Softplus INGARCH Models

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Supplementary Material

The Supplementary Material S1 provides the full set of tables with approximate moment calculations for different softplus INGARCH models. These are discussed in Section 3.2 of the main article.

The Supplementary Material S2 provides the full set of tables with simulation results about ML estimation for different softplus INGARCH models. These are discussed in Section 4.2 of the main article.

The Supplementary Material S3 presents further results about the real-data examples, see Section 5 of the main article.

The Supplementary Material S4 provides the proofs for Theorems 1–3.

S1. Tables with Approximate Moment Calculations

Table S1: Moment properties of a softplus INARCH(1) model ("sp") compared to those of a linear model with the same model parameters ("lin"), computed using Example 1: mean, dispersion ratio, and PACF at lags 1–3.

			μ		σ^2/μ		$\rho_{\mathrm{p}}(1)$		$\rho_{\mathrm{p}}(2)$		$\rho_{\mathrm{p}}(3)$	
#	α_0	α_1	sp	lin	sp	lin	sp	lin	sp	lin	sp	lin
$c=1$												
1	1.875	0.25	2.608	2.500	1.056	1.067	0.232	0.250	0.001	0.000	-0.002	0.000
2	3.750	0.25	5.012	5.000	1.068	1.067	0.249	0.250	-0.001	0.000	0.001	0.000
3	7.500	0.25	9.994	10.000	1.066	1.067	0.249	0.250	0.000	0.000	-0.001	0.000
4	3.125	-0.25	2.567	2.500	1.055	1.067	-0.226	-0.250	-0.002	0.000	0.000	0.000
5	6.250	-0.25	5.007	5.000	1.066	1.067	-0.246	-0.250	-0.002	0.000	0.000	0.000
6	12.500	-0.25	9.995	10.000	1.067	1.067	-0.251	-0.250	0.002	0.000	-0.002	0.000
7	1.250	0.50	2.692	2.500	1.278	1.333	0.465	0.500	0.001	0.000	0.000	0.000
8	2.500	0.50	5.027	5.000	1.329	1.333	0.496	0.500	0.000	0.000	-0.001	0.000
9	5.000	0.50	9.996	10.000	1.334	1.333	0.500	0.500	0.000	0.000	0.000	0.000
10	3.750	-0.50	2.579	2.500	1.234	1.333	-0.433	-0.500	0.006	0.000	0.003	0.000
11	7.500	-0.50	5.014	5.000	1.309	1.333	-0.488	-0.500	0.001	0.000	0.001	0.000
12	15.000	-0.50	9.999	10.000	1.334	1.333	-0.500	-0.500	-0.002	0.000	0.000	0.000
13	0.625	0.75	3.003	2.500	2.018	2.286	0.709	0.750	0.003	0.000	0.000	0.000
14	1.250	0.75	5.123	5.000	2.207	2.286	0.740	0.750	0.000	0.000	0.000	0.000
15	2.500	0.75	9.993	10.000	2.277	2.286	0.749	0.750	0.000	0.000	0.002	0.000
16	4.375	-0.75	2.620	2.500	1.563	2.286	-0.590	-0.750	0.035	0.000	0.006	0.000
17	8.750	-0.75	5.049	5.000	1.949	2.286	-0.694	-0.750	0.024	0.000	0.007	0.000
18	17.500	-0.75	10.011	10.000	2.231	2.286	-0.741	-0.750	0.006	0.000	0.001	0.000
$c=0.5$												
1	1.875	0.25	2.506	2.500	1.067	1.067	0.250	0.250	0.000	0.000	-0.001	0.000
2	3.750	0.25	4.998	5.000	1.069	1.067	0.250	0.250	-0.001	0.000	-0.001	0.000
3	7.500	0.25	9.994	10.000	1.069	1.067	0.249	0.250	-0.001	0.000	0.000	0.000
4	3.125	-0.25	2.504	2.500	1.065	1.067	-0.248	-0.250	0.000	0.000	0.001	0.000
5	6.250	-0.25	4.999	5.000	1.066	1.067	-0.251	-0.250	0.001	0.000	0.002	0.000
6	12.500	-0.25	10.000	10.000	1.069	1.067	-0.250	-0.250	0.001	0.000	0.001	0.000
7	1.250	0.50	2.520	2.500	1.319	1.333	0.492	0.500	0.000	0.000	0.001	0.000
8	2.500	0.50	4.999	5.000	1.333	1.333	0.501	0.500	-0.002	0.000	-0.001	0.000
9	5.000	0.50	10.002	10.000	1.333	1.333	0.499	0.500	0.001	0.000	-0.002	0.000
10	3.750	-0.50	2.515	2.500	1.289	1.333	-0.474	-0.500	0.005	0.000	0.002	0.000
11	7.500	-0.50	5.002	5.000	1.331	1.333	-0.498	-0.500	0.000	0.000	-0.002	0.000
12	15.000	-0.50	9.999	10.000	1.336	1.333	-0.501	-0.500	0.001	0.000	0.001	0.000
13	0.625	0.75	2.617	2.500	2.204	2.286	0.739	0.750	0.001	0.000	0.001	0.000
14	1.250	0.75	5.022	5.000	2.288	2.286	0.749	0.750	-0.002	0.000	0.000	0.000
15	2.500	0.75	9.993	10.000	2.288	2.286	0.750	0.750	0.002	0.000	0.000	0.000
16	4.375	-0.75	2.561	2.500	1.716	2.286	-0.634	-0.750	0.045	0.000	0.010	0.000
17	8.750	-0.75	5.026	5.000	2.046	2.286	-0.710	-0.750	0.021	0.000	0.008	0.000
18	17.500	-0.75	10.005	10.000	2.239	2.286	-0.744	-0.750	0.005	0.000	0.001	0.000

