\sum_{j=1}^{i+1}\sum_{k=1}^{j-1}\mathbb{E}\left[H_n(X_j,X_k)|X_1,\ldots,X_i\right]\\
&= \sum_{j=1}^{i}\varphi_n(X_j)+\sum_{j=1}^{i}\sum_{k=1}^{j-1}H_n(X_j,X_k)\\
&= S_i
\end{aligned}$$

because of the null expectations of $\mathbb{E}\left[\varphi_n(X)\right]$ and $\mathbb{E}\left[H_n\left(X_1,X_2\right)|X_1\right]$.

The main idea of the proof is to apply the martingale CLT of Brown (1971) (see also Theorem 3.2 of Hall and Heyde (1980)), in the same way as Hall (1984) did for the particular case where $\varphi_n\equiv 0$. Theorem 2 of Brown (1971) ensures that if the conditions

C1. $\lim_{n\to\infty} s_n^{-2}\sum_{i=1}^n \mathbb{E}\left[Y_{n_i}^2 \mathbb{1}_{\{|Y_{n_i}|>\varepsilon s_n\}}\right] = 0$, $\forall\varepsilon > 0$,

C2. $s_n^{-2}V_n^2 \xrightarrow{p} 1$,

are satisfied, with $s_n^2 = \mathbb{E}\left[U_n^2\right]$ and $V_n^2 = \sum_{i=1}^n \mathbb{E}\left[Y_{n_i}^2|X_1,\ldots,X_{i-1}\right]$, then $s_n^{-1}U_n \xrightarrow{d} \mathcal{N}(0,1)$. The aim of this proof is to prove separately both conditions. From now on, expectations will be taken with respect to the random variables $X_1,\ldots,X_n$, except otherwise is stated.

*Proof of **C1***. The key idea is to give bounds for $\mathbb{E}\left[Y_{n_i}^4\right]$ and prove that $s_n^{-4}\sum_{i=1}^n \mathbb{E}\left[Y_{n_i}^4\right] \to 0$ as $n\to\infty$. In that case, the Lindenberg's condition **C1** follows immediately:

$$\begin{aligned}\lim_{n\to\infty} s_n^{-2}\sum_{i=1}^n \mathbb{E}\left[Y_{n_i}^2 \mathbb{1}_{\{|Y_{n_i}|>\varepsilon s_n\}}\right] &\leq \lim_{n\to\infty} s_n^{-2}\sum_{i=1}^n \mathbb{E}\left[Y_{n_i}^4\varepsilon^{-2}s_n^{-2}\times 1\right]\\ &= \varepsilon^{-2}\lim_{n\to\infty} s_n^{-4}\sum_{i=1}^n \mathbb{E}\left[Y_{n_i}^4\right]\\ &= 0.\end{aligned}$$

In order to compute $s_n^2 = \mathbb{E}\left[U_n^2\right]$, it is needed

$$\mathbb{E}\left[Y_{n_i}^2\right] = \begin{cases} \mathbb{E}\left[\varphi_n^2(X_1)\right], & i=1,\\ \mathbb{E}\left[\varphi_n^2(X_i)\right] + (i-1)\mathbb{E}\left[H_n^2(X_1,X_2)\right], & 2\leq i\leq n,\end{cases}$$

where the second case holds because the independence of the variables, the tower property of the conditional expectation and (S1.1) ensure that

$$\mathbb{E}\left[\varphi_n(X_1)H_n(X_1,X_2)\right] = \mathbb{E}\left[H_n(X_1,X_2)H_n(X_1,X_3)\right] = 0.$$

Using these relations and the null expectation of $\varphi_n(X_1)$, it follows that for $j\neq k$,

$$\begin{aligned}\mathbb{E}\left[Y_{n_j}Y_{n_k}\right] =& \mathbb{E}\left[\varphi_n(X_j)\right]\mathbb{E}\left[\varphi_n(X_k)\right] + \sum_{l=1}^{k-1}\mathbb{E}\left[\varphi_n(X_j)H_n(X_k,X_l)\right]\\ &+\sum_{m=1}^{j-1}\mathbb{E}\left[\varphi_n(X_k)H_n(X_j,X_m)\right] + \sum_{l=1}^{k-1}\sum_{m=1}^{j-1}\mathbb{E}\left[H_n(X_k,X_l)H_n(X_j,X_m)\right]\\ =& 0.\end{aligned}$$

Then:

$$s_n^2 = n\mathbb{E}\left[\varphi_n^2(X_1)\right] + \sum_{j=1}^n (j-1)\mathbb{E}\left[H_n^2(X_1,X_2)\right] = \mathcal{O}\left(B_n\right). \tag{S1.2}$$

On the other hand,

$$\begin{aligned}\mathbb{E}\left[Y_{n_i}^4\right] =&\mathbb{E}\bigg[\bigg(\varphi_n(X_i)+\sum_{j=1}^{i-1}H_n(X_i,X_j)\bigg)^4\bigg]\\ =&\mathcal{O}\left(\mathbb{E}\left[\varphi_n^4(X_i)\right]\right)+\mathcal{O}\bigg(\mathbb{E}\bigg[\bigg(\sum_{j=1}^{i-1}H_n(X_i,X_j)\bigg)^4\bigg]\bigg)\\ =&\mathcal{O}\left(\mathbb{E}\left[\varphi_n^4(X_1)\right]\right)+(i-1)\mathcal{O}\left(\mathbb{E}\left[H_n^4(X_1,X_2)\right]\right)\\ &+3(i-1)(i-2)\mathcal{O}\left(\mathbb{E}\left[H_n^2(X_1,X_2)H_n^2(X_1,X_3)\right]\right),\end{aligned}$$

where the equalities are true in virtue of Lemma 12 and because

$$\mathbb{E}\big[H_n\left(X_1,X_2\right)H_n\left(X_1,X_3\right)H_n\left(X_1,X_4\right)H_n\left(X_1,X_5\right)\big]=\mathbb{E}\left[H_n^3\left(X_1,X_2\right)H_n\left(X_1,X_3\right)\right]=0.$$

Finally,

$$\begin{aligned}\sum_{i=1}^n\mathbb{E}\left[Y_{n_i}^4\right]=&n\mathcal{O}\left(\mathbb{E}\left[\varphi_n^4(X_1)\right]\right)+\frac{1}{2}n(n-1)\mathcal{O}\left(\mathbb{E}\left[H_n^4(X_1,X_2)\right]\right)\\ &+(n^3-n)\mathcal{O}\left(\mathbb{E}\left[G_n^2(X_1,X_2)\right]\right)\\ =&\mathcal{O}\left(A_n\right).\end{aligned}\tag{S1.3}$$

Then, joining (S1.2) and (S1.3),

$$s_n^{-4}\sum_{i=1}^n\mathbb{E}\left[Y_{n_i}^4\right]=\mathcal{O}\left(B_n^{-2}A_n\right)\xrightarrow[n\to\infty]{}0$$

and **C1** is satisfied.

*Proof of **C2**.* Now it is proved the convergence in squared mean of $s_n^{-2}V_n^2$ to 1, which implies that $s_n^{-2}V_n^2\xrightarrow{p}1$, by obtaining bounds for $\mathbb{E}\left[V_n^4\right]$.

First of all, let denote $V_n^2=\sum_{i=1}^n\nu_{n_i}$, where

$$\begin{aligned}\nu_{n_i}=&\mathbb{E}\left[Y_{n_i}^2|X_1,\ldots,X_{i-1}\right]\\ =&\mathbb{E}\bigg[\varphi_n^2(X_i)+2\varphi_n(X_i)\sum_{j=1}^{i-1}H_n(X_i,X_j)\\ &+\sum_{j=1}^{i-1}\sum_{k=1}^{i-1}H_n(X_i,X_j)H_n(X_i,X_k)\bigg|X_1,\ldots,X_{i-1}\bigg]\\ =&\mathbb{E}\left[\varphi_n^2(X_i)\right]+2\sum_{j=1}^{i-1}M_n(X_j)+\sum_{j=1}^{i-1}\sum_{k=1}^{i-1}\mathbb{E}\left[H_n(X_i,X_j)H_n(X_i,X_k)|X_j,X_k\right]\end{aligned}$$

$$= \mathbb{E}\left[\varphi_n^2(X_1)\right] + 2\sum_{j=1}^{i-1} M_n(X_j) + \sum_{j=1}^{i-1} G_n(X_j, X_j) + 2\sum_{1\leq j<k\leq i-1} G_n(X_j, X_k).$$

Using Lemma 12, the Jensen inequality and that for $j_1 \leq k_1$, $j_2 \leq k_2$,

$$\mathbb{E}\left[G_n(X_{j_1}, X_{k_1}) G_n(X_{j_2}, X_{k_2})\right] = \begin{cases} \mathbb{E}\left[G_n^2(X_1, X_1)\right], & j_1 = k_1 = j_2 = k_2, \\ \mathbb{E}\left[G_n(X_1, X_1)\right]^2, & j_1 = k_1 \neq j_2 = k_2, \\ \mathbb{E}\left[G_n^2(X_1, X_2)\right], & j_1 = j_2 < k_1 = k_2, \\ 0, & \text{otherwise,} \end{cases}$$

it follows:

$$\begin{aligned} \mathbb{E}\left[\nu_{n_i}^2\right] =& \mathcal{O}\left(\mathbb{E}\left[\varphi_n^2(X_1)\right]^2\right) + \sum_{j=1}^{i-1}\sum_{k=1}^{i-1} \mathcal{O}\left(\mathbb{E}\left[M_n(X_j) M_n(X_k)\right]\right) \\ &+ \sum_{j=1}^{i-1}\sum_{k=1}^{i-1} \mathcal{O}\left(\mathbb{E}\left[G_n(X_j, X_j) G_n(X_k, X_k)\right]\right) \\ &+ \sum_{\substack{1\leq j_1<k_1\leq i-1 \\ 1\leq j_2<k_2\leq i-1}} \mathcal{O}\left(\mathbb{E}\left[G_n(X_{j_1}, X_{k_1}) G_n(X_{j_2}, X_{k_2})\right]\right) \\ =& \mathcal{O}\left(\mathbb{E}\left[\varphi_n^4(X_1)\right]\right) + (i-1)\mathcal{O}\left(\mathbb{E}\left[M_n^2(X_1)\right]\right) \\ &+ (i-1)(i-2)\mathcal{O}\left(\mathbb{E}\left[M_n(X_1)\right]^2\right) \\ &+ (i-1)\mathcal{O}\left(\mathbb{E}\left[G_n^2(X_1, X_1)\right]\right) + (i-1)(i-2)\mathcal{O}\left(\mathbb{E}\left[G_n(X_1, X_1)\right]^2\right) \\ &+ (i-1)(i-2)\mathcal{O}\left(\mathbb{E}\left[G_n^2(X_1, X_2)\right]\right). \end{aligned}$$

Applying again the Lemma 12,

$$\mathbb{E}\left[V_n^4\right] = \mathbb{E}\left[\left(\sum_{i=1}^{n} \nu_{n_i}\right)^2\right] = \sum_{i=1}^{n} \mathcal{O}\left(\mathbb{E}\left[\nu_{n_i}^2\right]\right).$$

By the two previous computations and bearing in mind that $\mathbb{E}\left[G_n(X_1, X_1)\right]^2 = \mathcal{O}\left(\mathbb{E}\left[H_n^4(X_1, X_2)\right]\right)$ (by the Cauchy-Schwartz inequality) and $\mathbb{E}\left[M_n(X_1)\right] = 0$ (by the tower property), it yields:

$$\begin{aligned} \mathbb{E}\left[V_n^4\right] =& n\mathcal{O}\left(\mathbb{E}\left[\varphi_n^4(X_1)\right]\right) + n(n-1)\mathcal{O}\left(\mathbb{E}\left[M_n^2(X_1)\right]\right) \\ &+ n(n-1)(n-3)\mathcal{O}\left(\mathbb{E}\left[H_n^4(X_1, X_2)\right]\right) \\ &+ n(n-1)(n-3)\mathcal{O}\left(\mathbb{E}\left[G_n^2(X_1, X_2)\right]\right) \\ =& \mathcal{O}\left(A_n\right). \end{aligned}$$

Then, using the bound for $\mathbb{E}\left[V_n^4\right]$, that $s_n^2 = B_n$ and that $\mathbb{E}\left[V_n^2\right] = s_n^2$, it results

$$\mathbb{E}\left[\left(s_n^{-2} V_n^2 - 1\right)^2\right] = s_n^{-4}\mathbb{E}\left[\left(V_n^2 - s_n^2\right)^2\right] = s_n^{-4}\left(\mathbb{E}\left[V_n^4\right] - s_n^4\right) \leq s_n^{-4}\mathbb{E}\left[V_n^4\right] = \mathcal{O}\left(B_n^{-2} A_n\right).$$

Then $s_n^{-2}V_n^2$ converges to 1 in squared mean, which implies $s_n^{-2}V_n^2 \xrightarrow{p} 1$. □

Lemma 2. *Under **A1**–**A3**,*

$$n^{\frac{1}{2}}\phi(h,g)^{-\frac{1}{2}}I_{n,1} \xrightarrow{d} \mathcal{N}(0,1).$$

Proof of Lemma 2. The asymptotic normality of $I_{n,1}=\sum_{i=1}^n I_{n,1}^{(i)}$ will be derived checking the Lindenberg's condition. To that end, it is needed to prove the following relations:

$$\mathbb{E}\left[I_{n,1}^{(i)}\right]=0,\quad \mathbb{E}\left[\left(I_{n,1}^{(i)}\right)^2\right]=n^{-2}\phi(h,g)(1+o(1)),$$
$$\mathbb{E}\left[\left(I_{n,1}^{(i)}\right)^4\right]=\mathcal{O}\left(n^{-4}(h^8+g^8)\right),\quad s_n^4=\mathcal{O}\left(n^{-2}(h^8+g^8)\right),$$

where $s_n^2=\sum_{i=1}^n\mathbb{E}\big[\big(I_{n,1}^{(i)}\big)^2\big]$ and $\phi(h,g)$ is defined as in Theorem 1. If these relations hold, the Lindenberg's condition

$$\lim_{n\to\infty}s_n^{-2}\sum_{i=1}^n\mathbb{E}\left[\left(I_{n,1}^{(i)}\right)^2\mathbb{1}_{\left\{\left|I_{n,1}^{(i)}\right|>\varepsilon s_n\right\}}\right]=0,\quad \forall\varepsilon>0$$

is satisfied:

$$\begin{aligned}s_n^{-2}\sum_{i=1}^n\mathbb{E}\left[\left(I_{n,1}^{(i)}\right)^2\mathbb{1}_{\left\{\left|I_{n,1}^{(i)}\right|>\varepsilon s_n\right\}}\right]&\leq\sum_{i=1}^n\mathbb{E}\left[\left(I_{n,1}^{(i)}\right)^4\varepsilon^{-2}s_n^{-4}\times 1\right]\\&=\varepsilon^{-2}n\mathbb{E}\left[\left(I_{n,1}^{(i)}\right)^4\right]\mathcal{O}\left(n^2(h^8+g^8)^{-1}\right)\\&=\varepsilon^{-2}\mathcal{O}\left(n^{-1}\right).\end{aligned}$$

Therefore $s_n^{-1}I_{n,1}\xrightarrow{d}\mathcal{N}(0,1)$, which, by Slutsky's theorem, implies that

$$n^{\frac{1}{2}}\phi(h,g)^{-\frac{1}{2}}I_{n,1} \xrightarrow{d} \mathcal{N}(0,1).$$

In order to prove the moment relations for $I_{n,1}^{(i)}$ and bearing in mind the smoothing operator (5.2), let denote

$$\begin{aligned}\widetilde{I}_{n,1}^{(i)}&=2\frac{c_{h,q}(L)}{ng}\int_{\Omega_q\times\mathbb{R}}LK\left(\frac{1-\mathbf{x}^T\mathbf{X}_i}{h^2},\frac{z-Z_i}{g}\right)\left(\mathbb{E}\left[\hat{f}_{h,g}(\mathbf{x},z)\right]-f(\mathbf{x},z)\right)dz\,\omega_q(d\mathbf{x}),\\&=2n^{-1}LK_{h,g}\left(\mathbb{E}\left[\hat{f}_{h,g}(\mathbf{X}_i,Z_i)\right]-f(\mathbf{X}_i,Z_i)\right)\end{aligned}$$

so that $I_{n,1}^{(i)}=\widetilde{I}_{n,1}^{(i)}-\mathbb{E}\big[\widetilde{I}_{n,1}^{(i)}\big]$. Therefore, $\mathbb{E}\big[I_{n,1}^{(i)}\big]=0$ and $\widetilde{I}_{n,1}^{(i)}$ can be decomposed in two addends by virtue of Lemma 11:

$$\widetilde{I}_{n,1}^{(i)}=2n^{-1}LK_{h,g}\bigg(\frac{b_q(L)}{q}\mathrm{tr}\left[\boldsymbol{\mathcal{H}}_{\mathbf{x}}f(\mathbf{X}_i,Z_i)\right]h^2+\frac{\mu_2(K)}{2}\mathcal{H}_zf(\mathbf{X}_i,Z_i)g^2\bigg)+o\left(n^{-1}(h^2+g^2)\right)$$

$$= \widetilde{I}_{n,1}^{(i,1)} + \widetilde{I}_{n,1}^{(i,2)} + o\left(n^{-1}(h^2+g^2)\right),$$

where $\widetilde{I}_{n,1}^{(i,j)} = \delta_j LK_{h,g}\varphi_j(f,\mathbf{X}_i,Z_i)$ and

$$\varphi_j(f,\mathbf{x},z) = \begin{cases} \operatorname{tr}\left[\boldsymbol{\mathcal{H}}_{\mathbf{x}} f(\mathbf{x},z)\right], & j=1, \\ \mathcal{H}_z f(\mathbf{x},z), & j=2, \end{cases} \quad \delta_j = \begin{cases} \frac{2b_q(L)}{q} h^2 n^{-1}, & j=1, \\ \mu_2(K) g^2 n^{-1}, & j=2. \end{cases}$$

Note that as the order $o\left(h^2+g^2\right)$ is uniform in $(\mathbf{x},z)\in\Omega_q\times\mathbb{R}$, then it is possible to extract it from the integrand of $\widetilde{I}_{n,1}^{(i)}$. Applying Lemma 10 to the functions $\varphi_j(f,\cdot,\cdot)$, that by **A1** are uniformly continuous and bounded, it yields $LK_{h,g}\varphi_j(f,\mathbf{y},t)\to\varphi_j(f,\mathbf{y},t)$ uniformly in $(\mathbf{y},t)\in\Omega_q\times\mathbb{R}$ as $n\to\infty$. So, for any integers k_1 and k_2:

$$\begin{aligned} \lim_{n\to\infty} \delta_1^{-k_1}\delta_2^{-k_2}\mathbb{E}\left[\left(\widetilde{I}_{n,1}^{(i,1)}\right)^{k_1}\left(\widetilde{I}_{n,1}^{(i,2)}\right)^{k_2}\right] \\ &= \lim_{n\to\infty}\int_{\Omega_q\times\mathbb{R}} \left(LK_{h,g}\varphi_1(f,\mathbf{y},t)\right)^{k_1}\left(LK_{h,g}\varphi_2(f,\mathbf{y},t)\right)^{k_2} f(\mathbf{y},t)\,dt\,\omega_q(d\mathbf{y}) \\ &= \int_{\Omega_q\times\mathbb{R}} \varphi_1(f,\mathbf{y},t)^{k_1}\varphi_2(f,\mathbf{y},t)^{k_2} f(\mathbf{y},t)\,dt\,\omega_q(d\mathbf{y}) \\ &= \mathbb{E}\left[\varphi_1(f,\mathbf{X},Z)^{k_1}\varphi_2(f,\mathbf{X},Z)^{k_2}\right]. \end{aligned}$$

Here the limit can commute with the integral by the Dominated Convergence Theorem (DCT), since the functions $\left(LK_{h,g}\varphi_j(f,\mathbf{y},t)\right)^k$ are bounded by condition **A1** and the construction of the smoothing operator (5.2), being this dominating function integrable:

$$\begin{aligned} &\left(LK_{h,g}\varphi_1(f,\mathbf{y},t)\right)^{k_1}\left(LK_{h,g}\varphi_2(f,\mathbf{y},t)\right)^{k_2} f(\mathbf{y},t) \\ &\qquad \leq \sup_{(\mathbf{x},z)\in\Omega_q\times\mathbb{R}} \left|\varphi_1(f,\mathbf{x},z)^{k_1}\varphi_2(f,\mathbf{x},z)^{k_2}\right| f(\mathbf{y},t). \end{aligned}$$

Recapitulating, the relation obtained is:

$$\begin{aligned} \mathbb{E}\left[\left(\widetilde{I}_{n,1}^{(i,1)}\right)^{k_1}\left(\widetilde{I}_{n,1}^{(i,2)}\right)^{k_2}\right] &\sim 2^{k_1} n^{-(k_1+k_2)}\frac{b_q(L)^{k_1}}{q^{k_1}}\mu_2(K)^{k_2} h^{2k_1} g^{2k_2} \\ &\quad \times \mathbb{E}\left[\operatorname{tr}\left[\boldsymbol{\mathcal{H}}_{\mathbf{x}}(f,\mathbf{X},Z)\right]^{k_1}\mathcal{H}_z f(\mathbf{X},Z)^{k_2}\right]. \end{aligned}$$

Now it is easy to prove:

$$\begin{aligned} \mathbb{E}\left[\left(I_{n,1}^{(i)}\right)^2\right] &\sim \mathbb{E}\left[\left(\widetilde{I}_{n,1}^{(i,1)}+\widetilde{I}_{n,1}^{(i,2)}\right)^2\right] - \left(\mathbb{E}\left[\widetilde{I}_{n,1}^{(i,1)}\right]+\mathbb{E}\left[\widetilde{I}_{n,1}^{(i,2)}\right]\right)^2 \\ &= \mathbb{E}\left[\left(\widetilde{I}_{n,1}^{(i,1)}\right)^2\right] + \mathbb{E}\left[\left(\widetilde{I}_{n,1}^{(i,2)}\right)^2\right] - 2\mathbb{E}\left[\widetilde{I}_{n,1}^{(i,1)}\widetilde{I}_{n,1}^{(i,2)}\right] \\ &\quad - \mathbb{E}\left[\widetilde{I}_{n,1}^{(i,1)}\right]^2 - \mathbb{E}\left[\widetilde{I}_{n,1}^{(i,2)}\right]^2 - 2\mathbb{E}\left[\widetilde{I}_{n,1}^{(i,1)}\right]\mathbb{E}\left[\widetilde{I}_{n,1}^{(i,2)}\right] \\ &\sim n^{-2}\Bigg(\frac{4b_q(L)^2}{q^2}\mathbb{V}\mathrm{ar}\left[\operatorname{tr}\left[\boldsymbol{\mathcal{H}}_{\mathbf{x}}(f,\mathbf{X},Z)\right]\right]h^4 + \mu_2(K)^2\mathbb{V}\mathrm{ar}\left[\mathcal{H}_z f(\mathbf{X},Z)\right]g^4 \end{aligned}$$

$$+\frac{4b_q(L)\mu_2(K)}{q}\mathbb{C}\mathrm{ov}\left[\mathrm{tr}\left[\boldsymbol{\mathcal{H}}_{\mathbf{x}}(f,\mathbf{X},Z)\right],\mathcal{H}_z f(\mathbf{X},Z)\right]h^2g^2\Bigg)$$
$$= n^{-2}\phi(h,g).$$

With the previous expression, it follows $\mathbb{E}\big[\big(I_{n,1}^{(i)}\big)^2\big] = \mathcal{O}\left(n^{-2}(h^4+g^4)\right)$ (see the first point of Lemma 12) and $s_n^2 = n^{-1}\phi(h,g)(1+o(1)) = \mathcal{O}\left(n^{-1}(h^4+g^4)\right)$. Then by the fourth point of Lemma 12:

$$s_n^4 = \left(s_n^2\right)^2 = \mathcal{O}\left(n^{-2}(h^4+g^4)^2\right) = \mathcal{O}\left(n^{-2}(h^8+g^8)\right),$$
$$\mathbb{E}\left[\big(I_{n,1}^{(i)}\big)^4\right] = \mathcal{O}\left(\mathbb{E}\left[\big(\widetilde{I}_{n,1}^{(i)}\big)^4\right] + \mathbb{E}\left[\widetilde{I}_{n,1}^{(i)}\right]^4\right) = \mathcal{O}\left(\mathbb{E}\left[\big(\widetilde{I}_{n,1}^{(i)}\big)^4\right]\right),$$

where

$$\mathbb{E}\left[\big(\widetilde{I}_{n,1}^{(i)}\big)^4\right] = \mathcal{O}\left(\mathbb{E}\left[\big(\widetilde{I}_{n,1}^{(i)}\big)^4\right] + \mathbb{E}\left[\big(\widetilde{I}_{n,1}^{(i)}\big)^4\right]\right) = \mathcal{O}\left(n^{-4}(h^8+g^8)\right).$$

□

Lemma 3. *Under* ***A1***–***A3****,*

$$I_{n,2} = \mathbb{E}\left[I_{n,2}\right] + \mathcal{O}_{\mathbb{P}}\left(n^{-\frac{3}{2}}h^{-q}g^{-1}\right) = \frac{\lambda_q(L^2)\lambda_q(L)^{-2}R(K)}{nh^qg} + \mathcal{O}_{\mathbb{P}}\left(n^{-\frac{3}{2}}h^{-q}g^{-1}\right).$$

Proof of Lemma 3. To prove the result the Chebychev inequality will be used. To that end, the expectation and variance of $I_{n,2} = \frac{c_{h,q}(L)^2}{n^2g^2}\sum_{i=1}^n I_{n,2}^{(i)}$ have to be computed. But first recall that, by Lemma 10 and (2.1), for i and j naturals,

$$\int_{\Omega_q\times\mathbb{R}} LK^j\left(\frac{1-\mathbf{x}^T\mathbf{y}}{h^2},\frac{z-t}{g}\right)\varphi^i(\mathbf{y},t)\,dt\,\omega_q(d\mathbf{y}) \sim h^qg\lambda_q(L^j)\varphi^i(\mathbf{x},z), \tag{S1.4}$$

uniformly in $(\mathbf{x},z)\in\Omega_q\times\mathbb{R}$, with φ a uniformly continuous and bounded function and $\lambda_q(L^j) = \omega_{q-1}2^{\frac{q}{2}-1}\int_0^\infty L^j(r)r^{\frac{q}{2}-1}\,dr$. The following particular cases of this relation are useful to shorten the next computations:

i. $\mathbb{E}\big[LK\big(\frac{1-\mathbf{x}^T\mathbf{X}}{h^2},\frac{z-Z}{g}\big)\big] \sim h^qg\lambda_q(L)f(\mathbf{x},z)$,

ii. $\int_{\Omega_q\times\mathbb{R}} LK^2\big(\frac{1-\mathbf{x}^T\mathbf{y}}{h^2},\frac{z-t}{g}\big)\,dz\,\omega_q(d\mathbf{x}) \sim h^qg\lambda_q(L^2)R(K)$.