Table S1 *(continued)*: Moment properties of a softplus INARCH(1) model ("sp") compared to those of a linear model with the same model parameters ("lin"), computed using Example 1: mean, dispersion ratio, and PACF at lags 1–3. Boundary case "$c \to 0$" corresponds to the ReLU INARCH(1) model; see Remark 1.

			μ		σ^2/μ		$\rho_p(1)$		$\rho_p(2)$		$\rho_p(3)$	
#	α_0	α_1	sp	lin	sp	lin	sp	lin	sp	lin	sp	lin
$c = 0.25$												
1	1.875	0.25	2.499	2.500	1.065	1.067	0.250	0.250	0.001	0.000	-0.002	0.000
2	3.750	0.25	4.995	5.000	1.066	1.067	0.249	0.250	-0.002	0.000	0.000	0.000
3	7.500	0.25	10.003	10.000	1.065	1.067	0.250	0.250	0.001	0.000	0.001	0.000
4	3.125	-0.25	2.499	2.500	1.066	1.067	-0.251	-0.250	0.001	0.000	0.000	0.000
5	6.250	-0.25	5.003	5.000	1.069	1.067	-0.251	-0.250	0.000	0.000	0.000	0.000
6	12.500	-0.25	9.996	10.000	1.066	1.067	-0.251	-0.250	0.001	0.000	0.000	0.000
7	1.250	0.5	2.499	2.500	1.330	1.333	0.498	0.500	0.003	0.000	-0.001	0.000
8	2.500	0.5	5.001	5.000	1.336	1.333	0.500	0.500	0.000	0.000	0.000	0.000
9	5.000	0.5	10.004	10.000	1.335	1.333	0.499	0.500	-0.001	0.000	0.000	0.000
10	3.750	-0.5	2.505	2.500	1.309	1.333	-0.485	-0.500	0.007	0.000	0.001	0.000
11	7.500	-0.5	4.999	5.000	1.332	1.333	-0.499	-0.500	0.001	0.000	0.002	0.000
12	15.000	-0.5	10.004	10.000	1.331	1.333	-0.499	-0.500	0.000	0.000	0.001	0.000
13	0.625	0.75	2.524	2.500	2.280	2.286	0.749	0.750	0.002	0.000	0.002	0.000
14	1.250	0.75	5.001	5.000	2.293	2.286	0.751	0.750	0.000	0.000	0.000	0.000
15	2.500	0.75	9.982	10.000	2.284	2.286	0.749	0.750	0.000	0.000	0.001	0.000
16	4.375	-0.75	2.550	2.500	1.774	2.286	-0.647	-0.750	0.047	0.000	0.008	0.000
17	8.750	-0.75	5.021	5.000	2.073	2.286	-0.716	-0.750	0.020	0.000	0.007	0.000
18	17.500	-0.75	10.006	10.000	2.254	2.286	-0.745	-0.750	0.004	0.000	0.003	0.000
$c \to 0$												
1	1.875	0.25	2.499	2.500	1.065	1.067	0.249	0.250	-0.001	0.000	0.001	0.000
2	3.750	0.25	5.005	5.000	1.068	1.067	0.250	0.250	0.000	0.000	0.000	0.000
3	7.500	0.25	9.996	10.000	1.066	1.067	0.250	0.250	0.002	0.000	0.001	0.000
4	3.125	-0.25	2.500	2.500	1.066	1.067	-0.250	-0.250	-0.001	0.000	0.000	0.000
5	6.250	-0.25	4.997	5.000	1.066	1.067	-0.250	-0.250	-0.002	0.000	-0.001	0.000
6	12.500	-0.25	9.999	10.000	1.065	1.067	-0.250	-0.250	-0.001	0.000	0.001	0.000
7	1.250	0.5	2.498	2.500	1.333	1.333	0.500	0.500	-0.002	0.000	0.000	0.000
8	2.500	0.5	4.994	5.000	1.336	1.333	0.501	0.500	-0.001	0.000	-0.001	0.000
9	5.000	0.5	9.991	10.000	1.330	1.333	0.499	0.500	0.000	0.000	0.000	0.000
10	3.750	-0.5	2.504	2.500	1.318	1.333	-0.490	-0.500	0.000	0.000	0.000	0.000
11	7.500	-0.5	5.000	5.000	1.334	1.333	-0.499	-0.500	0.002	0.000	0.000	0.000
12	15.000	-0.5	10.002	10.000	1.333	1.333	-0.500	-0.500	0.000	0.000	0.000	0.000
13	0.625	0.75	2.500	2.500	2.279	2.286	0.749	0.750	0.001	0.000	0.000	0.000
14	1.250	0.75	4.994	5.000	2.273	2.286	0.749	0.750	0.000	0.000	0.002	0.000
15	2.500	0.75	9.984	10.000	2.282	2.286	0.749	0.750	0.000	0.000	0.000	0.000
16	4.375	-0.75	2.546	2.500	1.807	2.286	-0.653	-0.750	0.048	0.000	0.011	0.000
17	8.750	-0.75	5.021	5.000	2.081	2.286	-0.717	-0.750	0.021	0.000	0.006	0.000
18	17.500	-0.75	10.001	10.000	2.244	2.286	-0.744	-0.750	0.002	0.000	0.002	0.000