Expectation of $I_{n,2}$. The expectation is divided in two addends, which can be computed by applying the relations *i*–*ii*:

$$\mathbb{E}\left[I_{n,2}^{(i)}\right] = \mathbb{E}\left[\int_{\Omega_q\times\mathbb{R}} LK_n^2\left((\mathbf{x},z),(\mathbf{X},Z)\right)\,dz\,\omega_q(d\mathbf{x})\right]$$

$$
\begin{aligned}
&= \int_{\Omega_q\times\mathbb{R}} \mathbb{E}\left[LK^2\left(\frac{1-\mathbf{x}^T\mathbf{X}}{h^2},\frac{z-Z}{g}\right)\right] dz\,\omega_q(d\mathbf{x}) \\
&\quad - \int_{\Omega_q\times\mathbb{R}} \mathbb{E}\left[LK\left(\frac{1-\mathbf{x}^T\mathbf{X}}{h^2},\frac{z-Z}{g}\right)\right]^2 dz\,\omega_q(d\mathbf{x}) \\
&= \mathbb{E}\left[\int_{\Omega_q\times\mathbb{R}} LK^2\left(\frac{1-\mathbf{x}^T\mathbf{X}}{h^2},\frac{z-Z}{g}\right) dz\,\omega_q(d\mathbf{x})\right] - h^{2q}g^2\lambda_q(L)^2R(f)(1+o(1)) \\
&= h^q g\lambda_q(L^2)R(K) + \mathcal{O}\left(h^{2q}g^2\right).
\end{aligned}
$$

Therefore, the expectation of $I_{n,2}$ is

$$
\mathbb{E}\left[I_{n,2}\right] = \frac{\lambda_q(L)^{-2}}{nh^{2q}g^2}\left(h^q g\lambda_q(L^2)R(K) + \mathcal{O}\left(h^{2q}g^2\right)\right) = \frac{\lambda_q(L^2)\lambda_q(L)^{-2}R(K)}{nh^q g} + \mathcal{O}\left(n^{-1}\right).
$$

Variance of $I_{n,2}$. For the variance it suffices to compute its order, which follows considering the third point of Lemma 12:

$$
\begin{aligned}
\mathbb{E}\left[\left(I_{n,2}^{(i)}\right)^2\right] &= \int_{\Omega_q\times\mathbb{R}}\left\{\int_{\Omega_q\times\mathbb{R}} LK_n^2\left((\mathbf{x},z),(\mathbf{y},t)\right)\,dz\,\omega_q(d\mathbf{x})\right\}^2 f(\mathbf{y},t)\,dt\,\omega_q(d\mathbf{y}) \\
&= \mathcal{O}\left(I_{n,2}^{(i,1)} + I_{n,2}^{(i,2)}\right),
\end{aligned}
$$

where the involved terms are

$$
\begin{aligned}
I_{n,2}^{(i,1)} &= \int_{\Omega_q\times\mathbb{R}}\left\{\int_{\Omega_q\times\mathbb{R}} LK^2\left(\frac{1-\mathbf{x}^T\mathbf{y}}{h^2},\frac{z-t}{g}\right) dz\,\omega_q(d\mathbf{x})\right\}^2 f(\mathbf{y},t)\,dt\,\omega_q(d\mathbf{y}), \\
I_{n,2}^{(i,2)} &= \int_{\Omega_q\times\mathbb{R}}\left\{\int_{\Omega_q\times\mathbb{R}} \mathbb{E}\left[LK\left(\frac{1-\mathbf{x}^T\mathbf{X}}{h^2},\frac{z-Z}{g}\right)\right]^2 dz\,\omega_q(d\mathbf{x})\right\}^2 f(\mathbf{y},t)\,dt\,\omega_q(d\mathbf{y}).
\end{aligned}
$$

Using relations *i*–*ii* the orders of the addends $I_{n,2}^{(i,k)}$, $k=1,2$, follow easily:

$$
\begin{aligned}
I_{n,2}^{(i,1)} &\sim \int_{\Omega_q\times\mathbb{R}}\left\{h^q g\lambda_q(L^2)R(K)\right\}^2 f(\mathbf{y},t)\,dt\,\omega_q(d\mathbf{y}) \\
&= h^{2q}g^2\lambda_q(L^2)^2R(K)^2, \\
I_{n,2}^{(i,2)} &\sim \int_{\Omega_q\times\mathbb{R}}\left\{\int_{\Omega_q\times\mathbb{R}}\left(h^q g\lambda_q(L)f(\mathbf{x},z)\right)^2\,dz\,\omega_q(d\mathbf{x})\right\}^2 f(\mathbf{y},t)\,dt\,\omega_q(d\mathbf{y}) \\
&= h^{4q}g^4\lambda_q(L)^4R(f)^2.
\end{aligned}
$$

Therefore $I_{n,2}^{(i,1)} = \mathcal{O}\left(h^{2q}g^2\right)$, $I_{n,2}^{(i,2)} = \mathcal{O}\left(h^{4q}g^4\right)$ and $\mathbb{E}\big[\big(I_{n,2}^{(i)}\big)^2\big] = \mathcal{O}\big(I_{n,2}^{(i,1)}\big) + \mathcal{O}\big(I_{n,2}^{(i,2)}\big) = \mathcal{O}\left(h^{2q}g^2\right)$. The variance of $I_{n,2}$ is

$$
\mathbb{V}\mathrm{ar}\left[I_{n,2}\right] \leq n^{-4}c_{h,q}(L)^4g^{-4}\sum_{i=1}^n \mathbb{E}\left[\left(I_{n,2}^{(i)}\right)^2\right] = \mathcal{O}\left(n^{-3}h^{-2q}g^{-2}\right),
$$

so by Chebychev's inequality

$$\mathbb{P}\left\{\left|I_{n,2}-\mathbb{E}\left[I_{n,2}\right]\right|\geq kn^{-\frac{3}{2}}h^{-q}g^{-1}\right\}\leq\frac{1}{k^2},\quad\forall k>0,$$

which, by definition, is

$$I_{n,2}=\mathbb{E}\left[I_{n,2}\right]+\mathcal{O}_{\mathbb{P}}\left(n^{-\frac{3}{2}}h^{-q}g^{-1}\right)=\frac{\lambda_q(L^2)\lambda_q(L)^{-2}R(K)}{nh^qg}+\mathcal{O}_{\mathbb{P}}\left(n^{-\frac{3}{2}}h^{-q}g^{-1}\right),$$

because $\mathcal{O}\left(n^{-1}\right)=\mathcal{O}_{\mathbb{P}}\big(n^{-\frac{3}{2}}h^{-q}g^{-1}\big)$. □

Lemma 4. *Let be*

$$\begin{aligned}
H_n\left((\mathbf{x},z),(\mathbf{y},t)\right)&=\int_{\Omega_q\times\mathbb{R}}LK_n\left((\mathbf{u},v),(\mathbf{x},z)\right)LK_n\left((\mathbf{u},v),(\mathbf{y},t)\right)\,dv\,\omega_q(d\mathbf{u}),\\
G_n\left((\mathbf{x},z),(\mathbf{y},t)\right)&=\mathbb{E}\left[H_n\left((\mathbf{X},Z),(\mathbf{x},z)\right)H_n\left((\mathbf{X},Z),(\mathbf{y},t)\right)\right],\\
M_n(\mathbf{X}_1,Z_1)&=2\frac{c_{h,q}(L)^2}{n^2g^2}\mathbb{E}\left[I_{n,1}^{(2)}H_n\left((\mathbf{X}_1,Z_1),(\mathbf{X}_2,Z_2)\right)|(\mathbf{X}_1,Z_1)\right].
\end{aligned}$$

*Then, under **A1**–**A3**,*

$$\mathbb{E}\left[H_n^2\left((\mathbf{X}_1,Z_1),(\mathbf{X}_2,Z_2)\right)\right]=h^{3q}g^3\lambda_q(L)^4\sigma^2\left(1+o\left(1\right)\right),\tag{S1.5}$$

$$\mathbb{E}\left[H_n^4\left((\mathbf{X}_1,Z_1),(\mathbf{X}_2,Z_2)\right)\right]=\mathcal{O}\left(h^{5q}g^5\right),\tag{S1.6}$$

$$\mathbb{E}\left[G_n^2\left((\mathbf{X}_1,Z_1),(\mathbf{X}_2,Z_2)\right)\right]=\mathcal{O}\left(h^{7q}g^7\right),\tag{S1.7}$$

$$\mathbb{E}\left[M_n^2(\mathbf{X}_1,Z_1)\right]=\mathcal{O}\left(n^{-6}(h^4+g^4)h^{-\frac{3q}{2}}g^{-\frac{3}{2}}\right).\tag{S1.8}$$

Proof of Lemma 4. The proof is divided in four sections.

Proof of (S1.5). $\mathbb{E}\big[H_n^2\big(\mathbf{X}_1,Z_1),(\mathbf{X}_2,Z_2)\big)\big]$ can be split into three addends:

$$\begin{aligned}
&\mathbb{E}\big[H_n^2\big(\mathbf{X}_1,Z_1),(\mathbf{X}_2,Z_2)\big)\big]\\
&\quad=\mathbb{E}\left[\left(\int_{\Omega_q\times\mathbb{R}}LK_n\left((\mathbf{x},z),(\mathbf{X}_1,Z_1)\right)LK_n\left((\mathbf{x},z),(\mathbf{X}_2,Z_2)\right)\,dz\,\omega_q(d\mathbf{x})\right)^2\right]\\
&\quad=\mathbb{E}\Bigg[\int_{\Omega_q\times\mathbb{R}}\int_{\Omega_q\times\mathbb{R}}LK_n\left((\mathbf{x},z),(\mathbf{X}_1,Z_1)\right)LK_n\left((\mathbf{x},z),(\mathbf{X}_2,Z_2)\right)\\
&\qquad\times LK_n\left((\mathbf{y},t),(\mathbf{X}_1,Z_1)\right)LK_n\left((\mathbf{y},t),(\mathbf{X}_2,Z_2)\right)\,dz\,\omega_q(d\mathbf{x})\,dt\,\omega_q(d\mathbf{y})\Bigg]\\
&\quad=\int_{\Omega_q\times\mathbb{R}}\int_{\Omega_q\times\mathbb{R}}\mathbb{E}\left[LK_n\left((\mathbf{x},z),(\mathbf{X},Z)\right)LK_n\left((\mathbf{y},t),(\mathbf{X},Z)\right)\right]^2\,dz\,\omega_q(d\mathbf{x})\,dt\,\omega_q(d\mathbf{y})\\
&\quad=\int_{\Omega_q\times\mathbb{R}}\int_{\Omega_q\times\mathbb{R}}\left(E_1((\mathbf{x},z),(\mathbf{y},t))-E_2((\mathbf{x},z),(\mathbf{y},t))\right)^2\,dz\,\omega_q(d\mathbf{x})\,dt\,\omega_q(d\mathbf{y})
\end{aligned}$$

$$= A_1 - 2A_2 + A_3,$$

where:

$$E_1((\mathbf{x},z),(\mathbf{y},t)) = \mathbb{E}\left[LK\left(\frac{1-\mathbf{x}^T\mathbf{X}}{h^2},\frac{z-Z}{g}\right)LK\left(\frac{1-\mathbf{y}^T\mathbf{X}}{h^2},\frac{t-Z}{g}\right)\right],$$
$$E_2((\mathbf{x},z),(\mathbf{y},t)) = \mathbb{E}\left[LK\left(\frac{1-\mathbf{x}^T\mathbf{X}}{h^2},\frac{z-Z}{g}\right)\right]\mathbb{E}\left[LK\left(\frac{1-\mathbf{y}^T\mathbf{X}}{h^2},\frac{t-Z}{g}\right)\right].$$

The dominant term of the three is A_1, which has order $\mathcal{O}\left(h^{3q}g^3\right)$, as it will be seen. The terms A_2 and A_3 have order $\mathcal{O}\left(h^{4q}g^4\right)$, which can be seen applying iteratively the relation (S1.4):

$$\begin{aligned}
A_2 &= \int_{\Omega_q\times\mathbb{R}}\int_{\Omega_q\times\mathbb{R}} E_1((\mathbf{x},z),(\mathbf{y},t))E_2((\mathbf{x},z),(\mathbf{y},t))\,dt\,\omega_q(d\mathbf{y})\,dz\,\omega_q(d\mathbf{x})\\
&\sim \int_{\Omega_q\times\mathbb{R}}\int_{\Omega_q\times\mathbb{R}}\left(h^q g\lambda_q(L)LK\left(\frac{1-\mathbf{x}^T\mathbf{u}}{h^2},\frac{z-t}{g}\right)f(\mathbf{y},t)\right)\\
&\quad\times\left(h^{2q}g^2\lambda_q(L)^2 f(\mathbf{x},z)f(\mathbf{y},t)\right)\,dt\,\omega_q(d\mathbf{y})\,dz\,\omega_q(d\mathbf{x})\\
&\sim h^{4q}g^4\lambda_q(L)^4\int_{\Omega_q\times\mathbb{R}} f(\mathbf{x},z)^3\,dz\,\omega_q(d\mathbf{x}),\\
A_3 &= \int_{\Omega_q\times\mathbb{R}}\int_{\Omega_q\times\mathbb{R}} L_2^2((\mathbf{x},z),(\mathbf{y},t))\,dt\,\omega_q(d\mathbf{y})\,dz\,\omega_q(d\mathbf{x})\\
&\sim \int_{\Omega_q\times\mathbb{R}}\int_{\Omega_q\times\mathbb{R}} h^{4q}g^4\lambda_q(L)^4 f(\mathbf{x},z)^2 f(\mathbf{y},t)^2\,dt\,\omega_q(d\mathbf{y})\,dz\,\omega_q(d\mathbf{x})\\
&= h^{4q}g^4\lambda_q(L)^4R(f)^2.
\end{aligned}$$

Let recall now on the term A_1. In order to clarify the following computations, let denote by $(\mathbf{x},x)$, $(\mathbf{y},y)$ and $(\mathbf{z},z)$ the three variables in $\Omega_q\times\mathbb{R}$ that play the role of $(\mathbf{x},z)$, $(\mathbf{y},t)$ and $(\mathbf{u},v)$, respectively. The addend A_1 in this new notation is:

$$\begin{aligned}
A_1 = \int_{\Omega_q\times\mathbb{R}}\int_{\Omega_q\times\mathbb{R}}\Bigg[\int_{\Omega_q\times\mathbb{R}} LK\left(\frac{1-\mathbf{x}^T\mathbf{z}}{h^2},\frac{x-z}{g}\right)LK\left(\frac{1-\mathbf{y}^T\mathbf{z}}{h^2},\frac{y-z}{g}\right)\\
\times f(\mathbf{z},z)\,dz\,\omega_q(d\mathbf{z})\Bigg]^2\,dy\,\omega_q(d\mathbf{y})\,dx\,\omega_q(d\mathbf{x}).
\end{aligned}$$

The computation of A_1 will be divided in the cases $q\geq 2$ and $q=1$. There are several changes of variables involved, which will be detailed in *i*–*iv*. To begin with, let suppose $q\geq 2$:

$$A_1 \overset{i}{=} \int_{\Omega_q\times\mathbb{R}}\int_{\Omega_{q-1}}\int_{-1}^{1}\int_{\mathbb{R}}\Bigg[\int_{\Omega_{q-2}}\iint_{t^2+\tau^2<1}\int_{\mathbb{R}}$$

$$
\begin{aligned}
&\times LK\left(\frac{1-t}{h^2},\frac{x-z}{g}\right)LK\left(\frac{1-st-\tau(1-s^2)^{\frac{1}{2}}}{h^2},\frac{y-z}{g}\right)\\
&\times f\left(t\mathbf{x}+\tau\mathbf{B}_q\boldsymbol{\xi}+\left(1-t^2-\tau^2\right)^{\frac{1}{2}}\mathbf{A}_{\boldsymbol{\xi}}\boldsymbol{\eta},z\right)\left(1-t^2-\tau^2\right)^{\frac{q-3}{2}}\\
&\times\left.dz\,dt\,d\tau\,\omega_{q-2}(d\boldsymbol{\eta})\right]^2(1-s^2)^{\frac{q}{2}-1}\,dy\,ds\,\omega_{q-1}(d\boldsymbol{\xi})\,dx\,\omega_q(d\mathbf{x})\\
\overset{ii}{=}&\int_{\Omega_q\times\mathbb{R}}\int_{\Omega_{q-1}}\int_0^{2h^{-2}}\int_{\mathbb{R}}\left[\int_{\Omega_{q-2}}\int_0^{2h^{-2}}\int_{-1}^1\int_{\mathbb{R}}LK\left(\rho,\frac{x-z}{g}\right)\right.\\
&\times LK\left(r+\rho-h^2r\rho-\theta\left[r\rho(2-h^2r)(2-h^2\rho)\right]^{\frac{1}{2}},\frac{y-z}{g}\right)\\
&\times f\left((1-h^2\rho)\mathbf{x}+h\left[\rho(2-h^2\rho)\right]^{\frac{1}{2}}\left[\theta\mathbf{B}_{\mathbf{x}}\boldsymbol{\xi}+(1-\theta^2)^{\frac{1}{2}}\mathbf{A}_{\boldsymbol{\xi}}\boldsymbol{\eta}\right],z\right)\\
&\times\left.(1-\theta^2)^{\frac{q-3}{2}}h^{q-3}\left[\rho(2-h^2\rho)\right]^{\frac{q-3}{2}}h^3\left[\rho(2-h^2\rho)\right]^{\frac{1}{2}}\,dz\,d\theta\,d\rho\,\omega_{q-2}(d\boldsymbol{\eta})\right]^2\\
&\times h^{q-2}r^{\frac{q}{2}-1}(2-h^2r)^{\frac{q}{2}-1}h^2\,dy\,dr\,\omega_{q-1}(d\boldsymbol{\xi})\,dx\,\omega_q(d\mathbf{x})\\
\overset{iii}{=}&h^{3q}g^3\int_{\Omega_q\times\mathbb{R}}\int_{\Omega_{q-1}}\int_0^{2h^{-2}}\int_{\mathbb{R}}\left[\int_{\Omega_{q-2}}\int_0^{2h^{-2}}\int_{-1}^1\int_{\mathbb{R}}LK\left(\rho,u\right)\right.\\
&\times LK\left(r+\rho-h^2r\rho-\theta\left[r\rho(2-h^2r)(2-h^2\rho)\right]^{\frac{1}{2}},u+v\right)\\
&\times f\left((1-h^2\rho)\mathbf{x}+h\left[\rho(2-h^2\rho)\right]^{\frac{1}{2}}\left[\theta\mathbf{B}_{\mathbf{x}}\boldsymbol{\xi}+(1-\theta^2)^{\frac{1}{2}}\mathbf{A}_{\boldsymbol{\xi}}\boldsymbol{\eta}\right],x-ug\right)\\
&\times\left.(1-\theta^2)^{\frac{q-3}{2}}\left[\rho(2-h^2\rho)\right]^{\frac{q}{2}-1}\,du\,d\theta\,d\rho\,\omega_{q-2}(d\boldsymbol{\eta})\right]^2r^{\frac{q}{2}-1}(2-h^2r)^{\frac{q}{2}-1}\\
&\times dv\,dr\,\omega_{q-1}(d\boldsymbol{\xi})\,dx\,\omega_q(d\mathbf{x})\\
\overset{iv}{\sim}&h^{3q}g^3\int_{\Omega_q\times\mathbb{R}}\int_{\Omega_{q-1}}\int_0^{\infty}\int_{\mathbb{R}}\left[\int_{\Omega_{q-2}}\int_0^{\infty}\int_{-1}^1\int_{\mathbb{R}}LK\left(\rho,u\right)\right.\\
&\times LK\left(r+\rho-2\theta\left(r\rho\right)^{\frac{1}{2}},u+v\right)f\left(\mathbf{x},x\right)\left(1-\theta^2\right)^{\frac{q-3}{2}}\left(2\rho\right)^{\frac{q}{2}-1}\\
&\times\left.du\,d\theta\,d\rho\,\omega_{q-2}(d\boldsymbol{\eta})\right]^2(2r)^{\frac{q}{2}-1}\,dv\,dr\,\omega_{q-1}(d\boldsymbol{\xi})\,dx\,\omega_q(d\mathbf{x})\\
=&h^{3q}g^3R(f)\omega_{q-1}\omega_{q-2}^2 2^{\frac{3q}{2}-1}\int_0^\infty r^{\frac{q}{2}-1}\int_{\mathbb{R}}\left[\int_{\mathbb{R}}\int_0^\infty\rho^{\frac{q}{2}-1}LK(\rho,u)\right.\\
&\times\left.\int_{-1}^1(1-\theta^2)^{\frac{q-3}{2}}LK\left(r+\rho-2\theta(r\rho)^{\frac{1}{2}},u+v\right)\,d\theta\,d\rho\,du\right]^2\,dv\,dr\\
=&h^{3q}g^3\lambda_q(L)^4\sigma^2.
\end{aligned}
$$

The steps for the computation of the case $q\geq 2$ are the following:

i. Let $\mathbf{x}$ a fixed point in Ω_q, $q\geq 2$. Let be the change of variables:

$$\mathbf{y}=s\mathbf{x}+(1-s^2)^{\frac{1}{2}}\mathbf{B}_{\mathbf{x}}\boldsymbol{\xi},\quad \omega_q(d\mathbf{y})=(1-s^2)^{\frac{q}{2}-1}\,ds\,\omega_{q-1}(d\boldsymbol{\xi}),$$

where $s \in (-1,1)$, $\boldsymbol{\xi} \in \Omega_{q-1}$ and $\mathbf{B_x} = (\mathbf{b}_1, \ldots, \mathbf{b}_q)_{(q+1)\times q}$ is the semi-orthonormal matrix ($\mathbf{B_x}^T\mathbf{B_x} = \mathbf{I}_q$ and $\mathbf{B_x}\mathbf{B_x}^T = \mathbf{I}_{q+1} - \mathbf{x}\mathbf{x}^T$) resulting from the completion of $\mathbf{x}$ to the orthonormal basis $\{\mathbf{x}, \mathbf{b}_1, \ldots, \mathbf{b}_q\}$ of $\mathbb{R}^{q+1}$. Here $\mathbf{I}_q$ represents the identity matrix with dimension q. See Lemma 2 of García-Portugués et al. (2013b) for further details. Consider also the other change of variables

$$\mathbf{z} = t\mathbf{x} + \tau\mathbf{B_x}\boldsymbol{\xi} + (1-t^2-\tau^2)^{\frac{1}{2}}\mathbf{A}_{\boldsymbol{\xi}}\boldsymbol{\eta}, \quad \omega_q(d\mathbf{z}) = (1-t^2-\tau^2)^{\frac{q-3}{2}}\,dt\,d\tau\,\omega_{q-2}(d\boldsymbol{\eta}),$$

where $t, \tau \in (-1,1)$, $t^2+\tau^2 < 1$, $\boldsymbol{\eta} \in \Omega_{q-2}$ and $\mathbf{A}_{\boldsymbol{\xi}} = (\mathbf{a}_1, \ldots, \mathbf{a}_q)_{(q+1)\times(q-1)}$ is the semi-orthonormal matrix ($\mathbf{A}_{\boldsymbol{\xi}}^T\mathbf{A}_{\boldsymbol{\xi}} = \mathbf{I}_q$ and $\mathbf{A}_{\boldsymbol{\xi}}\mathbf{A}_{\boldsymbol{\xi}}^T = \mathbf{I}_{q+1} - \mathbf{x}\mathbf{x}^T - \mathbf{B_x}\boldsymbol{\xi}\boldsymbol{\xi}^T\mathbf{B_x}^T$) resulting from the completion of $\{\mathbf{x}, \mathbf{B_x}\boldsymbol{\xi}\}$ to the orthonormal basis $\{\mathbf{x}, \mathbf{B_x}\boldsymbol{\xi}, \mathbf{a}_1, \ldots, \mathbf{a}_{q-1}\}$ of $\mathbb{R}^{q+1}$. This change of variables can be obtained by replicating the proof of Lemma 2 in García-Portugués et al. (2013b) with an extra step for the case $q \geq 2$. With these two changes of variables,

$$\mathbf{y}^T\mathbf{z} = st + \tau(1-s^2)^{\frac{1}{2}}, \quad \mathbf{x}^T(\mathbf{B_x}\boldsymbol{\xi}) = \mathbf{x}^T(\mathbf{A}_{\boldsymbol{\xi}}\boldsymbol{\eta}) = (\mathbf{B_x}\boldsymbol{\xi})^T(\mathbf{A}_{\boldsymbol{\xi}}\boldsymbol{\eta}) = 0.$$

ii. Consider first the change of variables $r = \frac{1-s}{h^2}$ and then

$$\begin{cases} \rho = \frac{1-t}{h^2}, \\ \theta = \frac{\tau}{h[\rho(2-h^2\rho)]^{\frac{1}{2}}}, \end{cases} \qquad \left|\frac{\partial(t,\tau)}{\partial(\rho,\theta)}\right| = h^3\left[\rho(2-h^2\rho)\right]^{\frac{1}{2}}.$$

With this last change of variables, $\tau = h\theta\left[\rho(2-h^2\rho)\right]^{\frac{1}{2}}$, $t = 1 - h^2\rho$ and, as a result:

$$\begin{aligned} 1-s^2 &= h^2r(2-h^2r), \\ 1-t^2 &= h^2\rho(2-h^2\rho), \\ 1-t^2-\tau^2 &= (1-\theta^2)h^2\rho(2-h^2\rho), \\ \frac{1-st-\tau(1-s^2)^{\frac{1}{2}}}{h^2} &= r+\rho-h^2r\rho-\theta\left[r\rho(2-h^2r)(2-h^2\rho)\right]^{\frac{1}{2}}. \end{aligned}$$

iii. Use $u = \frac{x-z}{g}$ and $v = \frac{y-x}{g}$.

iv. By expanding the square, A_1 can be written as

$$\begin{aligned} A_1 = h^{3q}g^3 &\int_{\Omega_q\times\mathbb{R}}\int_{\Omega_{q-1}}\int_0^\infty\int_{\mathbb{R}}\Bigg[\int_{\Omega_{q-2}}\int_0^\infty\int_{-1}^1\int_{\mathbb{R}}\int_{\Omega_{q-2}}\int_0^\infty\int_{-1}^1\int_{\mathbb{R}} \\ &\times \varphi_n(\mathbf{x},x,r,\rho_1,\theta_1,u_1,v,\boldsymbol{\xi},\boldsymbol{\eta}_1)\varphi_n(\mathbf{x},x,r,\rho_2,\theta_2,u_2,v,\boldsymbol{\xi},\boldsymbol{\eta}_2) \\ &\times du_1\,d\theta_1\,d\rho_1\,\omega_{q-2}(d\boldsymbol{\eta}_1)\,du_2\,d\theta_2\,d\rho_2\,\omega_{q-2}(d\boldsymbol{\eta}_2)\Bigg] \\ &\times dv\,dr\,\omega_{q-1}(d\boldsymbol{\xi})\,dx\,\omega_q(d\mathbf{x}), \end{aligned} \tag{S1.9}$$

where

$$\begin{aligned}\varphi_n(\mathbf{x},x,r,\rho_i,&\theta_i,u_i,v,\boldsymbol{\xi},\boldsymbol{\eta}_i)\\&=L\left(\rho_i\right)L\left(r+\rho_i-h^2r\rho_i-\theta\left[r\rho_i(2-h^2r)(2-h^2\rho_i)\right]^{\frac{1}{2}}\right)\\&\quad\times K\left(u_i\right)K\left(u_i+v\right)f\left((\mathbf{x},x)+\boldsymbol{\alpha}_{h,g}\right)(1-\theta_i^2)^{\frac{q-3}{2}}\\&\quad\times\rho_i^{\frac{q}{2}-1}(2-h^2\rho_i)^{\frac{q}{2}-1}r^{\frac{q}{4}-\frac{1}{2}}(2-h^2r)^{\frac{q}{4}-\frac{1}{2}}\mathbb{1}_{[0,2h^{-2})}(r)\mathbb{1}_{[0,2h^{-2})}(\rho_i),\end{aligned}$$

with $\boldsymbol{\alpha}_{h,g}=\Big(-h^2\rho_i\mathbf{x}+h\left[\rho_i(2-h^2\rho_i)\right]^{\frac{1}{2}}\left[\theta\mathbf{B}_{\mathbf{x}}\boldsymbol{\xi}+(1-\theta_i^2)^{\frac{1}{2}}\mathbf{A}_{\boldsymbol{\xi}}\boldsymbol{\eta}_i\right],-u_ig\Big)$ and $i=1,2$. A first step to apply the DCT is to see that by the Taylor's theorem,

$$f\left((\mathbf{x},x)+\boldsymbol{\alpha}_{h,g}\right)=f(\mathbf{x},x)+\mathcal{O}\left(\boldsymbol{\alpha}_{h,g}^T\boldsymbol{\nabla}f(\mathbf{x},x)\right),$$

where the remaining order is $\mathcal{O}\big((h^2\rho_i+g^2u_i^2)^{\frac{1}{2}}\left|\left|\boldsymbol{\nabla}f(\mathbf{x},x)\right|\right|\big)$ because $\left|\left|\boldsymbol{\alpha}_{h,g}\right|\right|^2=2h^2\rho_i+g^2u_i^2$. Furthermore, the order is uniform for all points $(\mathbf{x},x)$ because of the boundedness assumption of the second derivative given by **A1** (see the proof of Lemma 11). Next, as $h,g\to0$, then the order becomes $\boldsymbol{o}\left((\sqrt{\rho_i}+u_i)\left|\left|\boldsymbol{\nabla}f(\mathbf{x},x)\right|\right|\right)$.