Table S2: Moment properties of a softplus INGARCH(1, 1) model ("sp") compared to those of a linear model with the same model parameters ("lin"), computed using Example 1: mean, dispersion ratio, and ACF at lags 1–3.

				μ		σ^2/μ		$\rho(1)$		$\rho(2)$		$\rho(3)$	
#	α_0	α_1	β_1	sp	lin	sp	lin	sp	lin	sp	lin	sp	lin
$c = 1$													
1	0.750	0.25	0.45	2.655	2.500	1.106	1.123	0.276	0.299	0.188	0.209	0.130	0.147
2	1.500	0.25	0.45	5.015	5.000	1.120	1.123	0.295	0.299	0.204	0.209	0.145	0.147
3	3.000	0.25	0.45	10.007	10.000	1.122	1.123	0.299	0.299	0.210	0.209	0.146	0.147
4	3.000	0.25	-0.45	2.599	2.500	1.057	1.065	0.208	0.222	-0.044	-0.044	0.010	0.009
5	6.000	0.25	-0.45	5.011	5.000	1.065	1.065	0.221	0.222	-0.045	-0.044	0.008	0.009
6	12.000	0.25	-0.45	10.007	10.000	1.067	1.065	0.223	0.222	-0.045	-0.044	0.009	0.009
7	2.000	-0.25	0.45	2.562	2.500	1.053	1.065	-0.204	-0.222	-0.047	-0.044	-0.010	-0.009
8	4.000	-0.25	0.45	5.005	5.000	1.063	1.065	-0.221	-0.222	-0.044	-0.044	-0.009	-0.009
9	8.000	-0.25	0.45	10.000	10.000	1.063	1.065	-0.222	-0.222	-0.044	-0.044	-0.011	-0.009
10	4.250	-0.25	-0.45	2.576	2.500	1.093	1.123	-0.265	-0.299	0.183	0.209	-0.123	-0.147
11	8.500	-0.25	-0.45	5.008	5.000	1.119	1.123	-0.294	-0.299	0.205	0.209	-0.142	-0.147
12	17.000	-0.25	-0.45	10.001	10.000	1.122	1.123	-0.300	-0.299	0.210	0.209	-0.147	-0.147
13	0.750	0.45	0.25	2.744	2.500	1.328	1.397	0.484	0.521	0.324	0.365	0.217	0.255
14	1.500	0.45	0.25	5.029	5.000	1.387	1.397	0.517	0.521	0.359	0.365	0.251	0.255
15	3.000	0.45	0.25	10.018	10.000	1.398	1.397	0.521	0.521	0.365	0.365	0.255	0.255
16	2.000	0.45	-0.25	2.642	2.500	1.181	1.211	0.381	0.406	0.067	0.081	0.013	0.016
17	4.000	0.45	-0.25	5.020	5.000	1.209	1.211	0.404	0.406	0.079	0.081	0.013	0.016
18	8.000	0.45	-0.25	9.997	10.000	1.214	1.211	0.407	0.406	0.082	0.081	0.016	0.016
19	3.000	-0.45	0.25	2.566	2.500	1.163	1.211	-0.362	-0.406	0.051	0.081	-0.008	-0.016
20	6.000	-0.45	0.25	5.006	5.000	1.204	1.211	-0.401	-0.406	0.078	0.081	-0.015	-0.016
21	12.000	-0.45	0.25	9.998	10.000	1.210	1.211	-0.406	-0.406	0.081	0.081	-0.017	-0.016
22	4.250	-0.45	-0.25	2.588	2.500	1.260	1.397	-0.440	-0.521	0.291	0.365	-0.187	-0.255
23	8.500	-0.45	-0.25	5.012	5.000	1.370	1.397	-0.505	-0.521	0.350	0.365	-0.241	-0.255
24	17.000	-0.45	-0.25	10.000	10.000	1.395	1.397	-0.519	-0.521	0.363	0.365	-0.254	-0.255
$c = 0.5$													
1	0.750	0.25	0.45	2.510	2.500	1.122	1.123	0.296	0.299	0.206	0.209	0.144	0.147
2	1.500	0.25	0.45	4.996	5.000	1.122	1.123	0.299	0.299	0.208	0.209	0.144	0.147
3	3.000	0.25	0.45	10.010	10.000	1.121	1.123	0.299	0.299	0.208	0.209	0.146	0.147
4	3.000	0.25	-0.45	2.506	2.500	1.064	1.065	0.221	0.222	-0.044	-0.044	0.011	0.009
5	6.000	0.25	-0.45	5.003	5.000	1.064	1.065	0.223	0.222	-0.045	-0.044	0.007	0.009
6	12.000	0.25	-0.45	9.997	10.000	1.066	1.065	0.222	0.222	-0.045	-0.044	0.008	0.009
7	2.000	-0.25	0.45	2.502	2.500	1.065	1.065	-0.219	-0.222	-0.046	-0.044	-0.009	-0.009
8	4.000	-0.25	0.45	5.001	5.000	1.066	1.065	-0.224	-0.222	-0.045	-0.044	-0.008	-0.009
9	8.000	-0.25	0.45	10.002	10.000	1.062	1.065	-0.222	-0.222	-0.045	-0.044	-0.010	-0.009
10	4.250	-0.25	-0.45	2.505	2.500	1.118	1.123	-0.293	-0.299	0.203	0.209	-0.142	-0.147
11	8.500	-0.25	-0.45	4.999	5.000	1.124	1.123	-0.301	-0.299	0.213	0.209	-0.147	-0.147
12	17.000	-0.25	-0.45	9.999	10.000	1.122	1.123	-0.298	-0.299	0.209	0.209	-0.146	-0.147
13	0.750	0.45	0.25	2.525	2.500	1.383	1.397	0.515	0.521	0.356	0.365	0.248	0.255
14	1.500	0.45	0.25	4.996	5.000	1.397	1.397	0.521	0.521	0.365	0.365	0.255	0.255
15	3.000	0.45	0.25	10.005	10.000	1.403	1.397	0.523	0.521	0.366	0.365	0.257	0.255
16	2.000	0.45	-0.25	2.513	2.500	1.206	1.211	0.404	0.406	0.081	0.081	0.018	0.016
17	4.000	0.45	-0.25	5.003	5.000	1.210	1.211	0.407	0.406	0.081	0.081	0.015	0.016
18	8.000	0.45	-0.25	10.006	10.000	1.213	1.211	0.408	0.406	0.083	0.081	0.016	0.016
19	3.000	-0.45	0.25	2.507	2.500	1.198	1.211	-0.393	-0.406	0.072	0.081	-0.013	-0.016
20	6.000	-0.45	0.25	4.999	5.000	1.209	1.211	-0.407	-0.406	0.082	0.081	-0.017	-0.016
21	12.000	-0.45	0.25	10.000	10.000	1.209	1.211	-0.406	-0.406	0.081	0.081	-0.014	-0.016
22	4.250	-0.45	-0.25	2.519	2.500	1.341	1.397	-0.488	-0.521	0.338	0.365	-0.229	-0.255
23	8.500	-0.45	-0.25	5.000	5.000	1.390	1.397	-0.517	-0.521	0.362	0.365	-0.252	-0.255
24	17.000	-0.45	-0.25	10.000	10.000	1.397	1.397	-0.522	-0.521	0.364	0.365	-0.254	-0.255