For bounding the directional kernel L, recall that by completing the square,

$$\begin{aligned}(2-h^2r)(2-h^2\rho_i)&=4-2h^2(r+\rho_i)+h^4\left((r+\rho_i)/2\right)^2-h^4\left(\left((r+\rho_i)/2)\right)^2-r\rho_i\right)\\&\leq\left(2-h^2\frac{r+\rho_i}{2}\right)^2.\end{aligned}$$

Using this, and the fact that $\theta\in(-1,1)$, for all $r,\rho_i\in[0,2h^{-2})$,

$$\begin{aligned}r+\rho_i-h^2r\rho_i-\theta&\left[r\rho_i(2-h^2r)(2-h^2\rho_i)\right]^{\frac{1}{2}}\\&\geq r+\rho_i-h^2r\rho_i-(r\rho_i)^{\frac{1}{2}}\left[(2-h^2r)(2-h^2\rho_i)\right]^{\frac{1}{2}}\\&\geq r+\rho_i-h^2r\rho_i-(r\rho_i)^{\frac{1}{2}}\left(2-h^2\frac{r+\rho_i}{2}\right)\\&=r+\rho_i-2(r\rho_i)^{\frac{1}{2}}+h^2(r\rho_i)^{\frac{1}{2}}\left(\frac{r+\rho_i}{2}-(r\rho_i)^{\frac{1}{2}}\right)\\&\geq r+\rho_i-2(r\rho_i)^{\frac{1}{2}},\end{aligned}$$

where the last inequality follows because the last addend is positive by the inequality of the geometric and arithmetic means. As L is a decreasing function by **A2**,

$$L\left(r+\rho_i-h^2r\rho_i-\theta\left[r\rho_i(2-h^2r)(2-h^2\rho_i)\right]^{\frac{1}{2}}\right)\leq L\left(r+\rho_i-2(r\rho_i)^{\frac{1}{2}}\right).$$

Then for all the variables in the integration domain of A_1,

$$\begin{aligned}\varphi_n(\mathbf{x}, x, r, \rho_i, \theta_i, u_i, v, \boldsymbol{\xi}, \boldsymbol{\eta}_i) \leq & \, L(\rho_i) \, L\left(r + \rho_i - 2(r\rho_i)^{\frac{1}{2}}\right) K(u_i) \, K(u_i + v) \\ & \times \left(f(\mathbf{x}, x) + o\left((\sqrt{\rho_i} + u_i) \, ||\boldsymbol{\nabla} f(\mathbf{x}, x)||\right)\right) (1 - \theta_i^2)^{\frac{q-3}{2}} \\ & \times \rho_i^{\frac{q}{2}-1} 2^{\frac{q}{2}-1} r^{\frac{q}{4}-\frac{1}{2}} 2^{\frac{q}{4}-\frac{1}{2}} \mathbb{1}_{[0,\infty)}(r) \mathbb{1}_{[0,\infty)}(\rho_i) \\ = & \, \Psi(\mathbf{x}, x, r, \rho_i, \theta_i, u_i, v).\end{aligned}$$

The product of functions φ_n in (S1.9) is bounded by the respective product of functions Ψ. The product is also integrable as a consequence of assumptions **A1** (integrability of f and $\boldsymbol{\nabla} f$), **A2** (integrability of kernels) and that the product of integrable functions is integrable. To prove it, recall that by the integral definition of the modified Bessel function of order $\frac{q}{2} - 1$ (see equation 10.32.2 of Olver et al. (2010)):

$$\int_{-1}^{1} (1 - \theta^2)^{\frac{q-3}{2}} \, d\theta = \frac{\sqrt{\pi}\Gamma\left(\frac{q-1}{2}\right)}{\Gamma\left(\frac{q}{2}\right)} < \infty, \quad \forall q \geq 2.$$

The integral of the linear kernel is proved to be finite using the Cauchy-Schwartz inequality and **A2**:

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} & K(u_1)K(u_1+v)K(u_2)K(u_2+v) \, du_1 \, du_2 \, dv \\ & = \int_{\mathbb{R}}\int_{\mathbb{R}} K(u_1)K(u_2) \left[\int_{\mathbb{R}} K(u_1+v)K(u_2+v) \, dv\right] du_1 \, du_2 \\ & \leq \int_{\mathbb{R}}\int_{\mathbb{R}} K(u_1)K(u_2)\mu_2(K)^{\frac{1}{2}}\mu_2(K)^{\frac{1}{2}} \, du_1 \, du_2 \\ & = \mu_2(K).\end{aligned}$$

For the directional situation, the following auxiliary result based on **A2** is needed:

$$\begin{aligned}\int_0^\infty L^2\left(\left(\sqrt{r} - \sqrt{\rho_i}\right)^2\right) r^{\frac{q}{2}-1} \, dr & \leq \int_0^\infty L^2(s) \left(\sqrt{s} + \sqrt{\rho_i}\right)^{q-1} s^{-\frac{1}{2}} \, dr \\ & = \int_0^\infty L^2(s) \sum_{k=0}^{q-1} s^{\frac{k-1}{2}} \rho_i^{\frac{q-1-k}{2}} \, dr \\ & = \sum_{k=0}^{q-1} \lambda_{k+1}(L^2)\rho_i^{\frac{q-1-k}{2}} \\ & = \mathcal{O}\left(\rho_i^{\frac{q-1}{2}}\right).\end{aligned}$$

Using this and that $\int_0^\infty L(\rho)\rho^{\frac{3q-5}{4}} \, dr \leq \lambda_{\lceil \frac{2q+1}{3} \rceil}(L) < \infty$, it follows:

$$\int_0^\infty \int_0^\infty \int_0^\infty L(\rho_1)L(\rho_2)L\left(r + \rho_1 - 2(r\rho_1)^{\frac{1}{2}}\right) L\left(r + \rho_2 - 2(r\rho_2)^{\frac{1}{2}}\right)$$

$$
\begin{aligned}
&\times \rho_1^{\frac{q}{2}-1}\rho_2^{\frac{q}{2}-1}r^{\frac{q}{2}-1}\,dr\,d\rho_1\,d\rho_2\\
&=\int_0^\infty\int_0^\infty L(\rho_1)L(\rho_2)\rho_1^{\frac{q}{2}-1}\rho_2^{\frac{q}{2}-1}\\
&\quad\times\left[\int_0^\infty L\left(\left(\sqrt{r}-\sqrt{\rho_1}\right)^2\right)L\left(\left(\sqrt{r}-\sqrt{\rho_2}\right)^2\right)r^{\frac{q}{2}-1}\,dr\right]d\rho_1\,d\rho_2\\
&\leq\int_0^\infty\int_0^\infty L(\rho_1)L(\rho_2)\rho_1^{\frac{q}{2}-1}\rho_2^{\frac{q}{2}-1}\mathcal{O}\left(\rho_1^{\frac{q-1}{4}}\right)\mathcal{O}\left(\rho_2^{\frac{q-1}{4}}\right)d\rho_1\,d\rho_2\\
&=\mathcal{O}(1).
\end{aligned}
$$

Then, by the DCT,

$$
\begin{aligned}
A_1\sim h^{3q}g^3&\int_{\Omega_q\times\mathbb{R}}\int_{\Omega_{q-1}}\int_0^\infty\int_{\mathbb{R}}\left[\int_{\Omega_{q-2}}\int_0^\infty\int_{-1}^1\int_{\mathbb{R}}\right.\\
&\times LK(\rho,u)\,LK\left(r+\rho-2\theta\,(r\rho)^{\frac{1}{2}},u+v\right)\\
&\left.\times f(\mathbf{x},x)\,(1-\theta^2)^{\frac{q-3}{2}}\,(2\rho)^{\frac{q}{2}-1}\,du\,d\theta\,d\rho\,\omega_{q-2}(d\boldsymbol{\eta})\right]^2\\
&\times(2r)^{\frac{q}{2}-1}\,dv\,dr\,\omega_{q-1}(d\boldsymbol{\xi})\,dx\,\omega_q(d\mathbf{x}),
\end{aligned}
$$

because all the functions involved are continuous almost everywhere.

Turn now to the case $q=1$. As before, the details of the case $q=1$ are explained in *vi*–*ix*:

$$
\begin{aligned}
A_1&\overset{vi}{=}\int_{\Omega_1\times\mathbb{R}}\int_{\Omega_0}\int_{-1}^1\int_{\mathbb{R}}\left[\int_{\Omega_0}\int_{-1}^1\int_{\mathbb{R}}LK\left(\frac{1-t}{h^2},\frac{x-z}{g}\right)\right.\\
&\quad\times LK\left(\frac{1-st-(1-t^2)^{\frac{1}{2}}(1-s^2)^{\frac{1}{2}}(\mathbf{B_x}\boldsymbol{\xi})^T(\mathbf{A_x}\boldsymbol{\eta})}{h^2},\frac{y-z}{g}\right)\\
&\quad\left.\times f\left(t\mathbf{x}+(1-t^2)^{\frac{1}{2}}\mathbf{A_x}\boldsymbol{\eta},z\right)(1-t^2)^{-\frac{1}{2}}\,dz\,dt\,\omega_0(d\boldsymbol{\eta})\right]^2\\
&\quad\times(1-s^2)^{-\frac{1}{2}}\,dy\,ds\,\omega_0(d\boldsymbol{\xi})\,dx\,\omega_1(d\mathbf{x})\\
&\overset{vii}{=}\int_{\Omega_1\times\mathbb{R}}\int_{\Omega_0}\int_0^{2h^{-2}}\int_{\mathbb{R}}\left[\int_{\Omega_0}\int_0^{2h^{-2}}\int_{\mathbb{R}}LK\left(\rho,\frac{x-z}{g}\right)\right.\\
&\quad\times LK\left(r+\rho-h^2r\rho-\left(r\rho(2-h^2r)(2-h^2\rho)\right)^{\frac{1}{2}}(\mathbf{B_x}\boldsymbol{\xi})^T\mathbf{A_x}\boldsymbol{\eta},\frac{y-z}{g}\right)\\
&\quad\times f\left((1-h^2\rho)\mathbf{x}+h\left[\rho(2-h^2\rho)\right]^{\frac{1}{2}}\mathbf{A_x}\boldsymbol{\eta},z\right)h^{-1}\rho^{-\frac{1}{2}}(2-h^2\rho)^{-\frac{1}{2}}h^2\\
&\quad\left.\times\,dz\,d\rho\,\omega_0(d\boldsymbol{\eta})\right]^2h^{-1}r^{-\frac{1}{2}}(2-h^2r)^{-\frac{1}{2}}h^2\,dy\,dr\,\omega_0(d\boldsymbol{\xi})\,dx\,\omega_1(d\mathbf{x})
\end{aligned}
$$

$$\stackrel{viii}{=} h^3g^3\int_{\Omega_1\times\mathbb{R}}\int_{\Omega_0}\int_0^{2h^{-2}}\int_{\mathbb{R}}\bigg[\int_{\Omega_0}\int_0^{2h^{-2}}\int_{\mathbb{R}} LK\,(\rho,u)$$
$$\times LK\left(r+\rho-h^2r\rho-\left(r\rho(2-h^2r)(2-h^2\rho)\right)^{\frac{1}{2}}(\mathbf{B_x}\boldsymbol{\xi})^T\mathbf{A_x}\boldsymbol{\eta},u+v\right)$$
$$\times f\left((1-h^2\rho)\mathbf{x}+h\left[\rho(2-h^2\rho)\right]^{\frac{1}{2}}\mathbf{A_x}\boldsymbol{\eta},x-ug\right)$$
$$\times \rho^{-\frac{1}{2}}(2-h^2\rho)^{-\frac{1}{2}}\,du\,d\rho\,\omega_0(d\boldsymbol{\eta})\bigg]^2 r^{-\frac{1}{2}}(2-h^2r)^{-\frac{1}{2}}$$
$$\times\ dv\,dr\,\omega_0(d\boldsymbol{\xi})\,dx\,\omega_1(d\mathbf{x})$$
$$\stackrel{vi}{=} h^3g^3\int_{\Omega_1\times\mathbb{R}}\int_{\Omega_0}\int_0^{2h^{-2}}\int_{\mathbb{R}}\bigg[\int_0^{2h^{-2}}\int_{\mathbb{R}} LK\,(\rho,u)$$
$$\times\bigg[LK\left(r+\rho-h^2r\rho+\left(r\rho(2-h^2r)(2-h^2\rho)\right)^{\frac{1}{2}},u+v\right)$$
$$\times f\left((1-h^2\rho)\mathbf{x}+h\left[\rho(2-h^2\rho)\right]^{\frac{1}{2}}\mathbf{B_x}\boldsymbol{\xi},x-ug\right)$$
$$+LK\left(r+\rho-h^2r\rho-\left(r\rho(2-h^2r)(2-h^2\rho)\right)^{\frac{1}{2}},u+v\right)$$
$$\times f\left((1-h^2\rho)\mathbf{x}-h\left[\rho(2-h^2\rho)\right]^{\frac{1}{2}}\mathbf{B_x}\boldsymbol{\xi},x-ug\right)\bigg]$$
$$\times \rho^{-\frac{1}{2}}(2-h^2\rho)^{-\frac{1}{2}}\,du\,d\rho\bigg]^2 r^{-\frac{1}{2}}(2-h^2r)^{-\frac{1}{2}}\,dv\,dr\,\omega_0(d\boldsymbol{\xi})\,dx\,\omega_1(d\mathbf{x})$$
$$\stackrel{ix}{\sim} h^3g^3\int_{\Omega_1\times\mathbb{R}}\int_{\Omega_0}\int_0^{\infty}\int_{\mathbb{R}}\bigg[\int_0^{\infty}\int_{\mathbb{R}} LK\,(\rho,u)$$
$$\times\left[LK\left(r+\rho+2(r\rho)^{\frac{1}{2}},u+v\right)+LK\left(r+\rho-2(r\rho)^{\frac{1}{2}},u+v\right)\right]$$
$$\times f\,(\mathbf{x},x)\,\rho^{-\frac{1}{2}}2^{-\frac{1}{2}}\,du\,d\rho\bigg]^2 r^{-\frac{1}{2}}2^{-\frac{1}{2}}\,dv\,dr\,\omega_0(d\boldsymbol{\xi})\,dx\,\omega_1(d\mathbf{x})$$
$$= h^3g^3R(f)2^{-\frac{1}{2}}\int_0^{\infty} r^{-\frac{1}{2}}\int_{\mathbb{R}}\bigg[\int_0^{\infty}\int_{\mathbb{R}}\rho^{-\frac{1}{2}}LK\,(\rho,u)$$
$$\times\left[LK\left(r+\rho+2(r\rho)^{\frac{1}{2}},u+v\right)+LK\left(r+\rho-2(r\rho)^{\frac{1}{2}},u+v\right)\right]$$
$$\times\ du\,d\rho\bigg]^2 dv\,dr$$
$$= h^{3q}g^3\lambda_q(L)^4\sigma^2.$$

The steps used for the computation are the following:

vi. Let $\mathbf{x}$ a fixed point in Ω_q. For $q=1$, let be the changes of variables

$$\mathbf{y}=s\mathbf{x}+(1-s^2)^{\frac{1}{2}}\mathbf{B_x}\boldsymbol{\xi},\quad \omega_1(d\mathbf{y})=(1-s^2)^{\frac{q}{2}-1}\,ds\,\omega_0(d\boldsymbol{\xi}),$$
$$\mathbf{z}=t\mathbf{x}+(1-t^2)^{\frac{1}{2}}\mathbf{A_x}\boldsymbol{\eta},\quad \omega_1(d\mathbf{z})=(1-t^2)^{\frac{q}{2}-1}\,dt\,\omega_0(d\boldsymbol{\eta}),$$

where $s,t\in(-1,1)$ and $\mathbf{B}_\mathbf{x}$ and $\mathbf{A}_\mathbf{x}$ are two semi-orthonormal matrices whose q columns are vectors that extend $\mathbf{x}$ to an orthonormal basis of $\mathbb{R}^{q+1}$. Note that as $q=1$ and $\mathbf{x}^T(\mathbf{B}_\mathbf{x}\boldsymbol{\xi})=\mathbf{x}^T(\mathbf{A}_\mathbf{x}\boldsymbol{\eta})=0$, then necessarily $\mathbf{B}_\mathbf{x}\boldsymbol{\xi}=\mathbf{A}_\mathbf{x}\boldsymbol{\eta}$ or $\mathbf{B}_\mathbf{x}\boldsymbol{\xi}=-\mathbf{A}_\mathbf{x}\boldsymbol{\eta}$.

vii. Let be the changes of variables $\rho=\frac{1-t}{h^2}$ and $r=\frac{1-s}{h^2}$. With this change, $t=1-h^2\rho$ and $s=1-h^2r$. Then $1-s^2=h^2r(2-h^2r)$, $1-t^2=h^2\rho(2-h^2\rho)$ and

$$\begin{aligned}&\frac{1-st-(1-s^2)^{\frac{1}{2}}(1-t^2)^{\frac{1}{2}}(\mathbf{B}_\mathbf{x}\boldsymbol{\xi})^T\mathbf{A}_\mathbf{x}\boldsymbol{\eta}}{h^2}\\&\qquad=r+\rho-h^2r\rho-\big(r\rho(2-h^2r)(2-h^2\rho)\big)^{\frac{1}{2}}(\mathbf{B}_\mathbf{x}\boldsymbol{\xi})^T\mathbf{A}_\mathbf{x}\boldsymbol{\eta}.\end{aligned}$$

viii. Use $u=\frac{x-z}{g}$ and $v=\frac{y-x}{g}$.

ix. A_1 can be written as

$$\begin{aligned}A_1=&h^{3q}g^3\int_{\Omega_q\times\mathbb{R}}\int_{\Omega_{q-1}}\int_0^\infty\int_\mathbb{R}\bigg[\int_0^\infty\int_\mathbb{R}\int_0^\infty\int_\mathbb{R}\varphi_n(\mathbf{x},x,r,\rho_1,u_1,v,\boldsymbol{\xi})\\&\times\varphi_n(\mathbf{x},x,r,\rho_2,u_2,v,\boldsymbol{\xi})\,du_1\,d\rho_1\,du_2\,d\rho_2\bigg]\,dv\,dr\,\omega_{q-1}(d\boldsymbol{\xi})\,dx\,\omega_q(d\mathbf{x}),\end{aligned}$$

where

$$\begin{aligned}&\varphi_n(\mathbf{x},x,r,\rho_i,u_i,v,\boldsymbol{\xi})\\&\quad=L(\rho_i)\,K(u_i)\Big[L\Big(r+\rho_i-h^2r\rho_i+\big[r\rho_i(2-h^2r)(2-h^2\rho_i)\big]^{\frac{1}{2}}\Big)\\&\qquad\times K(u_i+v)\,f\left((\mathbf{x},x)+\boldsymbol{\alpha}_{h,g}^{(1)}\right)+K(u_i+v)\,f\left((\mathbf{x},x)+\boldsymbol{\alpha}_{h,g}^{(2)}\right)\\&\qquad\times L\Big(r+\rho_i-h^2r\rho_i-\big[r\rho_i(2-h^2r)(2-h^2\rho_i)\big]^{\frac{1}{2}}\Big)\Big]\\&\qquad\times\rho_i^{-\frac{1}{2}}(2-h^2\rho_i)^{-\frac{1}{2}}r^{-\frac{1}{4}}(2-h^2r)^{-\frac{1}{4}}\mathbb{1}_{[0,2h^{-2})}(r)\mathbb{1}_{[0,2h^{-2})}(\rho_i),\end{aligned}$$

with $\boldsymbol{\alpha}_{h,g}^{(j)}=\Big(-h^2\rho_i\mathbf{x}+k_jh\left[\rho_i(2-h^2\rho_i)\right]^{\frac{1}{2}}\mathbf{B}_\mathbf{x}\boldsymbol{\xi},-u_ig\Big)$ and $k_1=1$, $k_2=-1$. As before, by the Taylor's theorem,

$$f\left((\mathbf{x},x)+\boldsymbol{\alpha}_{h,g}^{(k)}\right)=f(\mathbf{x},x)+\boldsymbol{o}\left((\sqrt{\rho_i}+u_i)\,||\boldsymbol{\nabla}f(\mathbf{x},x)||\right),$$

where the order is uniform for all points $(\mathbf{x},x)$. By analogous considerations as for the case $q\geq2$,

$$\begin{aligned}\varphi_n(\mathbf{x},x,r,\rho_i,u_i,v,\boldsymbol{\xi})\leq&\,2L(\rho_i)\,L\Big(r+\rho_i-2(r\rho_i)^{\frac{1}{2}}\Big)\,K(u_i)\,K(u_i+v)\Big(f(\mathbf{x},x)\\&+\boldsymbol{o}\left((\sqrt{\rho_i}+u_i)\,||\boldsymbol{\nabla}f(\mathbf{x},x)||\right)\Big)\rho_i^{-\frac{1}{2}}r^{-\frac{1}{4}}\mathbb{1}_{[0,\infty)}(r)\mathbb{1}_{[0,\infty)}(\rho_i)\end{aligned}$$

$$= \Psi(\mathbf{x}, x, r, \rho_i, u_i, v),$$

Then the product of functions φ_n is bounded by the respective product of functions Ψ, which is integrable, and by the DCT the limit commute with the integrals.

Proof of (S1.6). $\mathbb{E}\left[H_n^4\left((\mathbf{X}_1, Z_1), (\mathbf{X}_2, Z_2)\right)\right]$ can be decomposed in the sum of two terms:

$$\begin{aligned}
\mathbb{E}\big[H_n^4\big(\mathbf{X}_1, & Z_1), (\mathbf{X}_2, Z_2)\big)\big] \\
&= \mathbb{E}\left[\left(\int_{\Omega_q\times\mathbb{R}} LK_n\left((\mathbf{x}, z), (\mathbf{X}_1, Z_1)\right) LK_n\left((\mathbf{x}, z), (\mathbf{X}_2, Z_2)\right)\, dz\,\omega_q(d\mathbf{x})\right)^4\right] \\
&= \int_{\Omega_q\times\mathbb{R}}\int_{\Omega_q\times\mathbb{R}} \left(E_1((\mathbf{x}, z), (\mathbf{y}, t)) - E_2((\mathbf{x}, z), (\mathbf{y}, t))\right)^4\, dz\,\omega_q(d\mathbf{x})\, dt\,\omega_q(d\mathbf{y}) \\
&= \mathcal{O}\left(B_1 + B_2\right).
\end{aligned}$$

The computation of the orders of these terms is analogous to the ones of A_2 and A_3:

$$\begin{aligned}
B_1 &= \int_{\Omega_q\times\mathbb{R}}\int_{\Omega_q\times\mathbb{R}} \left(E_1((\mathbf{x}, z), (\mathbf{y}, t))\right)^4\, dt\,\omega_q(d\mathbf{y})\, dz\,\omega_q(d\mathbf{x}) \\
&\sim \int_{\Omega_q\times\mathbb{R}}\int_{\Omega_q\times\mathbb{R}} \left(h^q g\lambda_q(L) LK\left(\frac{1-\mathbf{x}^T\mathbf{u}}{h^2}, \frac{z-t}{g}\right) f(\mathbf{y}, t)\right)^4\, dt\,\omega_q(d\mathbf{y})\, dz\,\omega_q(d\mathbf{x}) \\
&\sim h^{5q} g^5 \lambda_q(L)^4 \lambda_q(L^4) R(f^2), \\
B_2 &= \int_{\Omega_q\times\mathbb{R}}\int_{\Omega_q\times\mathbb{R}} \left(E_2((\mathbf{x}, z), (\mathbf{y}, t))\right)^4\, dt\,\omega_q(d\mathbf{y})\, dz\,\omega_q(d\mathbf{x}) \\
&\sim \int_{\Omega_q\times\mathbb{R}}\int_{\Omega_q\times\mathbb{R}} h^{8q} g^8 \lambda_q(L)^8 f(\mathbf{x}, z)^4 f(\mathbf{y}, t)^4\, dt\,\omega_q(d\mathbf{y})\, dz\,\omega_q(d\mathbf{x}) \\
&= h^{8q} g^8 \lambda_q(L)^8 R(f^2)^2.
\end{aligned}$$

Then $\mathbb{E}\left[H_n^4\left((\mathbf{X}_1, Z_1), (\mathbf{X}_2, Z_2)\right)\right] = \mathcal{O}\left(h^{5q} g^5\right)$.

Proof of (S1.7). The notation $(\mathbf{x}, x)$, $(\mathbf{y}, y)$, $(\mathbf{z}, z)$ and $(\mathbf{u}, u)$ for variables in $\Omega_q\times\mathbb{R}$ will be employed again:

$$\begin{aligned}
G_n((\mathbf{x}, x), (\mathbf{y}, y)) &= \int_{\Omega_q\times\mathbb{R}} H_n\left((\mathbf{z}, z), (\mathbf{x}, x)\right) H_n\left((\mathbf{z}, z), (\mathbf{y}, y)\right) f(\mathbf{z}, z)\, dz\,\omega_q(d\mathbf{z}) \\
&= \int_{\Omega_q\times\mathbb{R}} \left\{\int_{\Omega_q\times\mathbb{R}} LK_n((\mathbf{u}, u), (\mathbf{x}, x)) LK_n((\mathbf{u}, u), (\mathbf{z}, z))\, du\,\omega_q(d\mathbf{u})\right\} \\
&\quad\times \left\{\int_{\Omega_q\times\mathbb{R}} LK_n((\mathbf{u}, u), (\mathbf{y}, y)) LK_n((\mathbf{u}, u), (\mathbf{z}, z))\, du\,\omega_q(d\mathbf{u})\right\} \\
&\quad\times f(\mathbf{z}, z)\, dz\,\omega_q(d\mathbf{z}).
\end{aligned}$$

Therefore:

$$\begin{aligned}
&\mathbb{E}\big[G_n^2((\mathbf{X}_1,Z_1),(\mathbf{X}_2,Z_2))\big]\\
&\quad=\int_{\Omega_q\times\mathbb{R}}\int_{\Omega_q\times\mathbb{R}}\Bigg\{\int_{\Omega_q\times\mathbb{R}}\left[\int_{\Omega_q\times\mathbb{R}} LK_n((\mathbf{u},u),(\mathbf{x},x))LK_n((\mathbf{u},u),(\mathbf{z},z))\,du\,\omega_q(d\mathbf{u})\right]\\
&\quad\quad\times\left[\int_{\Omega_q\times\mathbb{R}} LK_n((\mathbf{u},u),(\mathbf{y},y))LK_n((\mathbf{u},u),(\mathbf{z},z))\,du\,\omega_q(d\mathbf{u})\right]f(\mathbf{z},z)\,dz\,\omega_q(d\mathbf{z})\Bigg\}^2\\
&\quad\quad\times f(\mathbf{y},y)f(\mathbf{x},x)\,dy\,\omega_q(d\mathbf{y})\,dx\,\omega_q(d\mathbf{x}).
\end{aligned}$$

Then, according to the expression of LK_n, $\mathbb{E}\left[G_n^2\left((\mathbf{X}_1,Z_1),(\mathbf{X}_2,Z_2)\right)\right]$ can be decomposed in 16 summands, which, in view of the symmetric roles of $(\mathbf{x},x)$ and $(\mathbf{y},y)$ can be reduced to 9 different summands. The first of all, C_1, is the dominant and has order $\mathcal{O}\left(h^{7q}g^7\right)$. Again, the orders are computed using (S1.4) iteratively:

$$\begin{aligned}
C_1=&\int_{\Omega_q\times\mathbb{R}}\int_{\Omega_q\times\mathbb{R}}\Bigg\{\int_{\Omega_q\times\mathbb{R}}\\
&\times\left[\int_{\Omega_q\times\mathbb{R}} LK\left(\frac{1-\mathbf{u}^T\mathbf{x}}{h^2},\frac{u-x}{g}\right)LK\left(\frac{1-\mathbf{u}^T\mathbf{z}}{h^2},\frac{u-z}{g}\right)du\,\omega_q(d\mathbf{u})\right]\\
&\times\left[\int_{\Omega_q\times\mathbb{R}} LK\left(\frac{1-\mathbf{u}^T\mathbf{y}}{h^2},\frac{u-y}{g}\right)LK\left(\frac{1-\mathbf{u}^T\mathbf{x}}{h^2},\frac{u-x}{g}\right)du\,\omega_q(d\mathbf{u})\right]\\
&\times f(\mathbf{z},z)\,dz\,\omega_q(d\mathbf{z})\Bigg\}^2 f(\mathbf{y},y)f(\mathbf{x},x)\,dy\,\omega_q(d\mathbf{y})\,dx\,\omega_q(d\mathbf{x})\\
\sim&\int_{\Omega_q\times\mathbb{R}}\int_{\Omega_q\times\mathbb{R}}\Bigg\{\int_{\Omega_q\times\mathbb{R}}\left[\lambda_q(L)h^qgLK\left(\frac{1-\mathbf{x}^T\mathbf{z}}{h^2},\frac{x-z}{g}\right)\right]\\
&\times\left[\lambda_q(L)h^qgLK\left(\frac{1-\mathbf{y}^T\mathbf{z}}{h^2},\frac{y-z}{g}\right)\right]f(\mathbf{z},z)\,dz\,\omega_q(d\mathbf{z})\Bigg\}^2\\
&\times f(\mathbf{y},y)f(\mathbf{x},x)\,dy\,\omega_q(d\mathbf{y})\,dx\,\omega_q(d\mathbf{x})\\
\sim&\,\lambda_q(L)^4h^{4q}g^4\int_{\Omega_q\times\mathbb{R}}\int_{\Omega_q\times\mathbb{R}}\left\{\lambda_q(L)h^qgLK\left(\frac{1-\mathbf{y}^T\mathbf{x}}{h^2},\frac{y-x}{g}\right)f(\mathbf{x},x)\right\}^2\\
&\times f(\mathbf{y},y)f(\mathbf{x},x)\,dy\,\omega_q(d\mathbf{y})\,dx\,\omega_q(d\mathbf{x})\\
\sim&\,\lambda_q(L)^6h^{6q}g^6\int_{\Omega_q\times\mathbb{R}}\lambda_q(L^2)R(K)h^qgf(\mathbf{x},x)f(\mathbf{x},x)^3\,dx\,\omega_q(d\mathbf{x})\\
=&\,\lambda_q(L)^6\lambda_q(L^2)R(K)h^{7q}g^7R(f^2).
\end{aligned}$$

The rest of them have order $\mathcal{O}\left(h^{8q}g^8\right)$, something which can be seen by iteratively applying the Lemma 12 as before.