Table S2 *(continued)*: Moment properties of a softplus INGARCH$(1, 1)$ model ("sp") compared to those of a linear model with the same model parameters ("lin"), computed using Example 1: mean, dispersion ratio, and ACF at lags 1–3. Boundary case "$c \to 0$" corresponds to the ReLU INGARCH$(1, 1)$ model; see Remark 1.

				μ		σ^2/μ		$\rho(1)$		$\rho(2)$		$\rho(3)$	
#	α_0	α_1	β_1	sp	lin	sp	lin	sp	lin	sp	lin	sp	lin
$c = 0.25$													
1	0.750	0.25	0.45	2.501	2.500	1.123	1.123	0.297	0.299	0.209	0.209	0.145	0.147
2	1.500	0.25	0.45	5.003	5.000	1.123	1.123	0.298	0.299	0.211	0.209	0.148	0.147
3	3.000	0.25	0.45	9.997	10.000	1.122	1.123	0.298	0.299	0.209	0.209	0.144	0.147
4	3.000	0.25	-0.45	2.499	2.500	1.068	1.065	0.223	0.222	-0.045	-0.044	0.008	0.009
5	6.000	0.25	-0.45	5.005	5.000	1.064	1.065	0.223	0.222	-0.045	-0.044	0.009	0.009
6	12.000	0.25	-0.45	10.000	10.000	1.065	1.065	0.224	0.222	-0.043	-0.044	0.010	0.009
7	2.000	-0.25	0.45	2.501	2.500	1.065	1.065	-0.224	-0.222	-0.044	-0.044	-0.009	-0.009
8	4.000	-0.25	0.45	5.001	5.000	1.068	1.065	-0.222	-0.222	-0.045	-0.044	-0.010	-0.009
9	8.000	-0.25	0.45	9.999	10.000	1.067	1.065	-0.224	-0.222	-0.044	-0.044	-0.009	-0.009
10	4.250	-0.25	-0.45	2.500	2.500	1.123	1.123	-0.299	-0.299	0.211	0.209	-0.147	-0.147
11	8.500	-0.25	-0.45	4.999	5.000	1.125	1.123	-0.299	-0.299	0.209	0.209	-0.147	-0.147
12	17.000	-0.25	-0.45	9.997	10.000	1.122	1.123	-0.297	-0.299	0.209	0.209	-0.147	-0.147
13	0.750	0.45	0.25	2.502	2.500	1.399	1.397	0.519	0.521	0.363	0.365	0.254	0.255
14	1.500	0.45	0.25	5.009	5.000	1.402	1.397	0.523	0.521	0.368	0.365	0.258	0.255
15	3.000	0.45	0.25	10.002	10.000	1.389	1.397	0.519	0.521	0.363	0.365	0.253	0.255
16	2.000	0.45	-0.25	2.499	2.500	1.215	1.211	0.407	0.406	0.081	0.081	0.015	0.016
17	4.000	0.45	-0.25	4.994	5.000	1.211	1.211	0.406	0.406	0.081	0.081	0.016	0.016
18	8.000	0.45	-0.25	10.003	10.000	1.208	1.211	0.406	0.406	0.081	0.081	0.016	0.016
19	3.000	-0.45	0.25	2.502	2.500	1.206	1.211	-0.402	-0.406	0.079	0.081	-0.016	-0.016
20	6.000	-0.45	0.25	5.001	5.000	1.213	1.211	-0.406	-0.406	0.081	0.081	-0.015	-0.016
21	12.000	-0.45	0.25	9.998	10.000	1.209	1.211	-0.407	-0.406	0.082	0.081	-0.016	-0.016
22	4.250	-0.45	-0.25	2.507	2.500	1.361	1.397	-0.502	-0.521	0.349	0.365	-0.240	-0.255
23	8.500	-0.45	-0.25	5.000	5.000	1.393	1.397	-0.519	-0.521	0.363	0.365	-0.254	-0.255