Proof of (S1.8). It suffices to apply the tower property, the Cauchy-Schwartz inequality, result $\mathbb{E}\big[\big(I_{n,1}^{(2)}\big)^4\big] = \mathcal{O}\left(n^{-4}(h^8+g^8)\right)$ from Lemma 2 and (S1.6):

$$\begin{aligned}
\mathbb{E}\left[M_n^2(\mathbf{X}_1, Z_1)\right] =& 4\frac{c_{h,q}(L)^4}{n^4g^4}\mathbb{E}\left[\mathbb{E}\left[I_{n,1}^{(2)}H_n\left((\mathbf{X}_1,Z_1),(\mathbf{X}_2,Z_2)\right)\big|(\mathbf{X}_1,Z_1)\right]^2\right]\\
\leq& 4\frac{c_{h,q}(L)^4}{n^4g^4}\mathbb{E}\left[\big(I_{n,1}^{(2)}\big)^2H_n^2\left((\mathbf{X}_1,Z_1),(\mathbf{X}_2,Z_2)\right)\right]\\
\leq& 4\frac{c_{h,q}(L)^4}{n^4g^4}\mathbb{E}\left[\big(I_{n,1}^{(2)}\big)^4\right]^{\frac{1}{2}}\mathbb{E}\left[H_n^4\left((\mathbf{X}_1,Z_1),(\mathbf{X}_2,Z_2)\right)\right]^{\frac{1}{2}}\\
=& \mathcal{O}\left((nh^qg)^{-4}\right)\mathcal{O}\left(n^{-4}(h^8+g^8)\right)^{\frac{1}{2}}\mathcal{O}\left(h^{5q}g^5\right)^{\frac{1}{2}}\\
=& \mathcal{O}\left(n^{-6}(h^4+g^4)h^{-\frac{3q}{2}}g^{-\frac{3}{2}}\right).
\end{aligned}$$

□

S1.2 Testing independence with directional data

Lemma 5. *Under **A1**–**A3**,*

$$n(h^qg)^{\frac{1}{2}}\left(T_{n,1}-\frac{R(K)\lambda_q(L^2)\lambda_q(L)^{-2}}{nh^qg}\right)\xrightarrow{d}\mathcal{N}\left(0,2\sigma^2\right).$$

Proof of Lemma 5. By the decomposition of I_n in the proof of the Theorem 1, $T_{n,1} = I_{n,2}+I_{n,3}$ and therefore by (A.2) and (A.5),

$$T_{n,1} = \mathbb{E}\left[I_{n,2}\right]+\mathcal{O}_{\mathbb{P}}\left(n^{-\frac{3}{2}}h^{-q}g^{-1}\right)+2^{\frac{1}{2}}\sigma n^{-1}(h^qg)^{-\frac{1}{2}}N_n,$$

where N_n is asymptotically a normal. On the other hand, by (A.2),

$$\mathbb{E}\left[I_{n,2}\right] = \frac{\lambda_q(L^2)\lambda_q(L)^{-2}R(K)}{nh^qg}+\mathcal{O}\left(n^{-1}\right)$$

and then

$$T_{n,1} = \frac{\lambda_q(L^2)\lambda_q(L)^{-2}R(K)}{nh^qg}+2^{\frac{1}{2}}\sigma n^{-1}(h^qg)^{-\frac{1}{2}}N_n+\mathcal{O}_{\mathbb{P}}\left(n^{-\frac{3}{2}}h^{-q}g^{-1}\right),$$

because $\left(n^{\frac{3}{2}}h^qg\right)^{-1} = o\big((nh^{\frac{q}{2}}g^{\frac{1}{2}})^{-1}\big)$. As the last addend is asymptotically negligible compared with the second,

$$n(h^qg)^{\frac{1}{2}}\left(T_{n,1}-\frac{R(K)\lambda_q(L^2)\lambda_q(L)^{-2}}{nh^qg}\right)\xrightarrow{d}\mathcal{N}\left(0,2\sigma^2\right).$$

□

Lemma 6. *Under independence and **A1**–**A3**,*

$$\mathbb{E}\left[T_{n,2}\right] = \frac{\lambda_q(L^2)\lambda_q(L)^{-2}R(f_Z)}{nh^q} + \frac{R(K)R(f_{\mathbf{X}})}{ng} + o\left(n^{-1}(h^{-q}+g^{-1})\right),$$
$$\mathbb{V}\mathrm{ar}\left[T_{n,2}\right] = \mathcal{O}\left(n^{-2}(h^{-q}+g^{-1})\right).$$

Proof of Lemma 6. The term $T_{n,2}$ can be decomposed using the relation

$$\hat f_h(\mathbf{x})\hat f_g(z) - \mathbb{E}\left[\hat f_h(\mathbf{x})\right]\mathbb{E}\left[\hat f_g(z)\right] = S_1(\mathbf{x},z) + S_2(\mathbf{x},z) + S_3(\mathbf{x},z),$$

where:

$$\begin{aligned} S_1(\mathbf{x},z) &= \left(\hat f_h(\mathbf{x}) - \mathbb{E}\left[\hat f_h(\mathbf{x})\right]\right)\left(\hat f_g(z) - \mathbb{E}\left[\hat f_g(z)\right]\right),\\ S_2(\mathbf{x},z) &= \left(\hat f_h(\mathbf{x}) - \mathbb{E}\left[\hat f_h(\mathbf{x})\right]\right)\mathbb{E}\left[\hat f_g(z)\right],\\ S_3(\mathbf{x},z) &= \left(\hat f_g(z) - \mathbb{E}\left[\hat f_g(z)\right]\right)\mathbb{E}\left[\hat f_h(\mathbf{x})\right]. \end{aligned}$$

Hence,

$$\begin{aligned} T_{n,2} =& \int_{\Omega_q\times\mathbb{R}} S_1^2(\mathbf{x},z)\,dz\,\omega_q(d\mathbf{x}) + \int_{\Omega_q\times\mathbb{R}} S_2^2(\mathbf{x},z)\,dz\,\omega_q(d\mathbf{x})\\ &+ \int_{\Omega_q\times\mathbb{R}} S_3^2(\mathbf{x},z)\,dz\,\omega_q(d\mathbf{x}) + 2\int_{\Omega_q\times\mathbb{R}} S_1(\mathbf{x},z)S_2(\mathbf{x},z)\,dz\,\omega_q(d\mathbf{x})\\ &+ 2\int_{\Omega_q\times\mathbb{R}} S_1(\mathbf{x},z)S_3(\mathbf{x},z)\,dz\,\omega_q(d\mathbf{x}) + 2\int_{\Omega_q\times\mathbb{R}} S_2(\mathbf{x},z)S_3(\mathbf{x},z)\,dz\,\omega_q(d\mathbf{x})\\ =& T_{n,2}^{(1)} + T_{n,2}^{(2)} + T_{n,2}^{(3)} + T_{n,2}^{(4)} + T_{n,2}^{(5)} + T_{n,2}^{(6)}. \end{aligned}$$

To compute the expectation of each addend under independence, use the variance and expectation expansions for the directional and linear estimator (see for example García-Portugués et al. (2013b) for both) and relation (2.1). Recall that due to assumption **A1** it is possible to consider Taylor expansions on the marginal densities that have uniform remaining orders.

$$\begin{aligned} \mathbb{E}\left[T_{n,2}^{(1)}\right] &= \int_{\Omega_q} \mathbb{V}\mathrm{ar}\left[\hat f_h(\mathbf{x})\right]\omega_q(d\mathbf{x}) \int_{\mathbb{R}} \mathbb{V}\mathrm{ar}\left[\hat f_g(z)\right] dz\\ &= \mathcal{O}\left((n^2h^qg)^{-1}\right),\\ \mathbb{E}\left[T_{n,2}^{(2)}\right] &= \int_{\Omega_q} \mathbb{V}\mathrm{ar}\left[\hat f_h(\mathbf{x})\right]\omega_q(d\mathbf{x}) \int_{\mathbb{R}} \mathbb{E}\left[\hat f_g(z)\right]^2 dz\\ &= \left[\frac{\lambda_q(L^2)\lambda_q(L)^{-2}}{nh^q} + \mathcal{O}\left(n^{-1}\right)\right]\left[R(f_Z) + o(1)\right]\\ &= \frac{\lambda_q(L^2)\lambda_q(L)^{-2}R(f_Z)}{nh^q} + o\left((nh^q)^{-1}\right), \end{aligned}$$

$$\begin{aligned}\mathbb{E}\left[T_{n,2}^{(3)}\right] &= \int_{\mathbb{R}} \mathbb{V}\mathrm{ar}\left[\hat{f}_g(z)\right] dz \int_{\Omega_q} \mathbb{E}\left[\hat{f}_h(\mathbf{x})\right]^2 \omega_q(d\mathbf{x}) \\ &= \left[\frac{R(K)}{ng} + \mathcal{O}\left(n^{-1}\right)\right]\left[R(f_{\mathbf{X}}) + o\left(1\right)\right] \\ &= \frac{R(K)R(f_{\mathbf{X}})}{ng} + o\left((ng)^{-1}\right).\end{aligned}$$

The expectation of $T_{n,2}^{(4)}$, $T_{n,2}^{(5)}$ and $T_{n,2}^{(6)}$ is zero because of the separability of the directional and linear components. Joining these results,

$$\mathbb{E}\left[T_{n,2}\right] = \frac{\lambda_q(L^2)\lambda_q(L)^{-2}R(f_Z)}{nh^q} + \frac{R(K)R(f_{\mathbf{X}})}{ng} + o\left(n^{-1}(h^{-q} + g^{-1})\right),$$

because $\left(n^2h^qg\right)^{-1} = o\left(n^{-1}\left(h^{-q} + g^{-1}\right)\right)$.

Computing the variance is not so straightforward as the expectation and some extra results are needed. First of all, recall that by the formula of the variance of the sum, the Cauchy-Schwartz inequality and Lemma 12,

$$\mathbb{V}\mathrm{ar}\left[T_{n,2}\right] = \mathbb{V}\mathrm{ar}\left[\sum_{i=1}^{6} T_{n,2}^{(i)}\right] = \sum_{i=1}^{6} \mathcal{O}\left(\mathbb{V}\mathrm{ar}\left[T_{n,2}^{(i)}\right]\right).$$

Then the variance of each addend will be computed separately. For that purpose, recall that by the decomposition of the ISE given in Theorem 1,

$$\int_{\Omega_q\times\mathbb{R}} \left(\hat{f}_{h,g}(\mathbf{x},z) - \mathbb{E}\left[\hat{f}_{h,g}(\mathbf{x},z)\right]\right)^2 dz\,\omega_q(d\mathbf{x}) = I_{n,2} + I_{n,3},$$

so by equations (A.2) and (A.4),

$$\begin{aligned}\mathbb{V}\mathrm{ar}\left[I_{n,2} + I_{n,3}\right] &= \mathcal{O}\left(\mathbb{V}\mathrm{ar}\left[I_{n,2}\right] + \mathbb{V}\mathrm{ar}\left[I_{n,3}\right]\right) \\ &= \mathcal{O}\left((n^3h^qg)^{-1} + (n^2h^qg)^{-1}\right) \\ &= \mathcal{O}\left((n^2h^qg)^{-1}\right), \\ \mathbb{E}\left[(I_{n,2} + I_{n,3})^2\right] &= \mathbb{V}\mathrm{ar}\left[I_{n,2} + I_{n,3}\right] + \mathbb{E}\left[I_{n,2} + I_{n,3}\right]^2 \\ &= \mathcal{O}\left((n^2h^qg)^{-1} + (nh^qg)^{-2}\right) \\ &= \mathcal{O}\left((nh^qg)^{-2}\right).\end{aligned}$$

The marginal directional and linear versions of these relations will be required:

$$\mathbb{E}\left[\left(\int_{\Omega_q}\left(\hat{f}_h(\mathbf{x}) - \mathbb{E}\left[\hat{f}_h(\mathbf{x})\right]\right)^2 \omega_q(d\mathbf{x})\right)^2\right] = \mathcal{O}\left((nh^q)^{-2}\right),$$

$$\mathbb{E}\left[\left(\int_{\mathbb{R}}\left(\hat{f}_g(z)-\mathbb{E}\left[\hat{f}_g(z)\right]\right)^2 dz\right)^2\right]=\mathcal{O}\left((ng)^{-2}\right),$$
$$\mathbb{V}\mathrm{ar}\left[\int_{\Omega_q}\left(\hat{f}_h(\mathbf{x})-\mathbb{E}\left[\hat{f}_h(\mathbf{x})\right]\right)^2 \omega_q(d\mathbf{x})\right]=\mathcal{O}\left((n^2h^q)^{-1}\right),$$
$$\mathbb{V}\mathrm{ar}\left[\int_{\mathbb{R}}\left(\hat{f}_g(z)-\mathbb{E}\left[\hat{f}_g(z)\right]\right)^2 dz\right]=\mathcal{O}\left((n^2g)^{-1}\right).$$

Then:

$$\begin{aligned}
\mathbb{V}\mathrm{ar}\left[T_{n,2}^{(1)}\right] &\leq \mathbb{E}\left[\left(T_{n,2}^{(1)}\right)^2\right]\\
&=\mathbb{E}\left[\left(\int_{\Omega_q}\left(\hat{f}_h(\mathbf{x})-\mathbb{E}\left[\hat{f}_h(\mathbf{x})\right]\right)^2 \omega_q(d\mathbf{x})\right)^2\right]\\
&\quad\times\mathbb{E}\left[\left(\int_{\mathbb{R}}\left(\hat{f}_g(z)-\mathbb{E}\left[\hat{f}_g(z)\right]\right)^2 dz\right)^2\right]\\
&=\mathcal{O}\left((nh^q)^{-2}\right)\mathcal{O}\left((ng)^{-2}\right)\\
&=\mathcal{O}\left(n^{-4}h^{-2q}g^{-2}\right),\\
\mathbb{V}\mathrm{ar}\left[T_{n,2}^{(2)}\right] &=\mathbb{V}\mathrm{ar}\left[\left(\int_{\Omega_q}\left(\hat{f}_h(\mathbf{x})-\mathbb{E}\left[\hat{f}_h(\mathbf{x})\right]\right)^2 \omega_q(d\mathbf{x})\right)\left(\int_{\mathbb{R}}\mathbb{E}\left[\hat{f}_g(z)\right]^2 dz\right)\right]\\
&=\left(\int_{\mathbb{R}}\mathbb{E}\left[\hat{f}_g(z)\right]^2 dz\right)^2\mathbb{V}\mathrm{ar}\left[\int_{\Omega_q}\left(\hat{f}_h(\mathbf{x})-\mathbb{E}\left[\hat{f}_h(\mathbf{x})\right]\right)^2 \omega_q(d\mathbf{x})\right]\\
&=\mathcal{O}(1)\,\mathcal{O}\left((n^2h^q)^{-1}\right)\\
&=\mathcal{O}\left((n^2h^q)^{-1}\right),\\
\mathbb{V}\mathrm{ar}\left[T_{n,2}^{(3)}\right] &=\mathbb{V}\mathrm{ar}\left[\left(\int_{\mathbb{R}}\left(\hat{f}_g(z)-\mathbb{E}\left[\hat{f}_g(z)\right]\right)^2 dz\right)\left(\int_{\Omega_q}\mathbb{E}\left[\hat{f}_h(\mathbf{x})\right]^2 \omega_q(d\mathbf{x})\right)\right]\\
&=\left(\int_{\Omega_q}\mathbb{E}\left[\hat{f}_h(\mathbf{x})\right]^2 \omega_q(d\mathbf{x})\right)^2\mathbb{V}\mathrm{ar}\left[\int_{\mathbb{R}}\left(\hat{f}_g(z)-\mathbb{E}\left[\hat{f}_g(z)\right]\right)^2 dz\right]\\
&=\mathcal{O}(1)\,\mathcal{O}\left((n^2g)^{-1}\right)\\
&=\mathcal{O}\left((n^2g)^{-1}\right).
\end{aligned}$$

The next results follows from applying iteratively Cauchy-Schwartz and the previous orders:

$$\begin{aligned}
\mathbb{V}\mathrm{ar}\left[T_{n,2}^{(4)}\right] &\leq \mathbb{E}\left[\left(T_{n,2}^{(4)}\right)^2\right]\\
&\leq \mathbb{E}\left[\left(\int_{\Omega_q\times\mathbb{R}} S_1^2(\mathbf{x},z)\,dz\,\omega_q(d\mathbf{x})\right)\left(\int_{\Omega_q\times\mathbb{R}} S_2^2(\mathbf{y},t)\,dt\,\omega_q(d\mathbf{y})\right)\right]\\
&\leq \mathbb{E}\left[\left(T_{n,2}^{(1)}\right)^2\right]^{\frac{1}{2}}\mathbb{E}\left[\left(T_{n,2}^{(2)}\right)^2\right]^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
&= \mathcal{O}\left(n^{-3}h^{-2q}\right), \\
\mathbb{V}\mathrm{ar}\left[T_{n,2}^{(5)}\right] &\leq \mathbb{E}\left[\left(T_{n,2}^{(1)}\right)^2\right]^{\frac{1}{2}} \mathbb{E}\left[\left(T_{n,2}^{(3)}\right)^2\right]^{\frac{1}{2}} \\
&= \mathcal{O}\left(n^{-3}g^{-2}\right), \\
\mathbb{V}\mathrm{ar}\left[T_{n,2}^{(6)}\right] &\leq \mathbb{E}\left[\left(T_{n,2}^{(2)}\right)^2\right]^{\frac{1}{2}} \mathbb{E}\left[\left(T_{n,2}^{(3)}\right)^2\right]^{\frac{1}{2}} \\
&= \mathcal{O}\left(n^{-2}\right).
\end{aligned}
$$

Therefore, the order of $\mathbb{V}\mathrm{ar}\left[T_{n,2}\right]$ is $\mathcal{O}\left(n^{-2}(h^{-q}+g^{-1})\right)$ since it dominates $\mathcal{O}\left(n^{-4}h^{-2q}g^{-2}\right)$, $\mathcal{O}\left(n^{-3}(h^{-2q}+g^{-2})\right)$ and $\mathcal{O}\left(n^{-2}\right)$ by assumption **A3**. □

Lemma 7. *Under independence and **A1**–**A3**,* $\mathbb{E}\left[T_{n,3}\right] = -2\mathbb{E}\left[T_{n,2}\right]$ *and* $\mathbb{V}\mathrm{ar}\left[T_{n,3}\right] = \mathcal{O}\left(n^{-2}(h^{-q}+g^{-1})\right)$.

Proof of Lemma 7. The term $T_{n,3}$ can be split in a similar fashion to $T_{n,2}$. Let denote

$$
S_4(\mathbf{x},z) = \hat{f}_{h,g}(\mathbf{x},z) - \mathbb{E}\left[\hat{f}_{h,g}(\mathbf{x},z)\right].
$$

Then:

$$
\begin{aligned}
T_{n,3} =& -2\int_{\Omega_q\times\mathbb{R}} S_4(\mathbf{x},z)\left(S_1(\mathbf{x},z)+S_2(\mathbf{x},z)+S_3(\mathbf{x},z)\right)\,dz\,\omega_q(d\mathbf{x}) \\
=& -2\left(T_{n,3}^{(1)}+T_{n,3}^{(2)}+T_{n,3}^{(3)}\right).
\end{aligned}
$$

The key idea now is to use that, under independence,

$$
\begin{aligned}
LK_n\left((\mathbf{x},z),(\mathbf{X},Z)\right) =& L_n\left(\mathbf{x},\mathbf{X}\right)K_n\left(z,Z\right) + L_n\left(\mathbf{x},\mathbf{X}\right)\mathbb{E}\left[K\left(\frac{z-Z}{g}\right)\right] \\
&+ K_n\left(z,Z\right)\mathbb{E}\left[L\left(\frac{1-\mathbf{x}^T\mathbf{X}}{h^2}\right)\right],
\end{aligned}
\tag{S1.10}
$$

where L_n and K_n are the marginal versions of LK_n:

$$
\begin{aligned}
L_n\left(\mathbf{x},\mathbf{y}\right) &= L\left(\frac{1-\mathbf{x}^T\mathbf{y}}{h^2}\right) - \mathbb{E}\left[L\left(\frac{1-\mathbf{x}^T\mathbf{X}}{h^2}\right)\right], \\
K_n\left(z,t\right) &= K\left(\frac{z-t}{g}\right) - \mathbb{E}\left[K\left(\frac{z-Z}{g}\right)\right].
\end{aligned}
$$

By repeated use of (S1.10) in the integrands of $T_{n,3}$ and applying the Fubini theorem, it follows:

$$
\begin{aligned}
\mathbb{E}\left[T_{n,3}^{(1)}\right] =& \frac{c_{h,q}(L)^2}{n^2g^2}\int_{\Omega_q\times\mathbb{R}}\mathbb{E}\left[LK_n\left((\mathbf{x},z),(\mathbf{X},Z)\right)L_n\left(\mathbf{x},\mathbf{X}\right)K_n\left(z,Z\right)\right]\,dz\,\omega_q(d\mathbf{x}) \\
=& \frac{c_{h,q}(L)^2}{n^2g^2}\int_{\Omega_q\times\mathbb{R}}\mathbb{E}\Big[L_n\left(\mathbf{x},\mathbf{X}\right)^2K_n\left(z,Z\right)^2 + L_n\left(\mathbf{x},\mathbf{X}\right)^2K_n\left(z,Z\right)
\end{aligned}
$$

$$
\begin{aligned}
&\times \mathbb{E}\left[K\left(\frac{z-Z}{g}\right)\right] + L_n\left(\mathbf{x},\mathbf{X}\right)K_n\left(z,Z\right)^2\mathbb{E}\left[L\left(\frac{1-\mathbf{x}^T\mathbf{X}}{h^2}\right)\right]\Bigg] dz\,\omega_q(d\mathbf{x})\\
&=\frac{c_{h,q}(L)^2}{n^2g^2}\int_{\Omega_q\times\mathbb{R}}\mathbb{E}\left[L_n\left(\mathbf{x},\mathbf{X}\right)^2\right]\mathbb{E}\left[K_n\left(z,Z\right)^2\right]\,dz\,\omega_q(d\mathbf{x})\\
&=\int_{\Omega_q\times\mathbb{R}}\mathbb{E}\left[S_1(\mathbf{x},z)^2\right]\,dz\,\omega_q(d\mathbf{x})\\
&=\mathbb{E}\left[T_{n,2}^{(1)}\right],
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\left[T_{n,3}^{(2)}\right] &= \frac{c_{h,q}(L)^2}{n^2g^2}\int_{\Omega_q\times\mathbb{R}}\mathbb{E}\left[LK_n\left((\mathbf{x},z),(\mathbf{X},Z)\right)L_n\left(\mathbf{x},\mathbf{X}\right)\mathbb{E}\left[K\left(\frac{z-Z}{g}\right)\right]\right]dz\,\omega_q(d\mathbf{x})\\
&=\frac{c_{h,q}(L)^2}{n^2g^2}\int_{\Omega_q\times\mathbb{R}}\mathbb{E}\Bigg[L_n\left(\mathbf{x},\mathbf{X}\right)^2K_n\left(z,Z\right)\mathbb{E}\left[K\left(\frac{z-Z}{g}\right)\right]\\
&\quad + L_n\left(\mathbf{x},\mathbf{X}\right)K_n\left(z,Z\right)\mathbb{E}\left[K\left(\frac{z-Z}{g}\right)\right]\mathbb{E}\left[L\left(\frac{1-\mathbf{x}^T\mathbf{X}}{h^2}\right)\right]\\
&\quad + L_n\left(\mathbf{x},\mathbf{X}\right)^2\mathbb{E}\left[K\left(\frac{z-Z}{g}\right)\right]^2\Bigg]dz\,\omega_q(d\mathbf{x})\\
&=\frac{c_{h,q}(L)^2}{n^2g^2}\int_{\Omega_q\times\mathbb{R}}\mathbb{E}\left[L_n\left(\mathbf{x},\mathbf{X}\right)^2\right]\mathbb{E}\left[K\left(\frac{z-Z}{g}\right)\right]^2\,dz\,\omega_q(d\mathbf{x})\\
&=\int_{\Omega_q\times\mathbb{R}}\mathbb{E}\left[S_2(\mathbf{x},z)^2\right]\,dz\,\omega_q(d\mathbf{x})\\
&=\mathbb{E}\left[T_{n,2}^{(2)}\right],
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\left[T_{n,3}^{(3)}\right] &= \frac{c_{h,q}(L)^2}{n^2g^2}\int_{\Omega_q\times\mathbb{R}}\mathbb{E}\Bigg[LK_n\left((\mathbf{x},z),(\mathbf{X},Z)\right)K_n\left(z,Z\right)\\
&\quad\times\mathbb{E}\left[L\left(\frac{1-\mathbf{x}^T\mathbf{X}}{h^2}\right)\right]\Bigg]dz\,\omega_q(d\mathbf{x})\\
&=\frac{c_{h,q}(L)^2}{n^2g^2}\int_{\Omega_q\times\mathbb{R}}\mathbb{E}\Bigg[L_n\left(\mathbf{x},\mathbf{X}\right)K_n\left(z,Z\right)\mathbb{E}\left[L\left(\frac{1-\mathbf{x}^T\mathbf{X}}{h^2}\right)\right]\\
&\quad + L_n\left(\mathbf{x},\mathbf{X}\right)K_n\left(z,Z\right)\mathbb{E}\left[L\left(\frac{1-\mathbf{x}^T\mathbf{X}}{h^2}\right)\right]\mathbb{E}\left[K\left(\frac{z-Z}{g}\right)\right]\\
&\quad + K_n\left(z,Z\right)^2\mathbb{E}\left[L\left(\frac{1-\mathbf{x}^T\mathbf{X}}{h^2}\right)\right]^2\Bigg]dz\,\omega_q(d\mathbf{x})\\
&=\frac{c_{h,q}(L)^2}{n^2g^2}\int_{\Omega_q\times\mathbb{R}}\mathbb{E}\left[K_n\left(z,Z\right)^2\right]\mathbb{E}\left[L\left(\frac{1-\mathbf{x}^T\mathbf{X}}{h^2}\right)\right]^2\,dz\,\omega_q(d\mathbf{x})\\
&=\int_{\Omega_q\times\mathbb{R}}\mathbb{E}\left[S_3(\mathbf{x},z)^2\right]\,dz\,\omega_q(d\mathbf{x})
\end{aligned}
$$

$$= \mathbb{E}\left[T_{n,2}^{(3)}\right].$$

Then $\mathbb{E}\left[T_{n,3}\right] = -2\mathbb{E}\left[T_{n,2}\right]$.

Computing the variance is much more tedious: the order obtained by bounding the variances by repeated use of the Cauchy-Schwartz inequality is not enough. Instead of, a laborious decomposition of the term $T_{n,3}$ has to be done in order to compute separately the variance of each addend, by following the steps of Rosenblatt and Wahlen (1992). The first step is to split the variance using the Cauchy-Schwartz inequality and Lemma 12:

$$\mathbb{V}\mathrm{ar}\left[T_{n,3}\right] = \mathcal{O}\left(\mathbb{V}\mathrm{ar}\left[T_{n,3}^{(1)}\right] + \mathbb{V}\mathrm{ar}\left[T_{n,3}^{(2)}\right] + \mathbb{V}\mathrm{ar}\left[T_{n,3}^{(3)}\right]\right).$$

Each of the three terms will be also decomposed into other addends. To simplify their computation the following notation will be employed:

$$\begin{aligned} CLK_n(&(\mathbf{x}_1,z_1),(\mathbf{X}_1,Z_1);(\mathbf{x}_2,z_2),(\mathbf{X}_2,Z_2)) \\ &= \mathbb{C}\mathrm{ov}\left[LK\left(\frac{1-\mathbf{x}_1^T\mathbf{X}_1}{h^2},\frac{z_1-Z_1}{g}\right),LK\left(\frac{1-\mathbf{x}_2^T\mathbf{X}_2}{h^2},\frac{z_2-Z_2}{g}\right)\right] \end{aligned}$$

and also its marginal versions:

$$\begin{aligned} CL_n(\mathbf{x}_1,\mathbf{X}_1;\mathbf{x}_2,\mathbf{X}_2) &= \mathbb{C}\mathrm{ov}\left[L\left(\frac{1-\mathbf{x}_1^T\mathbf{X}_1}{h^2}\right),L\left(\frac{1-\mathbf{x}_2^T\mathbf{X}_2}{h^2}\right)\right], \\ CK_n(z_1,Z_1;z_2,Z_2) &= \mathbb{C}\mathrm{ov}\left[K\left(\frac{z_1-Z_1}{g}\right),K\left(\frac{z_2-Z_2}{g}\right)\right]. \end{aligned}$$

Term $T_{n,3}^{(2)}$. To begin with, let examine $T_{n,3}^{(2)}$ using the notation of LK_n, L_n and K_n:

$$T_{n,3}^{(2)} = \frac{c_{h,q}(L)^2}{n^2g^2}\sum_{i=1}^n\sum_{j=1}^n\int_{\Omega_q\times\mathbb{R}} LK_n((\mathbf{x},z),(\mathbf{X}_i,Z_i))L_n(\mathbf{x},\mathbf{X}_j)\mathbb{E}\left[K\left(\frac{z-Z}{g}\right)\right]\,dz\,\omega_q(d\mathbf{x}).$$

where the double summation can be split into two summations (a single sum plus the sum of the cross terms). Then,

$$\mathbb{V}\mathrm{ar}\left[T_{n,3}^{(2)}\right] = \frac{c_{h,q}(L)^4}{n^4g^4}\mathcal{O}\left(n\mathbb{V}\mathrm{ar}\left[T_{n,3}^{(2,1)}\right] + n^2\mathbb{V}\mathrm{ar}\left[T_{n,3}^{(2,2)}\right]\right),$$

where:

$$\begin{aligned} T_{n,3}^{(2,1)} &= \int_{\Omega_q\times\mathbb{R}} LK_n((\mathbf{x},z),(\mathbf{X},Z))L_n(\mathbf{x},\mathbf{X})\mathbb{E}\left[K\left(\frac{z-Z}{g}\right)\right]\,dz\,\omega_q(d\mathbf{x}), \\ T_{n,3}^{(2,2)} &= \int_{\Omega_q\times\mathbb{R}} LK_n((\mathbf{x},z),(\mathbf{X}_1,Z_1))L_n(\mathbf{x},\mathbf{X}_2)\mathbb{E}\left[K\left(\frac{z-Z}{g}\right)\right]\,dz\,\omega_q(d\mathbf{x}). \end{aligned}$$

The first term is computed by

$$
\begin{aligned}
\mathbb{V}\mathrm{ar}\left[T_{n,3}^{(2,1)}\right] \leq& \mathbb{E}\left[\left(T_{n,3}^{(2,1)}\right)^2\right] \\
=& \mathcal{O}\left(g^2\right)\mathbb{E}\left[\left(\int_{\Omega_q\times\mathbb{R}} LK_n((\mathbf{x},z),(\mathbf{X},Z))L_n(\mathbf{x},\mathbf{X})\,dz\,\omega_q(d\mathbf{x})\right)^2\right] \\
=& \mathcal{O}\left(g^2\right)\mathbb{E}\left[\left(\int_{\Omega_q\times\mathbb{R}}\left[LK\left(\frac{1-\mathbf{x}^T\mathbf{X}}{h^2},\frac{z-Z}{g}\right)-\mathcal{O}\left(h^q g\right)\right]\right.\right. \\
&\left.\left.\times\left[L\left(\frac{1-\mathbf{x}^T\mathbf{X}}{h^2}\right)-\mathcal{O}\left(h^q\right)\right]\,dz\,\omega_q(d\mathbf{x})\right)^2\right] \\
=& \mathcal{O}\left(g^2\right)\left(\int_{\Omega_{q-1}}\int_0^{2h^{-2}}\int_{\mathbb{R}}\left[LK\left(r,t\right)-\mathcal{O}\left(h^q g\right)\right]\left[L\left(r\right)-\mathcal{O}\left(h^q\right)\right]\right. \\
&\left.\times h^q(2-h^2r)^{\frac{q}{2}-1}r^{\frac{q}{2}-1}g\,dz\,dr\,\omega_{q-1}(d\boldsymbol{\xi})\right)^2 \\
=& \mathcal{O}\left(h^{2q}g^4\right),
\end{aligned}
$$

where the second equality follows from $\mathbb{E}\left[LK\left(\frac{1-\mathbf{x}^T\mathbf{X}}{h^2},\frac{z-Z}{g}\right)\right]=\mathcal{O}\left(h^q g\right)$ and $\mathbb{E}\left[L\left(\frac{1-\mathbf{x}^T\mathbf{X}}{h^2}\right)\right]=\mathcal{O}\left(h^q\right)$, and the third from applying the changes of variables of the proof of Lemma 4. The second addend is

$$
\begin{aligned}
\mathbb{V}\mathrm{ar}\left[T_{n,3}^{(2,2)}\right] \leq& \mathbb{E}\Bigg[\int_{\Omega_q\times\mathbb{R}}\int_{\Omega_q\times\mathbb{R}} LK_n((\mathbf{x}_1,z_1),(\mathbf{X}_1,Z_1))L_n(\mathbf{x}_1,\mathbf{X}_2) \\
&\times\mathbb{E}\left[K\left(\frac{z_1-Z}{g}\right)\right]LK_n((\mathbf{x}_2,z_2),(\mathbf{X}_1,Z_1))L_n(\mathbf{x}_2,\mathbf{X}_2) \\
&\times\mathbb{E}\left[K\left(\frac{z_2-Z}{g}\right)\right]\,dz_1\,\omega_q(d\mathbf{x}_1)\,dz_2\,\omega_q(d\mathbf{x}_2)\Bigg] \\
=& \int_{\Omega_q\times\mathbb{R}}\int_{\Omega_q\times\mathbb{R}} CLK_n((\mathbf{x}_1,z_1),(\mathbf{X}_1,Z_1);(\mathbf{x}_2,z_2),(\mathbf{X}_1,Z_1)) \\
&\times CL_n(\mathbf{x}_1,\mathbf{X}_2;\mathbf{x}_2,\mathbf{X}_2)\mathbb{E}\left[K\left(\frac{z_1-Z}{g}\right)\right]\mathbb{E}\left[K\left(\frac{z_2-Z}{g}\right)\right] \\
&\times\,dz_1\,\omega_q(d\mathbf{x}_1)\,dz_2\,\omega_q(d\mathbf{x}_2) \\
\leq& \left(\int_{\Omega_q\times\mathbb{R}}\int_{\Omega_q\times\mathbb{R}} CLK_n((\mathbf{x}_1,z_1),(\mathbf{X}_1,Z_1);(\mathbf{x}_2,z_2),(\mathbf{X}_1,Z_1))\right. \\
&\left.\times\,dz_1\,\omega_q(d\mathbf{x}_1)\,dz_2\,\omega_q(d\mathbf{x}_2)\right)\mathcal{O}\left(h^q g^2\right),
\end{aligned}
$$

because $CL_n(\mathbf{x}_1,\mathbf{X}_2;\mathbf{x}_2,\mathbf{X}_2)=\mathcal{O}\left(h^q\right)$ by Cauchy-Schwartz and the directional version of Lemma 11, and $\mathbb{E}\big[K\big(\frac{z-Z}{g}\big)\big] = \mathcal{O}(g)$. Also, the integral of the covariance is

$$\begin{aligned}
&\int_{\Omega_q\times\mathbb{R}}\int_{\Omega_q\times\mathbb{R}} CLK_n((\mathbf{x}_1,z_1),(\mathbf{X}_1,Z_1);(\mathbf{x}_2,z_2),(\mathbf{X}_1,Z_1))\,dz_1\,\omega_q(d\mathbf{x}_1)\,dz_2\,\omega_q(d\mathbf{x}_2)\\
&\quad=\int_{\Omega_q\times\mathbb{R}}\int_{\Omega_q\times\mathbb{R}}\mathbb{E}\left[LK\left(\frac{1-\mathbf{x}_1^T\mathbf{X}_1}{h^2},\frac{z_1-Z_1}{g}\right)LK\left(\frac{1-\mathbf{x}_2^T\mathbf{X}_1}{h^2},\frac{z_2-Z_1}{g}\right)\right]\\
&\quad\quad\times\,dz_1\,\omega_q(d\mathbf{x}_1)\,dz_2\,\omega_q(d\mathbf{x}_2)-\mathcal{O}\left(h^{2q}g^2\right)\\
&\quad=\int_{\Omega_q\times\mathbb{R}}\int_{\Omega_q\times\mathbb{R}}\int_{\Omega_q\times\mathbb{R}}LK\left(\frac{1-\mathbf{x}_1^T\mathbf{y}}{h^2},\frac{z_1-t}{g}\right)LK\left(\frac{1-\mathbf{x}_2^T\mathbf{y}}{h^2},\frac{z_2-t}{g}\right)\\
&\quad\quad\times\,f(\mathbf{y},t)\,dt\,\omega_q(d\mathbf{y})\,dz_1\,\omega_q(d\mathbf{x}_1)\,dz_2\,\omega_q(d\mathbf{x}_2)-\mathcal{O}\left(h^{2q}g^2\right)\\
&\quad=\mathcal{O}\left(h^{2q}g^2\right),
\end{aligned}$$

as it follows that the order of the first addend is $\mathcal{O}\left(h^{2q}g^2\right)$ by applying *i–ix* in the same way as in the computation of A_1 in Lemma 4 (recall that the square in A_1 is not present here and therefore the order is larger). Then $\mathbb{V}\mathrm{ar}\big[T_{n,3}^{(2,2)}\big] = \mathcal{O}\left(h^{3q}g^4\right)$ and as a consequence,

$$\mathbb{V}\mathrm{ar}\left[T_{n,3}^{(2)}\right] = \frac{c_{h,q}(L)^4}{n^4g^4}\mathcal{O}\left(nh^{2q}g^4+n^2h^{3q}g^4\right) = \mathcal{O}\left(n^{-2}h^{-q}\right). \tag{S1.11}$$

Term $T_{n,3}^{(3)}$. This addend follows analogously from $T_{n,3}^{(2)}$, as the only difference is the swapping of the roles of the directional and linear components:

$$T_{n,3}^{(3)} = \frac{c_{h,q}(L)^2}{n^2g^2}\sum_{i=1}^n\sum_{j=1}^n\int_{\Omega_q\times\mathbb{R}} LK_n((\mathbf{x},z),(\mathbf{X}_i,Z_i))K_n(z,Z_j)\mathbb{E}\left[L\left(\frac{1-\mathbf{x}^T\mathbf{X}}{h^2}\right)\right]dz\,\omega_q(d\mathbf{x}),$$

with the same decomposition that gives

$$\mathbb{V}\mathrm{ar}\left[T_{n,3}^{(3)}\right] = \frac{c_{h,q}(L)^4}{n^4g^4}\mathcal{O}\left(n\mathbb{V}\mathrm{ar}\left[T_{n,3}^{(3,1)}\right]+n^2\mathbb{V}\mathrm{ar}\left[T_{n,3}^{(3,2)}\right]\right),$$

where:

$$\begin{aligned}
T_{n,3}^{(3,1)} &= \int_{\Omega_q\times\mathbb{R}} LK_n((\mathbf{x},z),(\mathbf{X},Z))K_n(z,Z)\mathbb{E}\left[L\left(\frac{1-\mathbf{x}^T\mathbf{X}}{h^2}\right)\right]dz\,\omega_q(d\mathbf{x}),\\
T_{n,3}^{(3,2)} &= \int_{\Omega_q\times\mathbb{R}} LK_n((\mathbf{x},z),(\mathbf{X}_1,Z_1))K_n(z,Z_2)\mathbb{E}\left[L\left(\frac{1-\mathbf{x}^T\mathbf{X}}{h^2}\right)\right]dz\,\omega_q(d\mathbf{x}).
\end{aligned}$$

Then, by similar computations to those of $T_{n,3}^{(3)}$, $\mathbb{V}\mathrm{ar}\big[T_{n,3}^{(3,1)}\big] = \mathcal{O}\left(h^{4q}g^2\right)$, $\mathbb{V}\mathrm{ar}\big[T_{n,3}^{(3,2)}\big] = \mathcal{O}\left(h^{4q}g^3\right)$ and

$$\mathbb{V}\mathrm{ar}\left[T_{n,3}^{(3)}\right] = \frac{c_{h,q}(L)^4}{n^4g^4}\mathcal{O}\left(nh^{4q}g^2+n^2h^{4q}g^3\right) = \mathcal{O}\left(n^{-2}g^{-1}\right). \tag{S1.12}$$

Term $T_{n,3}^{(1)}$. This is the hardest part, as it presents more combinations. As with the previous terms,

$$T_{n,3}^{(1)} = \frac{c_{h,q}(L)^2}{n^3g^2} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \int_{\Omega_q\times\mathbb{R}} LK_n((\mathbf{x},z),(\mathbf{X}_i,Z_i))L_n(\mathbf{x},\mathbf{X}_j)K_n(z,Z_k)\,dz\,\omega_q(d\mathbf{x}).$$

and now the triple summation can be split into five summations

$$\begin{aligned}\mathbb{V}\mathrm{ar}\left[T_{n,3}^{(1)}\right] = \frac{c_{h,q}(L)^4}{n^6g^4}\mathcal{O}\Big(&n\mathbb{V}\mathrm{ar}\left[T_{n,3}^{(1,1)}\right] + n^2\left(\mathbb{V}\mathrm{ar}\left[T_{n,3}^{(1,2a)}\right] + \mathbb{V}\mathrm{ar}\left[T_{n,3}^{(1,2b)}\right] + \mathbb{V}\mathrm{ar}\left[T_{n,3}^{(1,2c)}\right]\right)\\ &+ n^3\mathbb{V}\mathrm{ar}\left[T_{n,3}^{(1,3)}\right]\Big),\end{aligned}$$

where:

$$\begin{aligned}
T_{n,3}^{(1,1)} &= \int_{\Omega_q\times\mathbb{R}} LK_n((\mathbf{x},z),(\mathbf{X}_1,Z_1))L_n(\mathbf{x},\mathbf{X}_1)K_n(z,Z_1)\,dz\,\omega_q(d\mathbf{x}),\\
T_{n,3}^{(1,2a)} &= \int_{\Omega_q\times\mathbb{R}} LK_n((\mathbf{x},z),(\mathbf{X}_1,Z_1))L_n(\mathbf{x},\mathbf{X}_2)K_n(z,Z_2)\,dz\,\omega_q(d\mathbf{x}),\\
T_{n,3}^{(1,2b)} &= \int_{\Omega_q\times\mathbb{R}} LK_n((\mathbf{x},z),(\mathbf{X}_1,Z_1))L_n(\mathbf{x},\mathbf{X}_1)K_n(z,Z_2)\,dz\,\omega_q(d\mathbf{x}),\\
T_{n,3}^{(1,2c)} &= \int_{\Omega_q\times\mathbb{R}} LK_n((\mathbf{x},z),(\mathbf{X}_1,Z_1))L_n(\mathbf{x},\mathbf{X}_2)K_n(z,Z_1)\,dz\,\omega_q(d\mathbf{x}),\\
T_{n,3}^{(1,3)} &= \int_{\Omega_q\times\mathbb{R}} LK_n((\mathbf{x},z),(\mathbf{X}_1,Z_1))L_n(\mathbf{x},\mathbf{X}_2)K_n(z,Z_3)\,dz\,\omega_q(d\mathbf{x}).
\end{aligned}$$

The first term is computed by

$$\begin{aligned}
\mathbb{V}\mathrm{ar}\Big[T_{n,3}^{(1,1)}\Big] &\leq \mathbb{E}\left[\left(\int_{\Omega_q\times\mathbb{R}} LK_n((\mathbf{x},z),(\mathbf{X},Z))L_n(\mathbf{x},\mathbf{X})K_n(z,Z)\,dz\,\omega_q(d\mathbf{x})\right)^2\right]\\
&= \mathbb{E}\Bigg[\Bigg(\int_{\Omega_q\times\mathbb{R}} \left[LK\left(\frac{1-\mathbf{x}^T\mathbf{X}}{h^2},\frac{z-Z}{g}\right) - \mathcal{O}\left(h^qg\right)\right]\\
&\quad\times\left[L\left(\frac{1-\mathbf{x}^T\mathbf{X}}{h^2}\right) - \mathcal{O}\left(h^q\right)\right]\left[K\left(\frac{z-Z}{g}\right) - \mathcal{O}\left(g\right)\right]dz\,\omega_q(d\mathbf{x})\Bigg)^2\Bigg]\\
&= \Bigg(\int_{\Omega_{q-1}}\int_0^{2h^{-2}}\int_{\mathbb{R}} \left[LK\left(r,t\right) - \mathcal{O}\left(h^qg\right)\right]\left[L\left(r\right) - \mathcal{O}\left(h^q\right)\right]\\
&\quad\times\left[K\left(t\right) - \mathcal{O}\left(g\right)\right]h^q(2-h^2r)^{\frac{q}{2}-1}r^{\frac{q}{2}-1}g\,dr\,\omega_{q-1}(d\boldsymbol{\xi})\,dz\Bigg)^2\\
&= \mathcal{O}\left(h^{2q}g^2\right),
\end{aligned}$$

by the same arguments as for $T_{n,3}^{(2,1)}$. The fifth addend is

$$\mathbb{V}\mathrm{ar}\Big[T_{n,3}^{(1,3)}\Big] \leq \mathbb{E}\Bigg[\int_{\Omega_q\times\mathbb{R}}\int_{\Omega_q\times\mathbb{R}} LK_n((\mathbf{x}_1,z_1),(\mathbf{X}_1,Z_1))L_n(\mathbf{x}_1,\mathbf{X}_2)K_n(z_1,Z_3)$$

$$
\begin{aligned}
&\times LK_n((\mathbf{x}_2,z_2),(\mathbf{X}_1,Z_1))L_n(\mathbf{x}_2,\mathbf{X}_2)K_n(z_2,Z_3)\,dz_1\,\omega_q(d\mathbf{x}_1)\,dz_2\,\omega_q(d\mathbf{x}_2)\Big]\\
&=\int_{\Omega_q\times\mathbb{R}}\int_{\Omega_q\times\mathbb{R}} CLK_n((\mathbf{x}_1,z_1),(\mathbf{X}_1,Z_1);(\mathbf{x}_2,z_2),(\mathbf{X}_1,Z_1))\\
&\quad\times CL_n(\mathbf{x}_1,\mathbf{X}_2;\mathbf{x}_2,\mathbf{X}_2)CK_n(z_1,Z_3;z_2,Z_3)\,dz_1\,\omega_q(d\mathbf{x}_1)\,dz_2\,\omega_q(d\mathbf{x}_2)\\
&\le\mathcal{O}\left(h^q g\right)\int_{\Omega_q\times\mathbb{R}}\int_{\Omega_q\times\mathbb{R}} CLK_n((\mathbf{x}_1,z_1),(\mathbf{X}_1,Z_1);(\mathbf{x}_2,z_2),(\mathbf{X}_1,Z_1))\\
&\quad\times\ dz_1\,\omega_q(d\mathbf{x}_1)\,dz_2\,\omega_q(d\mathbf{x}_2)\\
&\le\mathcal{O}\left(h^{3q}g^3\right),
\end{aligned}
$$

again by the same arguments used for $T_{n,3}^{(2,2)}$. It only remains to obtain the variance of $T_{n,3}^{(1,2a)}$, $T_{n,3}^{(1,2b)}$ and $T_{n,3}^{(1,2c)}$. The first one arises from

$$
\begin{aligned}
\mathbb{V}\mathrm{ar}\Big[T_{n,3}^{(1,2a)}\Big]&\le\mathbb{E}\bigg[\int_{\Omega_q\times\mathbb{R}}\int_{\Omega_q\times\mathbb{R}} LK_n((\mathbf{x}_1,z_1),(\mathbf{X}_1,Z_1))L_n(\mathbf{x}_1,\mathbf{X}_2)K_n(z_1,Z_2)\\
&\quad\times LK_n((\mathbf{x}_2,z_2),(\mathbf{X}_1,Z_1))L_n(\mathbf{x}_2,\mathbf{X}_2)K_n(z_2,Z_2)\,dz_1\,\omega_q(d\mathbf{x}_1)\,dz_2\,\omega_q(d\mathbf{x}_2)\bigg]\\
&=\int_{\Omega_q\times\mathbb{R}}\int_{\Omega_q\times\mathbb{R}} CLK_n((\mathbf{x}_1,z_1),(\mathbf{X}_1,Z_1);(\mathbf{x}_2,z_2),(\mathbf{X}_1,Z_1))\\
&\quad\times CL_n(\mathbf{x}_1,\mathbf{X}_2;\mathbf{x}_2,\mathbf{X}_2)CK_n(z_1,Z_2;z_2,Z_2)\,dz_1\,\omega_q(d\mathbf{x}_1)\,dz_2\,\omega_q(d\mathbf{x}_2)\\
&=\mathcal{O}\left(h^{3q}g^3\right),
\end{aligned}
$$

in virtue of the assumption of independence and the computation of $\mathbb{V}\mathrm{ar}\big[T_{n,3}^{(1,1)}\big]$. The second one is

$$
\begin{aligned}
\mathbb{V}\mathrm{ar}\Big[T_{n,3}^{(1,2b)}\Big]&\le\mathbb{E}\bigg[\int_{\Omega_q\times\mathbb{R}}\int_{\Omega_q\times\mathbb{R}} LK_n((\mathbf{x}_1,z_1),(\mathbf{X}_1,Z_1))L_n(\mathbf{x}_1,\mathbf{X}_1)K_n(z_1,Z_2)\\
&\quad\times LK_n((\mathbf{x}_2,z_2),(\mathbf{X}_1,Z_1))L_n(\mathbf{x}_2,\mathbf{X}_1)K_n(z_2,Z_2)\,dz_1\,\omega_q(d\mathbf{x}_1)\,dz_2\,\omega_q(d\mathbf{x}_2)\bigg]\\
&=\int_{\Omega_q\times\mathbb{R}}\int_{\Omega_q\times\mathbb{R}} \mathbb{E}\big[LK_n((\mathbf{x}_1,z_1),(\mathbf{X}_1,Z_1))LK_n((\mathbf{x}_2,z_2),(\mathbf{X}_1,Z_1))\\
&\quad\times L_n(\mathbf{x}_1,\mathbf{X}_1)L_n(\mathbf{x}_2,\mathbf{X}_1)\big]CK_n(z_1,Z_2;z_2,Z_2)\,dz_1\,\omega_q(d\mathbf{x}_1)\,dz_2\,\omega_q(d\mathbf{x}_2)\\
&=\mathcal{O}\left(g\right)\mathbb{E}\left[\left(\int_{\Omega_q\times\mathbb{R}} LK_n((\mathbf{x},z),(\mathbf{X}_1,Z_1))L_n(\mathbf{x},\mathbf{X}_1)\,dz\,\omega_q(d\mathbf{x})\right)^2\right]\\
&=\mathcal{O}\left(h^{2q}g^3\right),
\end{aligned}
$$

where the order of the expectation is obtained again using the change of variables described in the proof of Lemma 10,

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_{\Omega_q\times\mathbb{R}} LK_n((\mathbf{x},z),(\mathbf{X}_1,Z_1))L_n(\mathbf{x},\mathbf{X}_1)\,dz\,\omega_q(d\mathbf{x})\right)^2\right]\\
&\quad=\mathbb{E}\left[\left(\int_{\Omega_q\times\mathbb{R}}\left[LK\left(\frac{1-\mathbf{x}^T\mathbf{X}_1}{h^2},\frac{z-Z_1}{g}\right)-\mathcal{O}\left(h^qg\right)\right]\right.\right.\\
&\quad\quad\left.\left.\times\left[L\left(\frac{1-\mathbf{x}^T\mathbf{X}_1}{h^2}\right)-\mathcal{O}\left(h^q\right)\right]\,dz\,\omega_q(d\mathbf{x})\right)^2\right]\\
&\quad=\mathbb{E}\left[\left(\int_{\Omega_{q-1}}\int_0^{2h^{-2}}\int_{\mathbb{R}}\left[LK\left(r,u\right)-\mathcal{O}\left(h^qg\right)\right]\left[L\left(r\right)-\mathcal{O}\left(h^q\right)\right]\right.\right.\\
&\quad\quad\left.\left.\times h^q(2-h^2r)^{\frac{q}{2}-1}r^{\frac{q}{2}-1}g\,du\,dr\,\omega_{q-1}(d\boldsymbol{\xi})\right)^2\right]\\
&\quad=\mathcal{O}\left(h^{2q}g^2\right).
\end{aligned}$$

The variance of $T_{n,3}^{(1,2c)}$ is obtained analogously:

$$\begin{aligned}
\mathbb{V}\mathrm{ar}\left[T_{n,3}^{(1,2c)}\right]&\leq\mathbb{E}\left[\int_{\Omega_q\times\mathbb{R}}\int_{\Omega_q\times\mathbb{R}}LK_n((\mathbf{x}_1,z_1),(\mathbf{X}_1,Z_1))L_n(\mathbf{x}_1,\mathbf{X}_2)K_n(z_1,Z_1)\right.\\
&\quad\left.\times LK_n((\mathbf{x}_2,z_2),(\mathbf{X}_1,Z_1))L_n(\mathbf{x}_2,\mathbf{X}_2)K_n(z_2,Z_1)\,dz_1\,\omega_q(d\mathbf{x}_1)\,dz_2\,\omega_q(d\mathbf{x}_2)\right]\\
&=\int_{\Omega_q\times\mathbb{R}}\int_{\Omega_q\times\mathbb{R}}\mathbb{E}\big[LK_n((\mathbf{x}_1,z_1),(\mathbf{X}_1,Z_1))LK_n((\mathbf{x}_2,z_2),(\mathbf{X}_1,Z_1))\\
&\quad\times K_n(z_1,Z_1)K_n(z_2,Z_1)\big]CL_n(\mathbf{x}_1,\mathbf{X}_2;\mathbf{x}_2,\mathbf{X}_2)\,dz_1\,\omega_q(d\mathbf{x}_1)\,dz_2\,\omega_q(d\mathbf{x}_2)\\
&=\mathcal{O}\left(h^q\right)\mathbb{E}\left[\left(\int_{\Omega_q\times\mathbb{R}}LK_n((\mathbf{x},z),(\mathbf{X}_1,Z_1))K_n(z,Z_1)\,dz\,\omega_q(d\mathbf{x})\right)^2\right]\\
&=\mathcal{O}\left(h^{3q}g^2\right).
\end{aligned}$$

Then, putting together the variances of $T_{n,3}^{(1,1)}$, $T_{n,3}^{(1,2a)}$, $T_{n,3}^{(1,2b)}$, $T_{n,3}^{(1,2c)}$ and $T_{n,3}^{(1,3)}$, it follows

$$\begin{aligned}
\mathbb{V}\mathrm{ar}\left[T_{n,3}^{(1)}\right]&=\frac{c_{h,q}(L)^4}{n^6g^4}\mathcal{O}\left(nh^{2q}g^2+n^2(h^{3q}g^3+h^{2q}g^3+h^{3q}g^2)+n^3h^{3q}g^3\right)\\
&=\frac{c_{h,q}(L)^4}{n^6g^4}\mathcal{O}\left(n^3h^{3q}g^3\right)\\
&=\mathcal{O}\left(n^{-3}h^{-q}g^{-1}\right).
\end{aligned}\tag{S1.13}$$

Finally, joining (S1.11), (S1.12) and (S1.13),

$$\mathbb{V}\mathrm{ar}\left[T_{n,3}\right]=\mathcal{O}\left(n^{-3}h^{-q}g^{-1}\right)+\mathcal{O}\left(n^{-2}h^{-q}\right)+\mathcal{O}\left(n^{-2}g^{-1}\right)=\mathcal{O}\left(n^{-2}(h^{-q}+g^{-1})\right),$$

which proves the lemma. □

S1.3 Goodness-of-fit test for models with directional data

Lemma 8. *Under $H_0 : f = f_{\boldsymbol{\theta}_0}$, with $\boldsymbol{\theta}_0 \in \Theta$ unknown and **A1**–**A3** and **A5**–**A6**, $n(h^q g)^{\frac{1}{2}} R_{n,1} \xrightarrow{p} 0$ and $n(h^q g)^{\frac{1}{2}} R_{n,4} \xrightarrow{p} 0$.*

Proof of Lemma 8. Under the null $f = f_{\boldsymbol{\theta}_0}$, for a known $\boldsymbol{\theta}_0 \in \Theta$.