24	17.000	-0.45	-0.25	10.001	10.000	1.401	1.397	-0.523	-0.521	0.365	0.365	-0.256	-0.255
$c \to 0$													
1	0.750	0.25	0.45	2.501	2.500	1.123	1.123	0.301	0.299	0.211	0.209	0.148	0.147
2	1.500	0.25	0.45	4.998	5.000	1.121	1.123	0.299	0.299	0.209	0.209	0.148	0.147
3	3.000	0.25	0.45	10.008	10.000	1.126	1.123	0.300	0.299	0.210	0.209	0.147	0.147
4	3.000	0.25	-0.45	2.499	2.500	1.065	1.065	0.222	0.222	-0.043	-0.044	0.011	0.009
5	6.000	0.25	-0.45	5.004	5.000	1.067	1.065	0.223	0.222	-0.044	-0.044	0.007	0.009
6	12.000	0.25	-0.45	9.995	10.000	1.066	1.065	0.222	0.222	-0.045	-0.044	0.010	0.009
7	2.000	-0.25	0.45	2.498	2.500	1.066	1.065	-0.222	-0.222	-0.045	-0.044	-0.008	-0.009
8	4.000	-0.25	0.45	4.999	5.000	1.064	1.065	-0.222	-0.222	-0.042	-0.044	-0.010	-0.009
9	8.000	-0.25	0.45	9.999	10.000	1.067	1.065	-0.221	-0.222	-0.046	-0.044	-0.010	-0.009
10	4.250	-0.25	-0.45	2.501	2.500	1.125	1.123	-0.299	-0.299	0.209	0.209	-0.147	-0.147
11	8.500	-0.25	-0.45	4.999	5.000	1.122	1.123	-0.299	-0.299	0.208	0.209	-0.146	-0.147
12	17.000	-0.25	-0.45	10.006	10.000	1.123	1.123	-0.301	-0.299	0.209	0.209	-0.147	-0.147
13	0.750	0.45	0.25	2.500	2.500	1.401	1.397	0.523	0.521	0.367	0.365	0.260	0.255
14	1.500	0.45	0.25	4.997	5.000	1.389	1.397	0.518	0.521	0.362	0.365	0.253	0.255
15	3.000	0.45	0.25	9.999	10.000	1.398	1.397	0.522	0.521	0.365	0.365	0.256	0.255
16	2.000	0.45	-0.25	2.498	2.500	1.210	1.211	0.406	0.406	0.082	0.081	0.017	0.016
17	4.000	0.45	-0.25	5.003	5.000	1.213	1.211	0.406	0.406	0.080	0.081	0.016	0.016
18	8.000	0.45	-0.25	10.006	10.000	1.212	1.211	0.407	0.406	0.081	0.081	0.016	0.016
19	3.000	-0.45	0.25	2.501	2.500	1.206	1.211	-0.403	-0.406	0.078	0.081	-0.014	-0.016
20	6.000	-0.45	0.25	5.000	5.000	1.210	1.211	-0.406	-0.406	0.083	0.081	-0.017	-0.016
21	12.000	-0.45	0.25	10.000	10.000	1.208	1.211	-0.404	-0.406	0.078	0.081	-0.015	-0.016
22	4.250	-0.45	-0.25	2.505	2.500	1.375	1.397	-0.507	-0.521	0.354	0.365	-0.244	-0.255
23	8.500	-0.45	-0.25	5.002	5.000	1.393	1.397	-0.521	-0.521	0.363	0.365	-0.254	-0.255
24	17.000	-0.45	-0.25	10.004	10.000	1.398	1.397	-0.523	-0.521	0.366	0.365	-0.258	-0.255

Table S3: Moment properties of a softplus INGARCH$(1,1)$ model ("sp") compared to those of a linear model with the same model parameters ("lin"), computed using Example 1: mean, dispersion ratio, and ACF at lags 1–3.