Term $R_{n,4}$. Using a first order Taylor expansion of $f_{\hat{\boldsymbol{\theta}}}$ in $\boldsymbol{\theta}_0$,

$$\begin{aligned} R_{n,4} =& \int_{\Omega_q \times \mathbb{R}} \left(LK_{h,g} \left(f_{\boldsymbol{\theta}_0}(\mathbf{x}, z) - f_{\hat{\boldsymbol{\theta}}}(\mathbf{x}, z) \right) \right)^2 dz\, \omega_q(d\mathbf{x}) \\ =& \int_{\Omega_q \times \mathbb{R}} \left(LK_{h,g} \left(\left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0 \right)^T \frac{\partial f_{\boldsymbol{\theta}}(\mathbf{x}, z)}{\partial \boldsymbol{\theta}} \Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}_n} \right) \right)^2 dz\, \omega_q(d\mathbf{x}) \\ \leq& \left|\left| \hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0 \right|\right|^2 \int_{\Omega_q \times \mathbb{R}} \left(LK_{h,g} \left(\left|\left| \frac{\partial f_{\boldsymbol{\theta}}(\mathbf{x}, z)}{\partial \boldsymbol{\theta}} \Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}_n} \right|\right| \right) \right)^2 dz\, \omega_q(d\mathbf{x}) \\ =& \mathcal{O}_{\mathbb{P}} \left(n^{-1} \right) \mathcal{O}_{\mathbb{P}} (1) \\ =& \mathcal{O}_{\mathbb{P}} \left(n^{-1} \right), \end{aligned}$$

where $\boldsymbol{\theta}_n \in \Theta$ is a certain parameter depending on the sample. The order holds because, on the one hand, $\left|\left| \hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0 \right|\right|^2 = \mathcal{O}_{\mathbb{P}} \left(n^{-1} \right)$ by assumption **A6** and on the other, by **A5** and Lemma 10,

$$\begin{aligned} &\int_{\Omega_q \times \mathbb{R}} \left(LK_{h,g} \left(\left|\left| \frac{\partial f_{\boldsymbol{\theta}}(\mathbf{x}, z)}{\partial \boldsymbol{\theta}} \Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}_n} \right|\right| \right) \right)^2 dz\, \omega_q(d\mathbf{x}) \\ &\qquad = \left(\int_{\Omega_q \times \mathbb{R}} \left|\left| \frac{\partial f_{\boldsymbol{\theta}}(\mathbf{x}, z)}{\partial \boldsymbol{\theta}} \Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}_n} \right|\right|^2 dz\, \omega_q(d\mathbf{x}) \right) (1 + o(1)) \\ &\qquad = \mathcal{O}_{\mathbb{P}} (1). \end{aligned}$$

Therefore, $R_{n,4} = \mathcal{O}_{\mathbb{P}} \left(n^{-1} \right)$ and, by assumption **A3**, $n(h^q g)^{\frac{1}{2}} R_{n,4} \xrightarrow{p} 0$.

Term $R_{n,1}$. It follows also by a Taylor expansion of second order centred at $\boldsymbol{\theta}_0$:

$$\begin{aligned} R_{n,1} =& 2 \frac{c_{h,q}(L)}{ng} \sum_{i=1}^n \int_{\Omega_q \times \mathbb{R}} LK_n \left((\mathbf{x}, z), (\mathbf{X}_i, Z_i) \right) LK_{h,g} \left(f_{\boldsymbol{\theta}_0}(\mathbf{x}, z) - f_{\hat{\boldsymbol{\theta}}}(\mathbf{x}, z) \right) dz\, \omega_q(d\mathbf{x}) \\ =& 2 \frac{c_{h,q}(L)}{ng} \sum_{i=1}^n \int_{\Omega_q \times \mathbb{R}} LK_n \left((\mathbf{x}, z), (\mathbf{X}_i, Z_i) \right) LK_{h,g} \bigg(\left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0 \right)^T \frac{\partial f(\mathbf{x}, z)}{\partial \boldsymbol{\theta}} \Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} \\ &+ \left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0 \right)^T \frac{\partial^2 f(\mathbf{x}, z)}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^T} \Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}_n} \left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0 \right) \bigg) dz\, \omega_q(d\mathbf{x}) \end{aligned}$$

$$
\begin{aligned}
&\leq 2\frac{c_{h,q}(L)}{ng}\sum_{i=1}^n\int_{\Omega_q\times\mathbb{R}} LK_n\left((\mathbf{x},z),(\mathbf{X}_i,Z_i)\right)\left[\big|\big|\hat{\boldsymbol\theta}-\boldsymbol\theta_0\big|\big| LK_{h,g}\left(\left|\left|\frac{\partial f(\mathbf{x},z)}{\partial\boldsymbol\theta}\Big|_{\boldsymbol\theta=\boldsymbol\theta_0}\right|\right|\right)\right.\\
&\quad\left.+\big|\big|\hat{\boldsymbol\theta}-\boldsymbol\theta_0\big|\big|^2 LK_{h,g}\left(\left|\left|\frac{\partial^2 f(\mathbf{x},z)}{\partial\boldsymbol\theta\partial\boldsymbol\theta^T}\Big|_{\boldsymbol\theta=\boldsymbol\theta_n}\right|\right|_F\right)\right]dz\,\omega_q(d\mathbf{x})\\
&=\big|\big|\hat{\boldsymbol\theta}-\boldsymbol\theta_0\big|\big| R_{n,1}^{(1)}+\big|\big|\hat{\boldsymbol\theta}-\boldsymbol\theta_0\big|\big|^2 R_{n,1}^{(2)},
\end{aligned}
$$

where $||A||_F$ stands for the Frobenious norm of the matrix A. By Lemma 10 and assumption of **A5**,

$$
R_{n,1}^{(i)}=\mathcal{O}_{\mathbb{P}}\left(\frac{c_{h,q}(L)}{ng}\sum_{i=1}^n\int_{\Omega_q\times\mathbb{R}} LK_n\left((\mathbf{x},z),(\mathbf{X}_i,Z_i)\right)\,dz\,\omega_q(d\mathbf{x})\right),
$$

for $i=1,2$. As a consequence of this and of assumption **A6**, the first addend of $R_{n,1}$ dominates the second. The proof now is based on proving that $R_{n,1}^{(1)}=\mathcal{O}_{\mathbb{P}}\big(n^{-\frac{1}{2}}\big)$ using the Chebychev inequality and the fact that the integrand of $R_{n,1}^{(1)}$ is deterministic. Now recall that $\mathbb{E}\big[R_{n,1}^{(i)}\big]=0$ and by the proof of (S1.5) in Lemma 4,

$$
\begin{aligned}
\mathbb{V}\mathrm{ar}\Big[R_{n,1}^{(1)}\Big]&=\frac{c_{h,q}(L)^2}{ng^2}\mathbb{E}\left[\left(\int_{\Omega_q\times\mathbb{R}} LK_n\left((\mathbf{x},z),(\mathbf{X},Z)\right)\,dz\,\omega_q(d\mathbf{x})\right)^2\right]\\
&=\frac{c_{h,q}(L)^2}{ng^2}\int_{\Omega_q\times\mathbb{R}}\int_{\Omega_q\times\mathbb{R}}\mathbb{E}\left[LK_n\left((\mathbf{x},z),(\mathbf{X},Z)\right)LK_n\left((\mathbf{y},t),(\mathbf{X},Z)\right)\right]\\
&\quad\times\,dz\,\omega_q(d\mathbf{x})\,dt\,\omega_q(d\mathbf{y})\\
&=\frac{c_{h,q}(L)^2}{ng^2}\int_{\Omega_q\times\mathbb{R}}\int_{\Omega_q\times\mathbb{R}}\left(E_1\left((\mathbf{x},z),(\mathbf{y},t)\right)-E_2\left((\mathbf{x},z),(\mathbf{y},t)\right)\right)\\
&\quad\times\,dz\,\omega_q(d\mathbf{x})\,dt\,\omega_q(d\mathbf{y})\\
&=\frac{c_{h,q}(L)^2}{ng^2}\mathcal{O}\left(h^{2q}g^2\right)\\
&=\mathcal{O}\left(n^{-1}\right),
\end{aligned}
$$

so by the Chebychev inequality, $R_{n,1}^{(1)}=\mathcal{O}_{\mathbb{P}}\big(n^{-\frac{1}{2}}\big)$ and as a consequence of **A5**, $R_{n,1}=\mathcal{O}_{\mathbb{P}}\left(n^{-1}\right)$ and $n(h^qg)^{\frac{1}{2}}R_{n,1}\xrightarrow{p}0$ follows. □

Lemma 9. *Under the alternative hypothesis (5.4) and **A1**–**A3**, **A5** and **A7**,* $n(h^qg)^{\frac{1}{2}}\widetilde{R}_{n,1}\xrightarrow{p}0$ *and* $n(h^qg)^{\frac{1}{2}}\widetilde{R}_{n,4}\xrightarrow{p}R(\Delta)$.

Proof of Lemma 9. The convergence in probability is obtained using the decompositions $\widetilde{R}_{n,1}=R_{n,1}+\widetilde{R}_{n,1}^{(1)}$ and $\widetilde{R}_{n,4}=R_{n,4}+\widetilde{R}_{n,4}^{(1)}+\widetilde{R}_{n,4}^{(2)}$.

Terms $R_{n,1}$ and $R_{n,4}$. The proofs of $n(h^qg)^{\frac{1}{2}}R_{n,1}\xrightarrow{p}0$ and $n(h^qg)^{\frac{1}{2}}R_{n,4}\xrightarrow{p}0$ are analogous to the ones of Lemma 8 and follow just replacing assumption **A6** by **A7** and

H_0 by H_{1P}.

Term $R_{n,1}^{(1)}$. Recall that $\mathbb{E}\big[\widetilde{R}_{n,1}^{(1)}\big] = 0$ and its variance, using the same steps as in the proof of $R_{n,1}^{(1)}$ in Lemma 8, is

$$\begin{aligned}\mathbb{V}\mathrm{ar}\left[\widetilde{R}_{n,1}^{(1)}\right] =& 4\frac{c_{h,q}(L)^2}{n^2h^{\frac{q}{2}}g^{\frac{3}{2}}}\int_{\Omega_q\times\mathbb{R}}\int_{\Omega_q\times\mathbb{R}}\left(E_1\left((\mathbf{x},z),(\mathbf{y},t)\right)-E_2\left((\mathbf{x},z),(\mathbf{y},t)\right)\right)\\ &\times LK_{h,g}\Delta(\mathbf{x},z)LK_{h,g}\Delta(\mathbf{y},t)\,dz\,\omega_q(d\mathbf{x})\,dt\,\omega_q(d\mathbf{y})\\ =& 4\frac{c_{h,q}(L)^2}{n^2h^{\frac{q}{2}}g^{\frac{3}{2}}}\mathcal{O}\left(h^{2q}g^2\right)\\ =& \mathcal{O}\left(\left(n^2h^{\frac{q}{2}}g^{\frac{1}{2}}\right)^{-1}\right).\end{aligned}$$

Then, $\widetilde{R}_{n,1}^{(1)} = \mathcal{O}_{\mathbb{P}}\big((nh^{\frac{q}{4}}g^{\frac{1}{4}})^{-1}\big)$ and $n(h^qg)^{\frac{1}{2}}\widetilde{R}_{n,1}^{(1)} \xrightarrow{p} 0$.

Term $R_{n,4}^{(1)}$. Applying Lemma 10,

$$\widetilde{R}_{n,4}^{(1)} = \frac{1}{n(h^qg)^{\frac{1}{2}}}\int_{\Omega_q\times\mathbb{R}}\left(LK_{h,g}\Delta(\mathbf{x},z)\right)^2\,dz\,\omega_q(d\mathbf{x}) = \frac{1}{n(h^qg)^{\frac{1}{2}}}R(\Delta)(1+o\left(1\right))$$

and as a consequence $n(h^qg)^{\frac{1}{2}}\widetilde{R}_{n,4}^{(1)} \xrightarrow{p} R(\Delta)$.

Term $R_{n,4}^{(2)}$. Applying the Cauchy-Schwartz inequality:

$$\frac{\sqrt{nh^{\frac{q}{2}}g^{\frac{1}{2}}}}{2}\widetilde{R}_{n,4}^{(2)} \le \left(R_{n,4}\right)^{\frac{1}{2}}\left(nh^{\frac{q}{2}}g^{\frac{1}{2}}\widetilde{R}_{n,4}^{(1)}\right)^{\frac{1}{2}} = \mathcal{O}_{\mathbb{P}}\left(n^{-\frac{1}{2}}\right)\mathcal{O}_{\mathbb{P}}\left(1\right) = \mathcal{O}_{\mathbb{P}}\left(n^{-\frac{1}{2}}\right),$$

Therefore, $\widetilde{R}_{n,4}^{(2)} = \mathcal{O}_{\mathbb{P}}\big((nh^{\frac{q}{4}}g^{\frac{1}{4}})^{-1}\big)$ and $n(h^qg)^{\frac{1}{2}}\widetilde{R}_{n,4}^{(2)} = \mathcal{O}_{\mathbb{P}}\big((h^qg)^{\frac{1}{4}}\big) \xrightarrow{p} 0$. □

S1.4 General purpose lemmas

For the proofs of some lemmas, three auxiliary lemmas have been used.

Lemma 10. *Under **A1**–**A3**, for any function* $\varphi:\Omega_q\times\mathbb{R}\to\mathbb{R}$ *that is uniformly continuous and bounded, the smoothing operator (5.2) satisfies*

$$\sup_{(\mathbf{x},z)\in\Omega_q\times\mathbb{R}}\left|LK_{h,g}\varphi(\mathbf{x},z)-\varphi(\mathbf{x},z)\right| \xrightarrow[n\to\infty]{} 0. \tag{S1.14}$$

Thus, $LK_{h,g}\varphi(\mathbf{x},z)$ *converges to* $\varphi(\mathbf{x},z)$ *uniformly in* $\Omega_q\times\mathbb{R}$*.*

Proof of Lemma 10. Let denote $D_n = |LK_{h,g}\varphi(\mathbf{x},z) - \varphi(\mathbf{x},z)|$. Since $\varphi(\mathbf{x},z)$ can be written as $\frac{c_{h,q}(L)}{g}\int_{\Omega_q\times\mathbb{R}} LK\big(\frac{1-\mathbf{x}^T\mathbf{y}}{h^2},\frac{z-t}{g}\big)\varphi(\mathbf{y},t)\,dz\,\omega_q(d\mathbf{x})$, then

$$\begin{aligned}
D_n &= \left|\frac{c_{h,q}(L)}{g}\int_{\Omega_q\times\mathbb{R}} LK\left(\frac{1-\mathbf{x}^T\mathbf{y}}{h^2},\frac{z-t}{g}\right)(\varphi(\mathbf{y},t)-\varphi(\mathbf{x},z))\,dt\,\omega_q(d\mathbf{y})\right| \\
&\leq \frac{c_{h,q}(L)}{g}\int_{\Omega_q\times\mathbb{R}} LK\left(\frac{1-\mathbf{x}^T\mathbf{y}}{h^2},\frac{z-t}{g}\right)|\varphi(\mathbf{y},t)-\varphi(\mathbf{x},z)|\,dt\,\omega_q(d\mathbf{y}) \\
&\leq D_{n,1} + D_{n,2},
\end{aligned}$$

where:

$$\begin{aligned}
D_{n,1} &= \frac{c_{h,q}(L)}{g}\int_{A_\delta} LK\left(\frac{1-\mathbf{x}^T\mathbf{y}}{h^2},\frac{z-t}{g}\right)|\varphi(\mathbf{y},t)-\varphi(\mathbf{x},z)|\,dt\,\omega_q(d\mathbf{y}), \\
D_{n,2} &= \frac{c_{h,q}(L)}{g}\int_{\bar{A}_\delta} LK\left(\frac{1-\mathbf{x}^T\mathbf{y}}{h^2},\frac{z-t}{g}\right)|\varphi(\mathbf{y},t)-\varphi(\mathbf{x},z)|\,dt\,\omega_q(d\mathbf{y}), \\
A_\delta &= \left\{(\mathbf{y},t)\in\Omega_q\times\mathbb{R} : \max\left(\sqrt{2(1-\mathbf{x}^T\mathbf{y})}, |z-t|\right) < \delta\right\}, \\
A_{1,\delta} &= \left\{(\mathbf{y},t)\in\Omega_q\times\mathbb{R} : 1-\mathbf{x}^T\mathbf{y} < \frac{\delta^2}{2}\right\}, \\
A_{2,\delta} &= \{(\mathbf{y},t)\in\Omega_q\times\mathbb{R} : |z-t| < \delta\}
\end{aligned}$$

and $\bar{A}_\delta$ denotes the complementary set to A_δ for a $\delta > 0$. Recall that $A_\delta = A_{1,\delta}\cap A_{2,\delta}$ and as a consequence $\bar{A}_\delta = \bar{A}_{1,\delta}\cup\bar{A}_{2,\delta}$.

As stated in assumption **A1**, the uniform continuity of the functions defined in $\Omega_q\times\mathbb{R}$ is understood with respect to the product Euclidean norm, that is

$$||(\mathbf{x},z)||_2 = \sqrt{||\mathbf{x}||_{\Omega_q}^2 + ||z||_{\mathbb{R}}^2}, \text{ where } ||\cdot||_{\Omega_q} = ||\cdot||_2 \text{ and } ||\cdot||_{\mathbb{R}} = |\cdot|.$$

Nevertheless, given the equivalence between the product 2-norm and the product ∞-norm, defined as $||(\mathbf{x},z)||_\infty = \max\big(||\mathbf{x}||_{\Omega_q}, ||z||_{\mathbb{R}}\big)$, and for the sake of simplicity, the second norm will be used in the proof. Then, by the uniform continuity of φ, it holds that for any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\forall(\mathbf{x},z),(\mathbf{y},t)\in\Omega_q\times\mathbb{R},\ ||(\mathbf{x},z)-(\mathbf{y},t)||_\infty < \delta \implies |\varphi(\mathbf{x},z)-\varphi(\mathbf{y},t)| < \varepsilon.$$

Therefore the first term is dominated by

$$D_{n,1} < \varepsilon\frac{c_{h,q}(L)}{g}\int_{A_\delta} LK\left(\frac{1-\mathbf{x}^T\mathbf{y}}{h^2},\frac{z-t}{g}\right)dt\,\omega_q(d\mathbf{y}) \leq \varepsilon,$$

for any $\varepsilon > 0$, so as a consequence $D_{n,1} = o(1)$ uniformly in $(\mathbf{x},z)\in\Omega_q\times\mathbb{R}$.

For the second term, let consider the change of variables introduced in the proof of Lemma 4 (see Lemma 2 of García-Portugués et al. (2013b) for a detailed derivation):

$$\begin{cases} \mathbf{y} = u\mathbf{x} + (1-u^2)^{\frac{1}{2}}\mathbf{B}_{\mathbf{x}}\boldsymbol{\xi}, \\ \omega_q(d\mathbf{y}) = (1-u^2)^{\frac{q}{2}-1}\,du\,\omega_{q-1}(d\boldsymbol{\xi}), \end{cases}$$

where $u \in (-1,1)$, $\boldsymbol{\xi} \in \Omega_{q-1}$ and $\mathbf{B}_{\mathbf{x}} = (\mathbf{b}_1,\ldots,\mathbf{b}_q)_{(q+1)\times q}$ is the semi-orthonormal matrix resulting from the completion of $\mathbf{x}$ to the orthonormal basis $\{\mathbf{x},\mathbf{b}_1,\ldots,\mathbf{b}_q\}$. Applying this change of variables and then using the standard changes of variables $r = \frac{1-u}{h^2}$ (for the first addend) and $s = \frac{z-t}{g}$ (second addend), it follows:

$$\begin{aligned}
D_{n,2} =& \frac{c_{h,q}(L)}{g}\int_{\bar{A}_\delta} LK\left(\frac{1-\mathbf{x}^T\mathbf{y}}{h^2},\frac{z-t}{g}\right)|\varphi(\mathbf{y},t)-\varphi(\mathbf{x},z)|\,dt\,\omega_q(d\mathbf{y})\\
\leq& \frac{c_{h,q}(L)}{g}\int_{\bar{A}_{1,\delta}} LK\left(\frac{1-\mathbf{x}^T\mathbf{y}}{h^2},\frac{z-t}{g}\right)|\varphi(\mathbf{y},t)-\varphi(\mathbf{x},z)|\,dt\,\omega_q(d\mathbf{y})\\
&+\frac{c_{h,q}(L)}{g}\int_{\bar{A}_{2,\delta}} LK\left(\frac{1-\mathbf{x}^T\mathbf{y}}{h^2},\frac{z-t}{g}\right)|\varphi(\mathbf{y},t)-\varphi(\mathbf{x},z)|\,dt\,\omega_q(d\mathbf{y})\\
\leq& 2\frac{c_{h,q}(L)}{g}\sup_{(\mathbf{y},t)\in\Omega_q\times\mathbb{R}}|\varphi(\mathbf{y},t)|\left\{\int_{\bar{A}_{1,\delta}} LK\left(\frac{1-\mathbf{x}^T\mathbf{y}}{h^2},\frac{z-t}{g}\right)dt\,\omega_q(d\mathbf{y})\right.\\
&\left.+\int_{\bar{A}_{2,\delta}} LK\left(\frac{1-\mathbf{x}^T\mathbf{y}}{h^2},\frac{z-t}{g}\right)dt\,\omega_q(d\mathbf{y})\right\}\\
\leq& 2\sup_{(\mathbf{y},t)\in\Omega_q\times\mathbb{R}}|\varphi(\mathbf{y},t)|\left\{c_{h,q}(L)\omega_{q-1}\int_{-1}^{1-\frac{\delta^2}{2}} L\left(\frac{1-u}{h^2}\right)(1-u^2)^{\frac{q}{2}-1}\,du\right.\\
&\left.+2\int_{\delta g^{-1}}^{\infty}K(s)\,ds\right\}\\
\leq& 2\sup_{(\mathbf{y},t)\in\Omega_q\times\mathbb{R}}|\varphi(\mathbf{y},t)|\left\{c_{h,q}(L)\omega_{q-1}\int_{-1}^{1}(1-u^2)^{\frac{q}{2}-1}\,du\times\sup_{r\geq\delta^2/(2h^2)}L(r)r^{\frac{q}{2}}r^{-\frac{q}{2}}\right.\\
&\left.+2\int_{\delta g^{-1}}^{\infty}K(s)\,ds\right\}\\
\leq& \mathcal{O}(1)\left\{\lambda_{h,q}(L)^{-1}\omega_{q-1}2^{-\frac{q}{2}}\delta^{-q}\int_{-1}^{1}(1-u^2)^{\frac{q}{2}-1}\,du\times\sup_{r\geq\delta^2/(2h^2)}L(r)r^{\frac{q}{2}}+o(1)\right\}\\
=& \mathcal{O}(1)\left(\mathcal{O}(1)\,o(1)+o(1)\right)\\
=& o(1),
\end{aligned}$$

by relation (2.1), the fact $\int_{-1}^{1}(1-u^2)^{\frac{q}{2}-1}\,du<\infty$ for all $q\geq 1$ and because by assumption **A2**, $\lambda_{q+2}(L)<\infty$, which implies that $\lim_{r\to\infty}L(r)r^{\frac{q}{2}}=0$.

Then, $D_n \to 0$ as $n \to \infty$ and this holds regardless the point $(\mathbf{x}, z)$, since φ is uniformly continuous, so (S1.14) is satisfied and $LK_{h,g}\varphi(\mathbf{x}, z)$ converges to $\varphi(\mathbf{x}, z)$ uniformly in $\Omega_q \times \mathbb{R}$. □

Lemma 11. *Under* ***A1***–***A3****, the bias and the variance for the directional-linear estimator in a point* $(\mathbf{x}, z) \in \Omega_q \times \mathbb{R}$ *is given by*

$$\mathbb{E}\left[\hat{f}_{h,g}(\mathbf{x}, z)\right] = f(\mathbf{x}, z) + \frac{b_q(L)}{q}\mathrm{tr}\left[\boldsymbol{\mathcal{H}}_\mathbf{x} f(\mathbf{x}, z)\right] h^2 + \frac{1}{2}\mu_2(K)\mathcal{H}_z f(\mathbf{x}, z) g^2 + o\left(h^2 + g^2\right),$$
$$\mathbb{V}\mathrm{ar}\left[\hat{f}_{h,g}(\mathbf{x}, z)\right] = \frac{\lambda_q(L^2)\lambda_q(L)^{-2}R(K)}{nh^q g} f(\mathbf{x}, z) + o\left((nh^q g)^{-1}\right),$$

where the remainder orders are uniform.

Proof of Lemma 11. The asymptotic expressions of the bias and the variance are given in García-Portugués et al. (2013b). Recalling the extension of f in condition **A1**, the partial derivative of f for the direction $\mathbf{x}$ and evaluated at $(\mathbf{x}, z)$, that is $\mathbf{x}^T\boldsymbol{\nabla}_\mathbf{x} f(\mathbf{x}, z)$, is null:

$$\mathbf{x}^T\boldsymbol{\nabla}_\mathbf{x} f(\mathbf{x}, z) = \lim_{h\to 0} \frac{f\left((1+h)\mathbf{x}, z\right) - f\left(\mathbf{x}, z\right)}{h} = \lim_{h\to 0} \frac{f\left(\mathbf{x}, z\right) - f\left(\mathbf{x}, z\right)}{h} = 0.$$

Using this fact, it also follows that $\mathbf{x}^T\boldsymbol{\mathcal{H}}_\mathbf{x} f(\mathbf{x}, z)\mathbf{x} = 0$, since

$$\mathbf{x}^T\left(\frac{\partial}{\partial\mathbf{x}}\mathbf{x}^T\boldsymbol{\nabla}_\mathbf{x} f(\mathbf{x}, z)\right) = \mathbf{x}^T\left(\boldsymbol{\nabla}_\mathbf{x} f(\mathbf{x}, z) + \boldsymbol{\mathcal{H}}_\mathbf{x} f(\mathbf{x}, z)\mathbf{x}\right) = 0.$$

Therefore, the operator $\boldsymbol{\Psi}_\mathbf{x}(f, \mathbf{x}, z)$ appearing in the bias expansion given in García-Portugués et al. (2013b) can be written in the simplified form

$$\boldsymbol{\Psi}_\mathbf{x}(f, \mathbf{x}, z) = -\mathbf{x}^T\boldsymbol{\nabla}_\mathbf{x} f(\mathbf{x}, z) + \frac{1}{q}\left(\nabla^2 f(\mathbf{x}, z) - \mathbf{x}^T\boldsymbol{\mathcal{H}}_\mathbf{x} f(\mathbf{x}, z)\mathbf{x}\right) = \frac{1}{q}\mathrm{tr}\left[\boldsymbol{\mathcal{H}}_\mathbf{x} f(\mathbf{x}, z)\right],$$

because $\nabla^2 f(\mathbf{x}, z)$ represents the directional Laplacian of f (the trace of $\boldsymbol{\mathcal{H}}_\mathbf{x} f(\mathbf{x}, z)$).