				μ		σ^2/μ		$\rho(1)$		$\rho(2)$		$\rho(3)$	
#	α_0	α_1	β_1	sp	lin	sp	lin	sp	lin	sp	lin	sp	lin
$c=1$													
1	0.125	0.20	0.75	2.900	2.500	1.300	1.410	0.362	0.418	0.340	0.397	0.318	0.377
2	0.250	0.20	0.75	5.081	5.000	1.396	1.410	0.412	0.418	0.390	0.397	0.370	0.377
3	0.500	0.20	0.75	9.986	10.000	1.405	1.410	0.416	0.418	0.396	0.397	0.376	0.377
4	3.875	0.20	-0.75	2.592	2.500	1.052	1.057	0.149	0.159	-0.083	-0.088	0.047	0.048
5	7.750	0.20	-0.75	5.006	5.000	1.056	1.057	0.160	0.159	-0.085	-0.088	0.047	0.048
6	15.500	0.20	-0.75	9.998	10.000	1.057	1.057	0.160	0.159	-0.088	-0.088	0.047	0.048
7	1.125	-0.20	0.75	2.550	2.500	1.046	1.057	-0.148	-0.159	-0.083	-0.088	-0.048	-0.048
8	2.250	-0.20	0.75	5.004	5.000	1.057	1.057	-0.159	-0.159	-0.088	-0.088	-0.048	-0.048
9	4.500	-0.20	0.75	9.997	10.000	1.058	1.057	-0.159	-0.159	-0.088	-0.088	-0.049	-0.048
10	4.875	-0.20	-0.75	2.596	2.500	1.236	1.410	-0.317	-0.418	0.301	0.397	-0.275	-0.377
11	9.750	-0.20	-0.75	5.014	5.000	1.371	1.410	-0.399	-0.418	0.379	0.397	-0.357	-0.377
12	19.500	-0.20	-0.75	9.999	10.000	1.408	1.410	-0.417	-0.418	0.396	0.397	-0.376	-0.377
13	0.125	0.75	0.20	4.592	2.500	5.263	6.769	0.896	0.920	0.838	0.874	0.783	0.831
14	0.250	0.75	0.20	6.211	5.000	5.808	6.769	0.907	0.920	0.854	0.874	0.804	0.831
15	0.500	0.75	0.20	10.325	10.000	6.427	6.769	0.916	0.920	0.868	0.874	0.822	0.831
16	1.125	0.75	-0.20	2.818	2.500	1.659	1.806	0.622	0.661	0.314	0.363	0.159	0.200
17	2.250	0.75	-0.20	5.061	5.000	1.776	1.806	0.654	0.661	0.354	0.363	0.191	0.200
18	4.500	0.75	-0.20	10.000	10.000	1.809	1.806	0.661	0.661	0.363	0.363	0.199	0.200
19	3.875	-0.75	0.20	2.594	2.500	1.444	1.806	-0.541	-0.661	0.224	0.363	-0.093	-0.200
20	7.750	-0.75	0.20	5.028	5.000	1.681	1.806	-0.628	-0.661	0.323	0.363	-0.167	-0.200
21	15.500	-0.75	0.20	10.002	10.000	1.794	1.806	-0.657	-0.661	0.360	0.363	-0.196	-0.200
22	4.875	-0.75	-0.20	2.682	2.500	1.877	6.769	-0.649	-0.920	0.546	0.874	-0.423	-0.831
23	9.750	-0.75	-0.20	5.131	5.000	2.776	6.769	-0.777	-0.920	0.698	0.874	-0.601	-0.831
24	19.500	-0.75	-0.20	10.083	10.000	4.090	6.769	-0.858	-0.920	0.795	0.874	-0.725	-0.831
$c=0.5$													
1	0.125	0.20	0.75	2.548	2.500	1.383	1.410	0.406	0.418	0.384	0.397	0.365	0.377
2	0.250	0.20	0.75	4.998	5.000	1.407	1.410	0.416	0.418	0.397	0.397	0.376	0.377
3	0.500	0.20	0.75	10.030	10.000	1.409	1.410	0.415	0.418	0.396	0.397	0.375	0.377
4	3.875	0.20	-0.75	2.505	2.500	1.055	1.057	0.158	0.159	-0.088	-0.088	0.050	0.048
5	7.750	0.20	-0.75	4.999	5.000	1.059	1.057	0.158	0.159	-0.089	-0.088	0.047	0.048
6	15.500	0.20	-0.75	10.002	10.000	1.055	1.057	0.159	0.159	-0.088	-0.088	0.048	0.048
7	1.125	-0.20	0.75	2.502	2.500	1.058	1.057	-0.158	-0.159	-0.087	-0.088	-0.050	-0.048
8	2.250	-0.20	0.75	5.001	5.000	1.059	1.057	-0.159	-0.159	-0.088	-0.088	-0.049	-0.048
9	4.500	-0.20	0.75	10.000	10.000	1.060	1.057	-0.159	-0.159	-0.088	-0.088	-0.047	-0.048
10	4.875	-0.20	-0.75	2.517	2.500	1.342	1.410	-0.383	-0.418	0.367	0.397	-0.341	-0.377
11	9.750	-0.20	-0.75	5.002	5.000	1.399	1.410	-0.412	-0.418	0.391	0.397	-0.372	-0.377
12	19.500	-0.20	-0.75	9.997	10.000	1.413	1.410	-0.419	-0.418	0.399	0.397	-0.379	-0.377
13	0.125	0.75	0.20	3.542	2.500	5.815	6.769	0.906	0.920	0.853	0.874	0.803	0.831
14	0.250	0.75	0.20	5.483	5.000	6.369	6.769	0.915	0.920	0.867	0.874	0.821	0.831
15	0.500	0.75	0.20	10.074	10.000	6.671	6.769	0.919	0.920	0.873	0.874	0.829	0.831
16	1.125	0.75	-0.20	2.552	2.500	1.768	1.806	0.652	0.661	0.353	0.363	0.193	0.200
17	2.250	0.75	-0.20	4.996	5.000	1.811	1.806	0.660	0.661	0.363	0.363	0.198	0.200
18	4.500	0.75	-0.20	10.003	10.000	1.801	1.806	0.660	0.661	0.361	0.363	0.197	0.200
19	3.875	-0.75	0.20	2.538	2.500	1.555	1.806	-0.582	-0.661	0.273	0.363	-0.131	-0.200
20	7.750	-0.75	0.20	5.011	5.000	1.733	1.806	-0.641	-0.661	0.340	0.363	-0.181	-0.200
21	15.500	-0.75	0.20	10.001	10.000	1.808	1.806	-0.660	-0.661	0.363	0.363	-0.199	-0.200
22	4.875	-0.75	-0.20	2.630	2.500	2.129	6.769	-0.689	-0.920	0.606	0.874	-0.490	-0.831
23	9.750	-0.75	-0.20	5.108	5.000	3.002	6.769	-0.793	-0.920	0.719	0.874	-0.629	-0.831
24	19.500	-0.75	-0.20	10.082	10.000	4.273	6.769	-0.863	-0.920	0.803	0.874	-0.736	-0.831

Table S3 *(continued)*: Moment properties of a softplus INGARCH$(1,1)$ model ("sp") compared to those of a linear model with the same model parameters ("lin"), computed using Example 1: mean, dispersion ratio, and ACF at lags 1–3. Boundary case "$c \to 0$" corresponds to the ReLU INGARCH$(1,1)$ model; see Remark 1.

				μ		σ^2/μ		$\rho(1)$		$\rho(2)$		$\rho(3)$	
#	α_0	α_1	β_1	sp	lin	sp	lin	sp	lin	sp	lin	sp	lin
$c = 0.25$													
1	0.125	0.2	0.75	2.501	2.500	1.408	1.410	0.417	0.418	0.397	0.397	0.377	0.377
2	0.250	0.2	0.75	5.007	5.000	1.412	1.410	0.419	0.418	0.398	0.397	0.380	0.377
3	0.500	0.2	0.75	10.017	10.000	1.416	1.410	0.419	0.418	0.399	0.397	0.379	0.377
4	3.875	0.2	-0.75	2.498	2.500	1.057	1.057	0.160	0.159	-0.089	-0.088	0.048	0.048
5	7.750	0.2	-0.75	5.000	5.000	1.057	1.057	0.160	0.159	-0.087	-0.088	0.049	0.048
6	15.500	0.2	-0.75	10.005	10.000	1.057	1.057	0.159	0.159	-0.088	-0.088	0.048	0.048
7	1.125	-0.2	0.75	2.502	2.500	1.056	1.057	-0.159	-0.159	-0.090	-0.088	-0.046	-0.048
8	2.250	-0.2	0.75	5.002	5.000	1.059	1.057	-0.161	-0.159	-0.088	-0.088	-0.048	-0.048
9	4.500	-0.2	0.75	9.997	10.000	1.058	1.057	-0.161	-0.159	-0.086	-0.088	-0.048	-0.048
10	4.875	-0.2	-0.75	2.503	2.500	1.379	1.410	-0.402	-0.418	0.384	0.397	-0.361	-0.377
11	9.750	-0.2	-0.75	4.999	5.000	1.412	1.410	-0.418	-0.418	0.396	0.397	-0.377	-0.377
12	19.500	-0.2	-0.75	9.998	10.000	1.403	1.410	-0.416	-0.418	0.395	0.397	-0.375	-0.377
13	0.125	0.75	0.2	2.975	2.500	6.390	6.769	0.916	0.920	0.867	0.874	0.820	0.831
14	0.250	0.75	0.2	5.191	5.000	6.709	6.769	0.920	0.920	0.873	0.874	0.829	0.831
15	0.500	0.75	0.2	9.991	10.000	6.784	6.769	0.921	0.920	0.875	0.874	0.831	0.831
16	1.125	0.75	-0.2	2.503	2.500	1.800	1.806	0.659	0.661	0.362	0.363	0.199	0.200
17	2.250	0.75	-0.2	4.993	5.000	1.805	1.806	0.661	0.661	0.364	0.363	0.200	0.200
18	4.500	0.75	-0.2	9.996	10.000	1.802	1.806	0.660	0.661	0.364	0.363	0.201	0.200
19	3.875	-0.75	0.2	2.528	2.500	1.595	1.806	-0.596	-0.661	0.290	0.363	-0.142	-0.200
20	7.750	-0.75	0.2	5.009	5.000	1.750	1.806	-0.646	-0.661	0.345	0.363	-0.184	-0.200
21	15.500	-0.75	0.2	9.999	10.000	1.797	1.806	-0.659	-0.661	0.361	0.363	-0.198	-0.200
22	4.875	-0.75	-0.2	2.618	2.500	2.237	6.769	-0.701	-0.920	0.624	0.874	-0.511	-0.831
23	9.750	-0.75	-0.2	5.104	5.000	3.078	6.769	-0.797	-0.920	0.726	0.874	-0.638	-0.831
24	19.500	-0.75	-0.2	10.077	10.000	4.287	6.769	-0.864	-0.920	0.803	0.874	-0.737	-0.831