The uniformity of the orders, not considered in the above paper, can be obtained by using the extra-smoothness assumption **A1** and the integral form of the remainder in the Taylor's theorem on f:

$$f(\mathbf{y} + \boldsymbol{\alpha}) - f(\mathbf{y}) = \boldsymbol{\alpha}^T\boldsymbol{\nabla} f(\mathbf{y}) + \frac{1}{2}\boldsymbol{\alpha}^T\boldsymbol{\mathcal{H}} f(\mathbf{y})\boldsymbol{\alpha} + R,$$

with $\mathbf{y} \equiv (\mathbf{x}, z)$, $\boldsymbol{\alpha} \in \Omega_q \times \mathbb{R}$ and where the remainder has the exact form

$$R = \int_0^1 \frac{(1-t)^2}{2} \sum_{i,j,k=1}^{q+1} \frac{\partial^3}{\partial x_i \partial x_j \partial x_k} f(\mathbf{x} + t\boldsymbol{\alpha})\alpha_i\alpha_j\alpha_k\, dt \leq \frac{1}{6}M \sum_{i,j,k=1}^{q+1} \alpha_i\alpha_j\alpha_k = o\left(\boldsymbol{\alpha}^T\boldsymbol{\alpha}\right),$$

where M is the bound of the third derivatives of f and in the last equality it is used the second point of Lemma 12. Then the remainder does not depend on the point $\mathbf{y} \equiv (\mathbf{x}, z)$ and following the proofs of García-Portugués et al. (2013b) the convergence of the bias and variance is uniform on $\Omega_q \times \mathbb{R}$. □

Lemma 12. *Let a_n, b_n and c_n sequences of positive real numbers. Then:*

i. *If $a_n, b_n \to 0$, then $a_n b_n = o(a_n + b_n)$.*

ii. *If $a_n, b_n, c_n \to 0$, then $a_n b_n c_n = o\left(a_n^2 + b_n^2 + c_n^2\right)$.*

iii. *$a_n^i b_n^j = \mathcal{O}\big(a_n^k + b_n^k\big)$, for any integers $i, j \geq 0$ such that $i + j = k$.*

iv. *$(a_n + b_n)^k = \mathcal{O}\big(a_n^k + b_n^k\big)$, for any integer $k \geq 1$.*

Proof of Lemma 12. The first statement follows immediately from the definition of $o(\cdot)$,

$$a_n b_n = o(a_n + b_n) :\iff \lim_{n\to\infty} \frac{a_n b_n}{a_n + b_n} = \lim_{n\to\infty} \frac{1}{\frac{1}{b_n} + \frac{1}{a_n}} = \frac{1}{\infty} = 0.$$

For the second, suppose that, when $n \to \infty$, $a_n = \max(a_n, b_n, c_n)$ to fix notation. Then

$$\lim_{n\to\infty} \frac{a_n b_n c_n}{a_n^2 + b_n^2 + c_n^2} \leq \lim_{n\to\infty} \frac{a_n^3}{a_n^2 + b_n^2 + c_n^2} = \lim_{n\to\infty} \frac{1}{\frac{1}{a_n} + \frac{b_n^2}{a_n^3} + \frac{c_n^2}{a_n^3}} = \frac{1}{\infty} = 0.$$

Let C be a positive constant. The third statement follows from the definition of $\mathcal{O}(\cdot)$,

$$\lim_{n\to\infty} \frac{a_n^i b_n^j}{a_n^k + b_n^k} = \lim_{n\to\infty} \frac{1}{\left(\frac{a_n}{b_n}\right)^j + \left(\frac{b_n}{a_n}\right)^i} = \begin{cases} \frac{1}{0+\infty}, & a_n = o(b_n), \\ \frac{1}{\infty+0}, & b_n = o(a_n), \\ \frac{1}{C^j + C^{-i}}, & a_n \sim C b_n. \end{cases}$$

Then the limit is bounded and $a_n^i b_n^j = \mathcal{O}\big(a_n^k + b_n^k\big)$. The last statement arises as a consequence of this result and the Newton binomial:

$$(a_n + b_n)^k = \sum_{i=0}^k \binom{k}{i} a_n^{k-i} b_n^i = \sum_{i=0}^k \binom{k}{i} \mathcal{O}\left(a_n^k + b_n^k\right) = \mathcal{O}\left(a_n^k + b_n^k\right).$$

□

S2 Further results for the independence test

S2.1 Closed expressions

Consider K and L a normal and a von Mises kernel, respectively. In this case $R(K) = \left(2\pi^{\frac{1}{2}}\right)^{-1}$, $\lambda_q(L) = (2\pi)^{\frac{q}{2}}$ and $\lambda_q(L^2)\lambda_q(L)^{-2} = \left(2\pi^{\frac{1}{2}}\right)^{-q}$. Furthermore, it is possible to compute exactly the form of the contributions of these two kernels to the asymptotic variance, resulting:

$$\gamma_q\lambda_q(L)^{-4}\int_0^\infty r^{\frac{q}{2}-1}\left\{\int_0^\infty \rho^{\frac{q}{2}-1}L(\rho)\varphi_q(r,\rho)\,d\rho\right\}^2 dr = (8\pi)^{-\frac{q}{2}},$$
$$\int_{\mathbb{R}}\left\{\int_{\mathbb{R}} K(u)K(u+v)\,du\right\}^2 dv = (8\pi)^{-\frac{1}{2}}.$$

Corollary 4. *If $L(r)=e^{-r}$ and K is a normal density, then the asymptotic bias and variance in Theorem 2 are*

$$A_n = \frac{1}{2^{q+1}\pi^{\frac{q+1}{2}}nh^q g} - \frac{R(f_Z)}{2^q\pi^{\frac{q}{2}}nh^q} - \frac{R(f_{\mathbf{X}})}{2\pi^{\frac{1}{2}}ng},\quad \sigma_I^2 = (8\pi)^{-\frac{q+1}{2}}R(f_{\mathbf{X}})R(f_Z).$$

In addition, if $f_{\mathbf{X}} = f_{\mathrm{vM}}(\cdot;\boldsymbol{\mu},\kappa)$ and f_Z is the density of a $\mathcal{N}(m,\sigma^2)$, then $R(f_{\mathbf{X}}) = \left(2\pi^{\frac{q+1}{2}}\right)^{-1}\kappa^{\frac{q-1}{2}}\,\mathcal{I}_{\frac{q-1}{2}}(2\kappa)\mathcal{I}_{\frac{q-1}{2}}(\kappa)^{-2}$ and $R(f_Z)=\left(2\pi^{\frac{1}{2}}\sigma\right)^{-1}$.

Proof of Corollary 4. The expressions for $R(K)$, $R(f_Z)$ and $\int_{\mathbb{R}}\left\{\int_{\mathbb{R}} K(u)K(u+v)\,du\right\}^2 dv = (8\pi)^{-\frac{1}{2}}$ follow easily from the convolution properties of normal densities. The expressions for $\lambda_q(L)$ and $\lambda_q(L^2)$ can be derived from the definition of the Gamma function. Similarly,

$$R(f_{\mathbf{X}}) = C_q(\kappa)^2\int_{\Omega_q} e^{2\kappa\mathbf{x}^T\boldsymbol{\mu}}\,\omega_q(d\mathbf{x}) = \frac{C_q(\kappa)^2}{C_q(2\kappa)} = \frac{\kappa^{\frac{q-1}{2}}\mathcal{I}_{\frac{q-1}{2}}(2\kappa)}{2\pi^{\frac{q+1}{2}}\mathcal{I}_{\frac{q-1}{2}}(\kappa)^2},$$
$$\gamma_q^{-1}\lambda_q(L)^4 = \begin{cases} 2^{-\frac{5}{4}}\pi^2, & q=1,\\ 2^{\frac{q}{2}}\pi^{\frac{q}{2}+1}\Gamma\left(\frac{q}{2}\right)\Gamma\left(\frac{q-1}{2}\right)^2, & q>1.\end{cases} \tag{S2.1}$$

For $q=1$ the contribution of the directional kernel to the asymptotic variance can be computed using (S2.1) and

$$\int_0^\infty \rho^{-\frac{1}{2}}e^{-2\left(\rho\pm\sqrt{r\rho}\right)}\,d\rho = \sqrt{2\pi}e^{\frac{r}{2}}\left(1-\Phi\left(\mp\sqrt{r}\right)\right),$$

where Φ is the cumulative distribution function of a $\mathcal{N}(0,1)$. Then:

$$\begin{aligned}
&\gamma_1\lambda_1(L)^{-4}\int_0^\infty r^{-\frac{1}{2}}\left\{\int_0^\infty \rho^{-\frac{1}{2}}L(\rho)\varphi_1(r,\rho)\,d\rho\right\}^2 dr\\
&\quad=\gamma_1\lambda_1(L)^{-4}\int_0^\infty r^{-\frac{1}{2}}e^{-2r}\left\{\int_0^\infty \rho^{-\frac{1}{2}}e^{-2\rho-2(r\rho)^{\frac{1}{2}}}\,d\rho + \int_0^\infty \rho^{-\frac{1}{2}}e^{-2\rho+2(r\rho)^{\frac{1}{2}}}\,d\rho\right\}^2 dr\\
&\quad=2^{-\frac{1}{2}}\left(2\pi\right)^{-1}\int_0^\infty r^{-\frac{1}{2}}e^{-r}\,dr\\
&\quad=(8\pi)^{-\frac{1}{2}}.
\end{aligned}$$

For $q > 1$, the integral with respect to θ is computed from the definition of the modified Bessel function and the integral with respect to ρ is

$$\int_0^\infty \rho^{\frac{q}{4}-\frac{1}{2}} e^{-2\rho} \mathcal{I}_{\frac{q}{2}-1}\left(2\sqrt{r\rho}\right) \, d\rho = 2^{-\frac{q}{2}} r^{\frac{q}{4}-\frac{1}{2}} e^{\frac{r}{2}}.$$

Using these two facts, it results:

$$\begin{aligned}
&\gamma_q \lambda_q(L)^{-4} \int_0^\infty r^{\frac{q}{2}-1} \left\{ \int_0^\infty \rho^{\frac{q}{2}-1} L(\rho) \varphi_q(r,\rho) \, d\rho \right\}^2 dr \\
&\quad = 2^{-\frac{q}{2}} \pi^{-\left(\frac{q}{2}+1\right)} \Gamma\left(\frac{q}{2}\right)^{-1} \Gamma\left(\frac{q-1}{2}\right)^{-2} \\
&\quad \times \int_0^\infty r^{\frac{q}{2}-1} \left\{ \int_0^\infty \rho^{\frac{q}{2}-1} e^{-(r+2\rho)} \left[\pi^{\frac{1}{2}} \Gamma\left(\frac{q-1}{2}\right) (r\rho)^{-\frac{q-2}{4}} \mathcal{I}_{\frac{q}{2}-1}\left(2\sqrt{r\rho}\right) \right] d\rho \right\}^2 dr \\
&\quad = 2^{-\frac{q}{2}} \pi^{-\frac{q}{2}} \Gamma\left(\frac{q}{2}\right)^{-1} \int_0^\infty r^{\frac{q}{2}-1} e^{-2r} \left\{ 2^{-\frac{q}{2}} e^{\frac{r}{2}} \right\}^2 dr \\
&\quad = (8\pi)^{-\frac{q}{2}}.
\end{aligned}$$

□

S2.2 Extension to the directional-directional case

Under the directional-directional analogue of condition **A4**, that is, $h_{1,n}^{q_1} h_{2,n}^{-q_2} \to c$, with $0 < c < \infty$, the directional-linear independence test can be directly adapted to this setting, considering the following test statistic:

$$T_n = \int_{\Omega_{q_1} \times \Omega_{q_2}} \left(\hat{f}_{(\mathbf{X},\mathbf{Y});h_1,h_2}(\mathbf{x},\mathbf{y}) - \hat{f}_{\mathbf{X};h_1}(\mathbf{x}) \hat{f}_{\mathbf{Y};h_2}(\mathbf{y}) \right)^2 \omega_{q_2}(d\mathbf{y}) \, \omega_{q_1}(d\mathbf{x}).$$

Corollary 5 (Directional-directional independence test)**.** *Under the directional-directional analogues of **A1**–**A4** and the null hypothesis of independence,*

$$n(h_1^{q_1} h_2^{q_2})^{\frac{1}{2}} (T_n - A_n) \xrightarrow{d} \mathcal{N}(0, 2\sigma_I^2),$$

where

$$\begin{aligned}
A_n = &\frac{\lambda_{q_1}(L_1^2) \lambda_{q_1}(L_1)^{-2} \lambda_{q_2}(L_2^2) \lambda_{q_2}(L_2)^{-2}}{n h_1^{q_1} h_2^{q_2}} \\
&- \frac{\lambda_{q_1}(L_1^2) \lambda_{q_1}(L_1)^{-2} R(f_{\mathbf{Y}})}{n h_1^{q_1}} - \frac{\lambda_{q_2}(L_2^2) \lambda_{q_2}(L_2)^{-2} R(f_{\mathbf{X}})}{n h_2^{q_2}},
\end{aligned}$$

and σ_I^2 is defined as σ^2 in Corollary 2 but with $R(f) = R(f_{\mathbf{X}}) R(f_{\mathbf{Y}})$. Further, if L_1 and L_2 are the von Mises kernel,

$$A_n = \frac{1}{2^{q_1+q_2} \pi^{\frac{q_1+q_2}{2}} n h_1^{q_1} h_2^{q_2}} - \frac{R(f_{\mathbf{Y}})}{2^{q_1} \pi^{\frac{q_1}{2}} n h_1^{q_1}} - \frac{R(f_{\mathbf{X}})}{2^{q_2} \pi^{\frac{q_2}{2}} n h_2^{q_2}}$$

and $\sigma_I^2 = (8\pi)^{-\frac{q_1+q_2}{2}} R(f_{\mathbf{X}})R(f_{\mathbf{Y}})$. *If* $f_{\mathbf{X}}$ *and* $f_{\mathbf{Y}}$ *are von Mises densities,* $R(f_{\mathbf{X}})$ *and* $R(f_{\mathbf{Y}})$ *are given as in Corollary 4.*

Proof of Corollary 5. The proof follows from adapting the proofs of Theorem 2 and Corollary 4 to the directional-directional situation. □

S2.3 Some numerical experiments

The purpose of this subsection is to provide some numerical experiments to illustrate the degree of misfit between the true distribution of the standardized statistic (approximated by Monte Carlo) and its asymptotic distribution, for increasing sample sizes.

For simplicity, independence will be assessed in a circular-linear framework ($q = 1$), with a vM$((0,1),1)$ for the circular variable and a $\mathcal{N}(0,1)$ for the linear one. Kernel density estimation is done using von Mises and normal kernels, as in Corollary 4. Sample sizes considered are $n = 5^j \times 10^k$, $j = 0,1$, $k = 3,5$ (see supplementary material for $k = 1,2,4$). The sequence of bandwidths is taken as $h_n = g_n = 2n^{-\frac{1}{3}}$, as a compromise between fast convergence and numerical problems avoidance. Figure A.5 presents the histogram of 1000 values from $(nh_n^q g_n)^{\frac{1}{2}}(T_n - A_n)$ for different sample sizes, jointly with the p-values of the Kolmogorov-Smirnov test for the distribution $\mathcal{N}(0, 2\sigma_I^2)$ and of the Shapiro-Wilk test for normality. Both tests are significant, until a very large sample size (close to $500,000$ data) is reached.

It should be noted that, in practical problems, the use of the asymptotic distribution does not seem feasible, and a resampling mechanism for the calibration of the test is required. This issue is addressed in García-Portugués et al. (2014), considering a permutation approach. The reader is referred to the aforementioned paper for the details concerning the practical application.

S3 Extended simulation study

Some technical details concerning the simulation study and further results are provided in this section. First, the simulated models considered will be described. For constructing the test statistic, parametric estimators as well as simulation methods are required. Different Maximum Likelihood Estimators (MLE) and simulation approaches have been

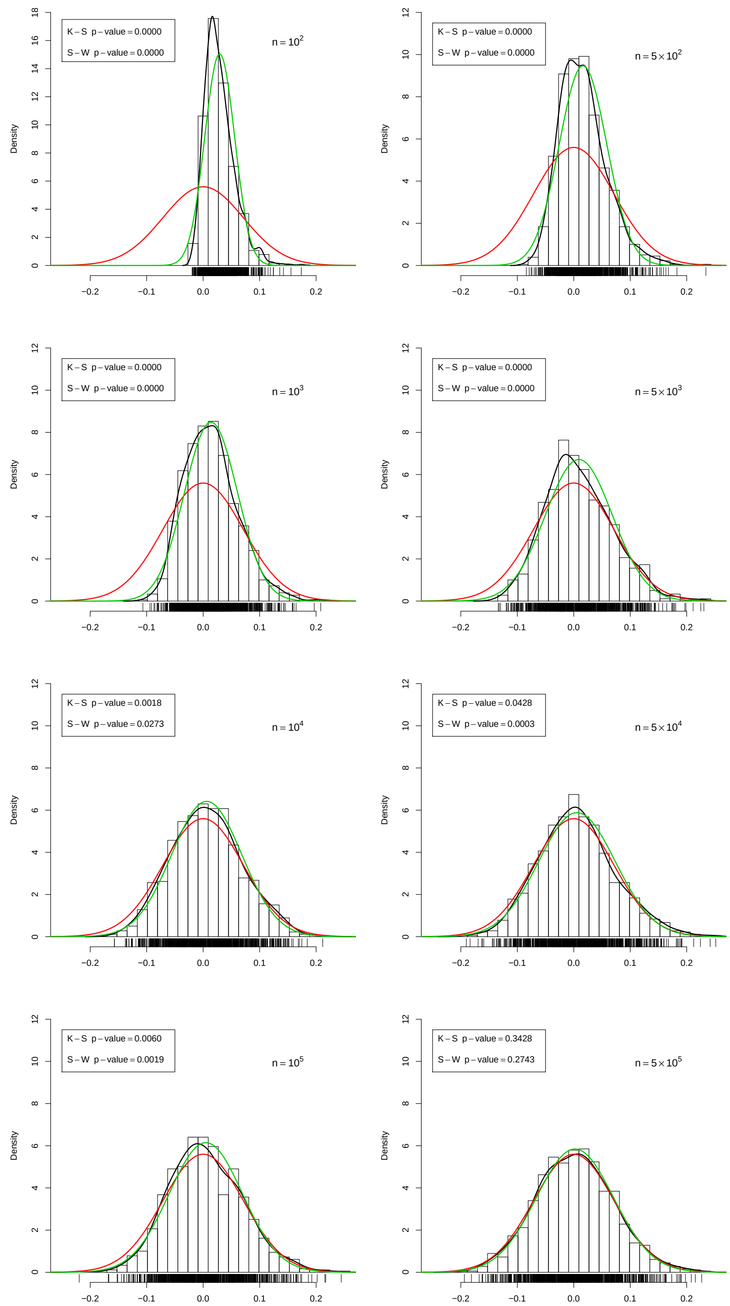

Figure A.5: Comparison of the asymptotic and empirical distributions of $(nh_n^q g_n)^{\frac{1}{2}}(T_n - A_n)$ for sample sizes $n = 5^j \times 10^k$, $j = 0, 1$, $k = 2, 3, 4, 5$. Black curves represent a kernel estimation from 1000 simulations, green curves represent a normal fit to the unknown density and red curves represent the theoretical asymptotic distribution.

considered, playing copulas a remarkable role in both problems (see Nelsen (2006) for a comprehensive review). Some details on the construction of alternative models and bandwidth choice will be also given, jointly with extended results showing the performance of the tests (for circular-linear and circular-circular cases) for different significance levels.

S3.1 Parametric models

Two collections of Circular-Linear (CL) and Circular-Circular (CC) parametric scenarios have been considered. The corresponding density contours can be seen in Figures 6.1 and 6.2 in the paper. For the circular-linear case, the first five models (CL1–CL5) contain parametric densities with independent components and different kinds of marginals, for which estimation and simulation are easily accomplished. The models are based on von Mises, wrapped Cauchy, wrapped normal, normal, log-normal, gamma and mixtures of these densities. Models CL6–CL7 represent two parametric choices of the model in Mardia and Sutton (1978) for cylindrical variables, which is constructed conditioning a normal density on a von Mises one. Models CL8–CL9 include two parametric densities of the semiparametric circular-linear model given in Theorem 5 of Johnson and Wehrly (1978). This family is indexed by a circular density g that defines the underlying circular-linear copula density, allowing for flexibility both in the specification of the link density and the marginals. CL10 is the model given in Theorem 1 of Johnson and Wehrly (1978), which considers an exponential density conditioned on a von Mises. CL11 is constructed considering the QS copula density of García-Portugués et al. (2013a) and cardioid and log-normal marginals. Finally, CL12 is an adaptation of the circular-circular copula density of Kato (2009) to the circular-linear scenario, using an identity matrix in the joint structure and von Mises and log-normal marginals.

The first models (CC1–CC5) of the circular-circular case include also parametric densities with independent components and different kinds of marginals (von Mises, wrapped Cauchy, cardioid and mixtures of them). Models CC6–CC7 represent two parametric choices of the sine model given by Singh et al. (2002). This model introduces elliptical contours for bivariate circular densities and also allows for certain multimodality. Models CC8–CC9 are two densities of the semiparametric models of Wehrly and Johnson (1979), which are based on the previous work of Johnson and Wehrly (1978) and comprise as a particular case the bivariate von Mises model of Shieh and Johnson (2005). Models CC10–CC11 are two parametric choices of the wrapped normal torus density given in Johnson and Wehrly (1977), a natural extension of the circular wrapped normal to the circular-circular setting. Finally, CC12 employs the copula density of Kato (2009) with von Mises marginals.

Density name	Expression
Normal	$f_{\mathcal{N}}(z;m,\sigma)=\frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{(z-m)^2}{2\sigma^2}\right\}$
Log-normal	$f_{\mathcal{LN}}(z;m,\sigma)=\frac{1}{z\sqrt{2\pi}\sigma}\exp\left\{-\frac{(\log z-m)^2}{2\sigma^2}\right\}\mathbb{1}_{(0,\infty)}(z)$
Gamma	$f_{\Gamma}(z;a,p)=\frac{a^p}{\Gamma(p)}z^{p-1}e^{-az}\mathbb{1}_{(0,\infty)}(z)$
Bivariate normal	$f_{\mathcal{N}}(z_1,z_2;m_1,m_2,\sigma_1,\sigma_2,\rho)=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}$
	$\times\exp\left\{-\frac{1}{2(1-\rho^2)}\left(\frac{(z_1-m_1)^2}{\sigma_1^2}+\frac{(z_2-m_2)^2}{\sigma_2^2}-\frac{2\rho(z_1-m_1)(z_2-m_2)}{\sigma_1\sigma_2}\right)\right\}$
Von Mises	$f_{\mathrm{vM}}(\theta;\mu,\kappa)=\frac{1}{2\pi\mathcal{I}_0(\kappa)}\exp\left\{\kappa\cos(\theta-\mu)\right\}$
Cardioid	$f_{\mathrm{Ca}}(\theta;\mu,\rho)=\frac{1}{2\pi}\left(1+2\rho\cos(\theta-\mu)\right)$
Wrapped Cauchy	$f_{\mathrm{WC}}(\theta;m,\sigma)=\frac{1-\rho^2}{2\pi\left(1+\rho^2-2\rho\cos(\theta-\mu)\right)}$
Wrapped Normal	$f_{\mathrm{WN}}(\theta;\mu,\rho)=\sum_{p=-\infty}^{\infty}f_{\mathcal{N}}(\theta+2\pi p;m,\sigma)$

Table A.2: Notation for the densities described in Tables A.3 and A.4.

The notation and density expressions used for the construction of the parametric models are collected in Table A.2, whereas Tables A.3 and A.4 show the explicit expressions and parameters for the circular-linear and circular-circular models displayed in Figure 6.1. Most of the circular densities considered in the simulation study are purely circular (and hence not directional) and their circular formulation has been used in order to simplify expressions. The directional notation can be obtained taking into account that $\mathbf{x}=(\cos\theta,\sin\theta)$, $\mathbf{y}=(\cos\psi,\sin\psi)$ and $\boldsymbol{\mu}=(\cos\mu,\sin\mu)$. The distribution function of a circular variable with density f, with $\theta\in[0,2\pi)$ will be denoted by $F(\theta)=\int_0^\theta f(\varphi)\,d\varphi$.

S3.2 Estimation

In the scenarios considered, for most of the marginal densities, MLE are available through specific libraries of `R`. For the normal and log-normal densities closed expressions are used and for the gamma density the `fitdistr` function of the `MASS` (Venables and Ripley (2002)) library is employed. The estimation of the von Mises parameters is done exactly for the mean and numerically for the concentration parameter, whereas for the wrapped Cauchy and wrapped normal densities the numerical routines of the `circular` (Agostinelli and Lund (2011)) package are used. The MLE for the cardioid density are obtained by numerical optimization. Finally, the fitting of mixtures of normals and von Mises was carried out using the Expectation-Maximization algorithms given in packages `nor1mix` (Mächler (2013)) and `movMF` (Hornik and Grün (2012)), respectively.

The fitting of the independent models CL1–CL5 and CC1–CC5 is easily accomplished by marginal fitting of each component. For models CL6–CL7, the closed expressions for the MLE given in Mardia and Sutton (1978) are used. For models CL8–CL9, CL11–CL12, CC8–CC9 and CC12 a two-step Maximum Likelihood (ML) estimation procedure based on the copula density decomposition is used: first, the marginals are fitted by ML and then the copula is estimated by ML using the pseudo-observations computed from the fitted marginals. This procedure is described in more detail in Section 3 of García-Portugués et al. (2013a). In models CL8–CL9 and CC8–CC9 the MLE for the copula are obtained by estimating univariate von Mises or mixtures of von Mises, whereas numerical optimization is required for the copula estimation. For models CC6–CC7 and CC10–CC11, MLE can be also carried out by numerical optimization. Finally, MLE for model CL10 in Johnson and Wehrly (1978) were obtained analytically: given the circular-linear sample $\{(\Theta_i, Z_i)\}_{i=1}^n$,

$$\hat{\lambda} = \frac{\bar{Z}}{(\bar{Z})^2 - (\bar{Z}_c)^2}, \quad \hat{\kappa} = \sqrt{\hat{\lambda}^2 - \hat{\lambda}\bar{Z}^{-1}} \quad \text{and} \quad \sum_{i=1}^n Z_i \sin(\Theta_i - \hat{\mu}) = 0,$$

with $\bar{Z} = \frac{1}{n}\sum_{i=1}^n Z_i$ and $\bar{Z}_c = \frac{1}{n}\sum_{i=1}^n Z_i \cos(\Theta_i - \hat{\mu})$.

S3.3 Simulation

Simulating from the linear marginals is easily accomplished by the built-in functions in `R`. The simulation of the wrapped Cauchy and wrapped normal is done with the `circular` library, the von Mises is sampled implementing the algorithm described in Wood (1994) and the cardioid by the inversion method, whose equation is solved numerically. Sampling from the independence models is straightforward. Conditioning on the circular variable, it is easy to sample from models CL6–CL7 (sample the circular observation from a von Mises and then the linear from a normal with mean depending on the circular), CL10 (von Mises marginal and exponential with varying rate) and CC6–CC7 (using the properties detailed in Singh et al. (2002) and the inversion method). Simulation in CC10–CC11 is straightforward: sample from a bivariate normal and then wrap around $[0, 2\pi)$ by applying a modulus of 2π. Finally, simulation in two steps using copulas was required for models CL8–CL9, CL12, CC8–CC9 and CC12, where first a pair of uniform random variables (U, V) is sampled from the copula of the density and then the inversion method is applied marginally. See Section 3.1 of García-Portugués et al. (2013a) for more details. The simulation of the pair (U, V) was done by the conditional and inversion methods and, specifically, for the models based on the densities given by Johnson and Wehrly (1978) and Wehrly and Johnson (1979), a transformation method was obtained. It is summarized in the following algorithm.

Algorithm 2. *Let g be a circular density. A pair (U, V) of uniform variables with joint density $c_g(u, v) = 2\pi g(2\pi(u \pm v))$ is obtained as follows:*

i. Sample Ψ, a random variable with circular density g.

ii. Sample V, a uniform variable in $[0, 1]$.

iii. Set $U = \frac{(\Psi \mp 2\pi V) \mod 2\pi}{2\pi}$.

S3.4 Alternative models

The alternative hypothesis for the goodness-of-fit test, both in the circular-linear and circular-circular cases, is stated as:

$$H_{k,\delta} : f = (1 - \delta) f^k_{\boldsymbol{\theta}_0} + \delta\Delta, \quad 0 \leq \delta \leq 1.$$

Three mixing densities Δ are considered, two for the circular-linear situation and one for the circular-circular:

$$\begin{aligned} \Delta_1(\theta, z) &= f_{\mathrm{vM}}(\theta; \mu_1, \kappa) \times f_{\mathcal{N}}(z; m_1, \sigma_1), \\ \Delta_2(\theta, z) &= f_{\mathrm{vM}}(\theta; \mu_1, \kappa) \times f_{\mathcal{LN}}(z; m_2, \sigma_2), \\ \Delta_3(\theta, \psi) &= f_{\mathrm{vM}}(\theta; \mu_2, \kappa) \times f_{\mathrm{vM}}(\psi; \mu_1, \kappa), \end{aligned}$$

where $\mu_1 = \pi$, $\mu_2 = 0$, $\kappa = 3$, $m_1 = 2$, $\sigma_1 = 1$ and $m_2 = \sigma_2 = \frac{1}{2}$. To account for similar ranges in the linear data obtained under $H_{k,0}$ and under $H_{k,\delta}$, Δ_1 is used in models CL1, CL4–CL11 and CL13, whereas Δ_2 in the other models. In the circular-circular case, the deviation for all models is Δ_3.

S3.5 Bandwidth choice

The delicate issue of the bandwidth choice for the testing procedure has been approached as follows. In the simulation results presented in Section 6, a fixed pair of bandwidths was chosen based on a Likelihood Cross Validation criterion. Ideally, one would like to run the test in a grid of several bandwidths to check how the test is affected by the bandwidth choice. This was done for six circular-linear and circular-circular models,

Model	Density	Parameters	Description
CL1	$f_{vM}(\theta;\mu,\kappa)\times f_{\mathcal{N}}(z;m,\sigma)$	$\mu=\frac{3\pi}{2}$, $\kappa=2$, $m=0$, $\sigma=1$	Indep. von Mises and normal
CL2	$f_{WC}(\theta;\mu,\rho)\times f_{\mathcal{LN}}(z;m,\sigma)$	$\mu=\frac{3\pi}{2}$, $\rho=\sigma=\frac{3}{4}$, $m=\frac{1}{2}$	Indep. wrapped Cauchy and log-normal
CL3	$\big(p_1 f_{vM}(\theta;\mu_1,\kappa_1)+p_2 f_{vM}(\theta;\mu_2,\kappa_2)\big)$ $\times f_{\Gamma}(z;a,p)$	$\mu_1=\frac{\pi}{4}$, $\mu_2=\frac{5\pi}{4}$, $\kappa_1=\kappa_2=2$, $p_1=p_2=\frac{1}{2}$, $a=\frac{1}{3}$, $p=3$	Indep. mixture of von Mises and gamma
CL4	$f_{WN}(\theta;m_1,\sigma_1)$ $\times\left(p_1 f_{\mathcal{N}}(z;m_2,\sigma_2)+p_2 f_{\mathcal{N}}(z;m_3,\sigma_3)\right)$	$m_1=\frac{3\pi}{2}$, $\sigma_1=\sigma_3=1$, $m_2=0$, $\sigma_2=\frac{1}{4}$, $m_3=2$, $p_1=p_2=\frac{1}{2}$	Indep. wrapped normal and mixture of normals
CL5	$\big(p_1 f_{vM}(\theta;\mu_1,\kappa_1)+p_2 f_{vM}(\theta;\mu_2,\kappa_2)\big)$ $\times\big(p_3 f_{\mathcal{N}}(z;m_1,\sigma_1)+p_4 f_{\mathcal{N}}(z;m_2,\sigma_2)\big)$	$\mu_1=\frac{5\pi}{4}$, $\mu_2=\frac{7\pi}{4}$, $\kappa_1=10$, $\kappa_2=3$, $m_1=-1$, $m_2=2$, $\sigma_1=1$, $\sigma_2=p_1=p_2=\frac{1}{2}$, $p_3=\frac{3}{4}$, $p_4=\frac{1}{4}$	Indep. mixture of von Mises and of normals
CL6	$f_{vM}(\theta;\mu,\kappa)\times f_{\mathcal{N}}\big(z;m(\theta),\sigma(1-\rho_1-\rho_2)\big)$, with $m(\theta)=m+\sigma\kappa^{\frac{1}{2}}\rho_1(\cos(\theta)-\cos(\mu))$ $+\rho_2(\sin(\theta)-\sin(\mu))$	$\mu=\frac{3\pi}{2}$, $\kappa=1$, $m=0$, $\rho_1=\rho_2=\sigma=\frac{1}{2}$	See equation (1.1) of Mardia and Sutton (1978)
CL7		$\mu=\frac{3\pi}{2}$, $\kappa=5$, $m=0$, $\rho_1=\frac{1}{2}$, $\rho_2=-\frac{3}{4}$, $\sigma=\frac{3}{2}$	
CL8	$f_{vM}\left(2\pi\big(\frac{\theta}{2\pi}+F_{\mathcal{N}}(z;m,\sigma)\big);\mu_g,\kappa_g\right)$ $\times f_{\mathcal{N}}(z;m,\sigma)$	$m=0$, $\sigma=1$, $\mu_g=\frac{5\pi}{4}$, $\kappa_g=\frac{3}{2}$	See Theorem 5 of Johnson and Wehrly (1978) considering a von Mises and a mixture of von Mises as the link functions
CL9	$g\big(2\pi(\frac{\theta}{2\pi}-F_{\mathcal{N}}(z;m,\sigma))\big)\times f_{\mathcal{N}}(z;m,\sigma)$, with $g(\theta)=p_{g_1}f_{vM}(\theta;\mu_{g_1},\kappa_{g_1})$ $+p_{g_2}f_{vM}(\theta;\mu_{g_2},\kappa_{g_2})$	$m=0$, $\sigma=p_{g_1}=p_{g_2}=\frac{1}{2}$, $\mu_{g_1}=\frac{\pi}{4}$, $\kappa_{g_1}=\kappa_{g_2}=3$, $\mu_{g_2}=\frac{5\pi}{4}$	
CL10	$\frac{(\lambda^2-\kappa^2)^{\frac{1}{2}}}{2\pi}\exp\{-\lambda z+\kappa z\cos(\theta-\mu)\}$	$\mu=\frac{3\pi}{2}$, $\kappa=2$, $\lambda=3$	See Theorem 1 of Johnson and Wehrly (1978)
CL11	$\big\{1+2\pi\alpha\cos(2\pi F_{Ca}(\theta;\mu,\rho))$ $(1-2F_{\mathcal{N}}(z;m,\sigma))\big\}\times f_{Ca}(\theta;\mu,\rho)f_{\mathcal{N}}(z;m,\sigma)$	$\mu=\frac{3\pi}{2}$, $\rho=\frac{9}{20}$, $m=1$, $\sigma=\frac{1}{2}$, $\alpha=\frac{1}{2\pi}$	See equation (7) of García-Portugués et al. (2013a)
CL12	$\frac{(1-\rho^2)\times f_{vM}(\theta;\mu,\kappa)f_{\mathcal{LN}}(z;m,\sigma)}{4\pi^2\left(1-2\rho F_{vM}(\theta;\mu,\kappa)F_{\mathcal{LN}}(z;m,\sigma)+\rho^2\right)}$	$\mu=\frac{3\pi}{2}$, $\kappa=1$, $m=\frac{1}{2}$, $\sigma=\rho=\frac{3}{4}$	See Section 4.1 in Kato (2009)

Table A.3: Circular-linear models.

Model	Density	Parameters	Description
CC1	$\frac{1}{2\pi} \times f_{vM}(\psi;\mu,\kappa)$	$\mu = 0$, $\kappa = 2$	Indep. uniform and von Mises
CC2	$f_{vM}(\theta;\mu_1,\kappa_1) \times f_{vM}(\psi;\mu_2,\kappa_2)$	$\mu_1 = \frac{3\pi}{2}$, $\kappa_1 = 1$, $\mu_2 = \pi$, $\kappa_2 = 3$	Indep. von Mises and von Mises
CC3	$f_{vM}(\theta;\mu_1,\kappa) \times f_{WC}(\psi;\mu_2,\rho)$	$\mu_1 = \frac{3\pi}{2}$, $\kappa = 2$, $\mu_2 = \frac{\pi}{4}$, $\rho = \frac{7}{10}$	Indep. von Mises and wrapped Cauchy
CC4	$\big(p_1 f_{vM}(\theta;\mu_1,\kappa_1) + p_2 f_{vM}(\theta;\mu_2,\kappa_2)\big) \times f_{Ca}(\psi;\mu_3,\rho)$	$\mu_1 = 0$, $\kappa_1 = \kappa_2 = 10$, $\mu_2 = \frac{3\pi}{2}$, $\mu_3 = 0$, $\rho = \frac{1}{4}$, $p_1 = p_2 = \frac{1}{2}$	Indep. mixture von Mises and cardioid
CC5	$\big(p_1 f_{vM}(\theta;\mu_1,\kappa_1) + p_2 f_{vM}(\theta;\mu_2,\kappa_2)\big) \times \big(p_3 f_{vM}(\psi;\mu_3,\kappa_3) + p_4 f_{vM}(\psi;\mu_4,\kappa_4)\big)$	$\mu_1 = 0$, $\kappa_1 = \kappa_2 = 3$, $\mu_2 = \frac{3\pi}{2}$, $\mu_3 = \frac{\pi}{4}$, $\kappa_3 = \kappa_4 = 5$, $\mu_4 = \frac{7\pi}{4}$, $p_1 = p_2 = p_3 = p_4 = \frac{1}{2}$	Indep. mixture of von Mises and of von Mises
CC6	$C \exp\big\{\kappa_1 \cos(\theta-\mu_1) + \kappa_2 \cos(\psi-\mu_2) + \lambda \sin(\theta-\mu_1)\sin(\psi-\mu_2)\big\}$	$\mu_1 = \frac{7\pi}{8}$, $\kappa_1 = \frac{1}{2}$, $\mu_2 = 0$, $\kappa_2 = 1$, $\lambda = -3$	See equation (1.1) of Singh et al. (2002)
CC7		$\mu_1 = \mu_2 = 0$, $\kappa_1 = 5$, $\kappa_2 = 1$, $\lambda = -5$	
CC8	$f_{vM}\Big(2\pi\big((F_{Ca}(\theta;\mu,\rho) - \frac{\psi}{2\pi}\big);\mu_g,\kappa_g\Big) \times f_{Ca}(\theta;\mu,\rho)$	$\mu = 0$, $\rho = \frac{1}{2}$, $\mu_g = \pi$, $\kappa_g = 7$	See equations (1) and (2) in Wehrly and Johnson (1979) with a von Mises and a mixture of von Mises as links
CC9	$\frac{1}{2\pi}\big(p_{g_1} f_{vM}(\theta+\psi;\mu_{g_1},\kappa_{g_1}) + p_{g_2} f_{vM}(\theta+\psi;\mu_{g_2},\kappa_{g_2})\big)$	$\mu_{g_1} = \frac{\pi}{4}$, $\kappa_{g_1} = \kappa_{g_2} = 10$, $\mu_{g_2} = \frac{7\pi}{4}$, $p_{g_1} = p_{g_2} = \frac{1}{2}$	
CC10	$\sum_{p_1=-\infty}^{\infty} \sum_{p_2=-\infty}^{\infty} f_{\mathcal{N}}(\theta + 2\pi p_1, \psi + 2\pi p_2; m_1, m_2, \sigma_1, \sigma_2, \rho)$	$m_1 = 0$, $m_2 = \frac{\pi}{6}$, $\sigma_1 = \frac{3}{2}$, $\sigma_2 = \frac{1}{4}$, $\rho = 0$	See Example 7.3 in Johnson and Wehrly (1978)
CC11		$m_1 = m_2 = 0$, $\sigma_1 = \sigma_2 = 1$, $\rho = -\frac{9}{10}$	
CC12	$\frac{(1-\rho^2) \times f_{vM}(\theta;\mu_1,\kappa_1) f_{vM}(\psi;\mu_2,\kappa_2)}{4\pi^2\left(1-2\rho F_{vM}(\theta;\mu_1,\kappa_1) F_{vM}(\psi,\mu_2,\kappa_2)+\rho^2\right)}$	$\mu_1 = \frac{3\pi}{4}$, $\kappa_1 = 5$, $\mu_2 = 0$, $\kappa_2 = 1$, $\rho = \frac{1}{2}$	See Section 4.1 of Kato (2009)

Table A.4: Circular-circular models.

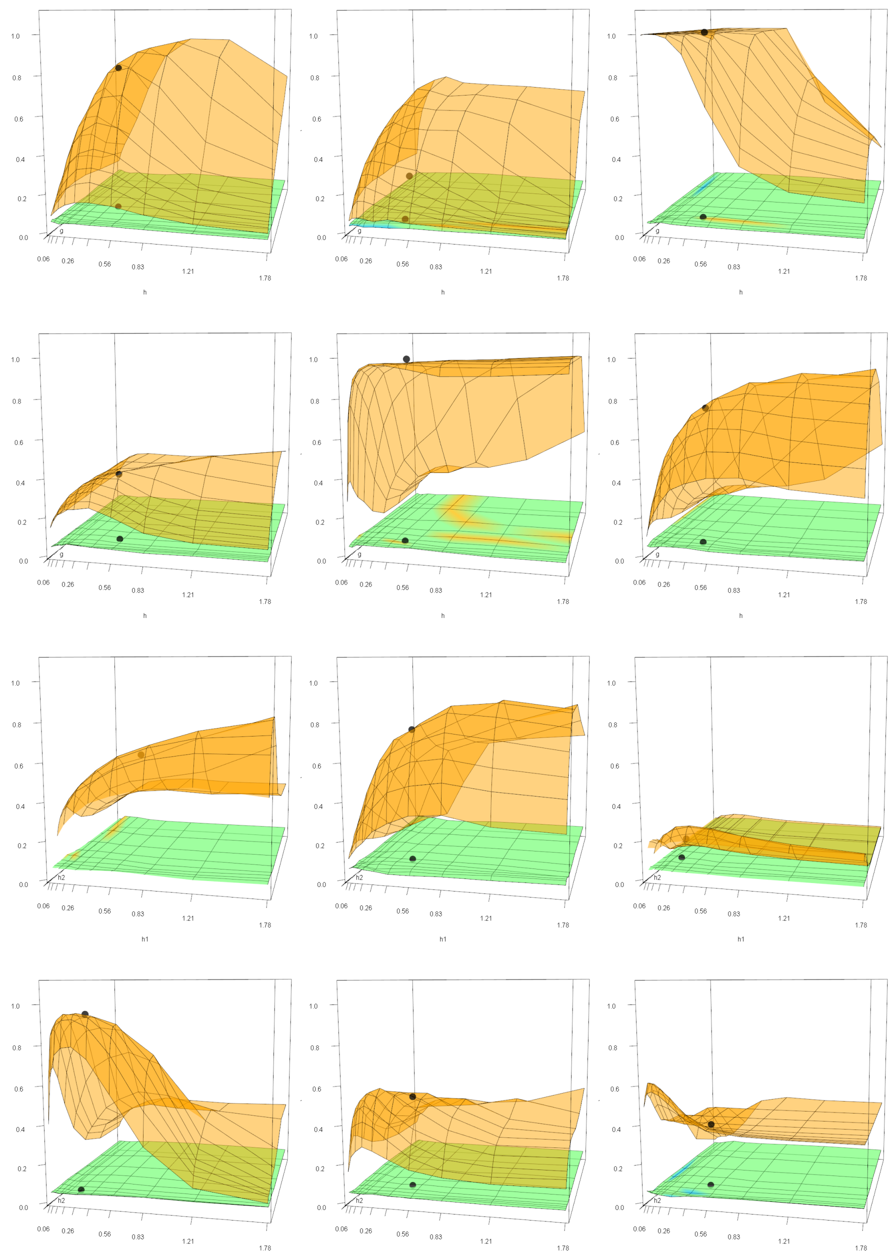

Figure A.6: Empirical size and power of the goodness-of-fit tests for a 10×10 grid of bandwidths. First two rows, from left to right and up to down: models CL1, CL5, CL7, CL8, CL9 and CL11. Last two rows: CC1, CC5, CC7, CC8, CC9 and CC11. Lower surface represents the empirical rejection rate under $H_{0.00}$ and upper surface under $H_{0.15}$. Green colour represent that the empirical rejection is in the 95% confidence interval of $\alpha = 0.05$, blue that is lower and orange that is larger. Black points represent the sized and powers obtained with the median of the LCV bandwidths (for model CC1 under H_0 is outside the grid).

Model	Sample size n and significance level α								
	$n = 100$			$n = 500$			$n = 1000$		
	α=0.10	α=0.05	α=0.01	α=0.10	α=0.05	α=0.01	α=0.10	α=0.05	α=0.01
$H_{1,0.00}$	0.111	0.051	0.010	0.107	0.052	0.013	0.102	0.048	0.013
$H_{2,0.00}$	0.094	0.051	0.013	0.096	0.049	0.010	0.107	0.050	0.009
$H_{3,0.00}$	0.095	0.048	0.014	0.101	0.046	0.014	0.090	0.050	0.009
$H_{4,0.00}$	0.102	0.045	0.009	0.096	0.039	0.011	0.102	0.045	0.008
$H_{5,0.00}$	0.094	0.049	0.009	0.102	0.049	0.009	0.101	0.041	0.009
$H_{6,0.00}$	0.095	0.039	0.010	0.104	0.043	0.010	0.110	0.050	0.015
$H_{7,0.00}$	0.086	0.042	0.013	0.093	0.043	0.008	0.091	0.049	0.016
$H_{8,0.00}$	0.095	0.049	0.011	0.108	0.050	0.003	0.108	0.044	0.006
$H_{9,0.00}$	0.106	0.062	0.016	0.086	0.043	0.010	0.104	0.064	0.015
$H_{10,0.00}$	0.094	0.045	0.007	0.103	0.056	0.018	0.097	0.045	0.005
$H_{11,0.00}$	0.102	0.059	0.009	0.104	0.056	0.010	0.113	0.056	0.013
$H_{12,0.00}$	0.120	0.073	0.020	0.113	0.054	0.013	0.109	0.051	0.010
$H_{1,0.10}$	0.665	0.552	0.355	1.000	0.997	0.981	1.000	1.000	1.000
$H_{2,0.10}$	0.361	0.244	0.107	0.885	0.805	0.579	0.995	0.982	0.898
$H_{3,0.10}$	0.185	0.107	0.032	0.502	0.362	0.166	0.775	0.659	0.421
$H_{4,0.10}$	0.255	0.172	0.060	0.687	0.568	0.322	0.927	0.868	0.697
$H_{5,0.10}$	0.416	0.272	0.087	0.987	0.972	0.894	1.000	1.000	0.999
$H_{6,0.10}$	0.997	0.996	0.988	1.000	1.000	1.000	1.000	1.000	1.000
$H_{7,0.10}$	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000	1.000
$H_{8,0.10}$	0.325	0.204	0.069	0.940	0.893	0.723	1.000	1.000	0.983
$H_{9,0.10}$	0.947	0.914	0.796	1.000	1.000	1.000	1.000	1.000	1.000
$H_{10,0.10}$	0.340	0.218	0.089	0.829	0.723	0.481	0.962	0.944	0.838
$H_{11,0.10}$	0.618	0.510	0.296	0.996	0.993	0.963	1.000	1.000	1.000
$H_{12,0.10}$	0.230	0.152	0.057	0.788	0.655	0.442	0.991	0.969	0.895
$H_{1,0.15}$	0.883	0.822	0.621	1.000	1.000	1.000	1.000	1.000	1.000
$H_{2,0.15}$	0.650	0.525	0.311	1.000	0.997	0.977	1.000	1.000	1.000
$H_{3,0.15}$	0.281	0.163	0.055	0.776	0.682	0.420	0.970	0.940	0.860
$H_{4,0.15}$	0.399	0.297	0.127	0.910	0.869	0.724	0.998	0.993	0.981
$H_{5,0.15}$	0.663	0.514	0.235	0.999	0.999	0.999	1.000	1.000	1.000
$H_{6,0.15}$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
$H_{7,0.15}$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
$H_{8,0.15}$	0.522	0.379	0.168	0.999	0.997	0.976	1.000	1.000	1.000
$H_{9,0.15}$	0.996	0.989	0.962	1.000	1.000	1.000	1.000	1.000	1.000
$H_{10,0.15}$	0.505	0.378	0.154	0.988	0.975	0.893	1.000	1.000	0.996
$H_{11,0.15}$	0.838	0.763	0.567	1.000	1.000	1.000	1.000	1.000	1.000
$H_{12,0.15}$	0.373	0.254	0.114	0.989	0.967	0.872	1.000	1.000	1.000

Table A.5: Empirical size and power of the circular-linear goodness-of-fit test for models CL1–CL12 with different sample sizes, deviations and significance levels.

Model	Sample size n and significance level α								
	$n = 100$			$n = 500$			$n = 1000$		
	α=0.10	α=0.05	α=0.01	α=0.10	α=0.05	α=0.01	α=0.10	α=0.05	α=0.01
$H_{1,0.00}$	0.102	0.061	0.016	0.094	0.047	0.004	0.103	0.048	0.008
$H_{2,0.00}$	0.094	0.054	0.007	0.100	0.043	0.011	0.096	0.056	0.012
$H_{3,0.00}$	0.103	0.061	0.009	0.096	0.042	0.011	0.113	0.058	0.011
$H_{4,0.00}$	0.094	0.049	0.010	0.089	0.048	0.008	0.108	0.052	0.016
$H_{5,0.00}$	0.117	0.059	0.011	0.091	0.050	0.003	0.090	0.051	0.009
$H_{6,0.00}$	0.101	0.069	0.055	0.082	0.045	0.009	0.074	0.034	0.009
$H_{7,0.00}$	0.095	0.048	0.010	0.100	0.059	0.014	0.105	0.044	0.005
$H_{8,0.00}$	0.094	0.043	0.014	0.100	0.054	0.013	0.097	0.050	0.011
$H_{9,0.00}$	0.094	0.043	0.009	0.104	0.057	0.017	0.098	0.042	0.012
$H_{10,0.00}$	0.095	0.047	0.005	0.096	0.041	0.006	0.088	0.042	0.010
$H_{11,0.00}$	0.088	0.041	0.008	0.096	0.047	0.010	0.108	0.053	0.013
$H_{12,0.00}$	0.117	0.062	0.023	0.116	0.058	0.013	0.092	0.048	0.016
$H_{1,0.10}$	0.587	0.456	0.240	0.996	0.995	0.961	1.000	1.000	1.000
$H_{2,0.10}$	0.634	0.506	0.300	0.998	0.994	0.976	1.000	1.000	1.000
$H_{3,0.10}$	0.786	0.706	0.466	1.000	1.000	1.000	1.000	1.000	1.000
$H_{4,0.10}$	0.890	0.837	0.665	1.000	1.000	1.000	1.000	1.000	1.000
$H_{5,0.10}$	0.601	0.431	0.176	1.000	1.000	0.999	1.000	1.000	1.000
$H_{6,0.10}$	0.237	0.123	0.059	0.875	0.759	0.503	0.982	0.958	0.859
$H_{7,0.10}$	0.210	0.112	0.025	0.838	0.724	0.429	0.996	0.989	0.916
$H_{8,0.10}$	0.794	0.693	0.480	1.000	1.000	1.000	1.000	1.000	1.000
$H_{9,0.10}$	0.471	0.325	0.112	1.000	1.000	1.000	1.000	1.000	1.000
$H_{10,0.10}$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
$H_{11,0.10}$	0.985	0.973	0.910	1.000	1.000	1.000	1.000	1.000	1.000
$H_{12,0.10}$	0.942	0.899	0.788	1.000	1.000	1.000	1.000	1.000	1.000
$H_{1,0.15}$	0.847	0.751	0.521	1.000	1.000	1.000	1.000	1.000	1.000
$H_{2,0.15}$	0.862	0.798	0.627	1.000	1.000	1.000	1.000	1.000	1.000
$H_{3,0.15}$	0.958	0.932	0.830	1.000	1.000	1.000	1.000	1.000	1.000
$H_{4,0.15}$	0.981	0.958	0.885	1.000	1.000	1.000	1.000	1.000	1.000
$H_{5,0.15}$	0.847	0.720	0.445	1.000	1.000	1.000	1.000	1.000	1.000
$H_{6,0.15}$	0.443	0.270	0.097	0.985	0.960	0.858	0.997	0.993	0.982
$H_{7,0.15}$	0.357	0.201	0.043	0.990	0.976	0.879	1.000	1.000	1.000
$H_{8,0.15}$	0.969	0.945	0.842	1.000	1.000	1.000	1.000	1.000	1.000
$H_{9,0.15}$	0.719	0.600	0.345	1.000	1.000	1.000	1.000	1.000	1.000
$H_{10,0.15}$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
$H_{11,0.15}$	1.000	1.000	0.993	1.000	1.000	1.000	1.000	1.000	1.000
$H_{12,0.15}$	0.999	0.993	0.975	1.000	1.000	1.000	1.000	1.000	1.000

Table A.6: Empirical size and power of the circular-circular goodness-of-fit test for models CC1–CC12 with different sample sizes, deviations and significance levels.

as shown in Figure A.6. Specifically, Figure A.6 shows percentages of rejections under the null ($\delta = 0.00$, green) and under the alternative ($\delta = 0.15$, orange), computed from $M = 1000$ Monte Carlo samples for each pair of bandwidths (the same collection of samples for each pair) on a logarithmic spaced 10×10 grid. The sample size considered was $n = 100$ and the number of bootstrap replicates was $B = 1000$.

As it can be seen, the test is correctly calibrated regardless the bandwidths value. In fact, for all the models explored, the rejection rates for each pair of bandwidths in the grid are inside the 95% confidence interval of the proportion $\alpha = 0.05$ (this happens for 95.75% of the bandwidths in the grid). However, the power is notably affected by the choice of the bandwidths, with rather different behaviours depending on the model and on the alternative. Reasonable choices of the bandwidths based on an estimation criterion such as the one obtained by the median of the LCV bandwidths (6.1) lead in general to a competitive power.

S3.6 Further results

Tables A.5 and A.6 collect the results of the simulation study for each combination of model (CL or CC), deviation (δ), sample size (n) and significance level (α). When the null hypothesis holds, the level of the test is correctly attained for all significance levels, sample sizes and models. Under the alternative, the tests perform satisfactorily, having both of them a quick detection of the alternative when only a 10% and a 15% of the data come from a density not belonging to the null parametric family.

S4 Extended data application

The analysis of the two real datasets presented in Section 7 has been complemented by exploring the effect of different bandwidths in the test. To that aim, Figure A.7 shows the p-values computed from $B = 1000$ bootstrap replicates for a logarithmic spaced 10×10 grid, as well as bandwidths obtained by LCV for each dataset. The graphs shows that there are no evidences against the model of Mardia and Sutton (1978) for modelling the wildfires data and that the model used to describe the proteins dataset is not adequate. This model employs the copula structure of Wehrly and Johnson (1979) with marginals and link function given by circular densities based on NNTS, specifying Fernández-Durán (2007) that the best fit in terms of BIC arises from considering three components for the NNTS's in the marginals and two for the link function. The fitting of the NNTS densities was performed using the `nntsmanifoldnewtonestimation` function of the package `CircNNTSR` (Fernández-Durán and Gregorio-Domínguez (2013)), which computes the MLE of the NNTS parameters using a Newton algorithm on the hypersphere. The two-step ML procedure described in Section S3 was employed to fit first the marginals and then the copula. The resulting contour levels of the parametric estimate are quite similar to the ones shown in Figure 5 of Fernández-Durán (2007). The dataset is available as `ProteinsAAA` in the `CircNNTSR` package.

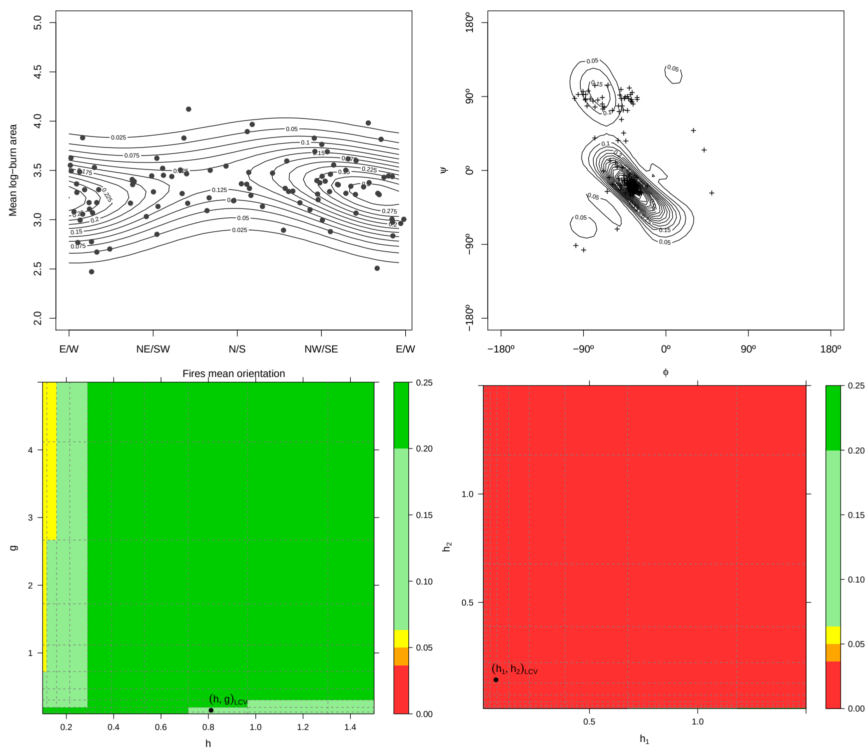

Figure A.7: Upper row, from left to right: parametric fit (model from Mardia and Sutton (1978)) to the circular mean orientation and mean log-burnt area of the fires in each of the 102 watersheds of Portugal; parametric fit (model from Fernández-Durán (2007)) for the dihedral angles of the alanine-alanine-alanine segments. Lower row: p-values of the goodness-of-fit tests for a 10×10 grid, with the LCV bandwidth for the data.

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