$c \to 0$													
1	0.125	0.2	0.75	2.489	2.500	1.404	1.410	0.416	0.418	0.395	0.397	0.376	0.377
2	0.250	0.2	0.75	5.003	5.000	1.410	1.410	0.418	0.418	0.397	0.397	0.377	0.377
3	0.500	0.2	0.75	10.015	10.000	1.405	1.410	0.417	0.418	0.395	0.397	0.375	0.377
4	3.875	0.2	-0.75	2.500	2.500	1.055	1.057	0.158	0.159	-0.086	-0.088	0.050	0.048
5	7.750	0.2	-0.75	4.997	5.000	1.060	1.057	0.160	0.159	-0.087	-0.088	0.048	0.048
6	15.500	0.2	-0.75	10.001	10.000	1.058	1.057	0.159	0.159	-0.089	-0.088	0.047	0.048
7	1.125	-0.2	0.75	2.499	2.500	1.058	1.057	-0.159	-0.159	-0.089	-0.088	-0.046	-0.048
8	2.250	-0.2	0.75	4.999	5.000	1.056	1.057	-0.160	-0.159	-0.088	-0.088	-0.048	-0.048
9	4.500	-0.2	0.75	10.000	10.000	1.057	1.057	-0.159	-0.159	-0.090	-0.088	-0.048	-0.048
10	4.875	-0.2	-0.75	2.504	2.500	1.393	1.410	-0.409	-0.418	0.390	0.397	-0.368	-0.377
11	9.750	-0.2	-0.75	5.001	5.000	1.410	1.410	-0.418	-0.418	0.397	0.397	-0.378	-0.377
12	19.500	-0.2	-0.75	10.002	10.000	1.409	1.410	-0.418	-0.418	0.396	0.397	-0.376	-0.377
13	0.125	0.75	0.2	2.449	2.500	6.613	6.769	0.919	0.920	0.872	0.874	0.827	0.831
14	0.250	0.75	0.2	5.029	5.000	6.811	6.769	0.921	0.920	0.875	0.874	0.832	0.831
15	0.500	0.75	0.2	10.126	10.000	6.895	6.769	0.922	0.920	0.877	0.874	0.833	0.831
16	1.125	0.75	-0.2	2.498	2.500	1.802	1.806	0.660	0.661	0.362	0.363	0.198	0.200
17	2.250	0.75	-0.2	5.005	5.000	1.818	1.806	0.662	0.661	0.365	0.363	0.200	0.200
18	4.500	0.75	-0.2	9.981	10.000	1.808	1.806	0.661	0.661	0.363	0.363	0.201	0.200
19	3.875	-0.75	0.2	2.525	2.500	1.612	1.806	-0.600	-0.661	0.294	0.363	-0.145	-0.200
20	7.750	-0.75	0.2	5.008	5.000	1.747	1.806	-0.646	-0.661	0.345	0.363	-0.185	-0.200
21	15.500	-0.75	0.2	10.001	10.000	1.802	1.806	-0.660	-0.661	0.363	0.363	-0.199	-0.200
22	4.875	-0.75	-0.2	2.615	2.500	2.269	6.769	-0.705	-0.920	0.629	0.874	-0.519	-0.831
23	9.750	-0.75	-0.2	5.104	5.000	3.114	6.769	-0.799	-0.920	0.729	0.874	-0.641	-0.831
24	19.500	-0.75	-0.2	10.070	10.000	4.317	6.769	-0.865	-0.920	0.805	0.874	-0.738	-0.831

S2. Tables with Maximum Likelihood Estimations

Table S4: ML estimation for simulated softplus INARCH(1) processes: mean of estimates, and s. e. of estimates compared to mean of approximate s. e.

					Mean of		Simulated s. e. and mean approx. s. e. for			
μ	c	α_0	α_1	n	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\alpha}_0$		$\hat{\alpha}_1$	
2	1	3	-0.5	50	3.010	-0.509	0.422	0.429	0.159	0.161
				100	3.005	-0.505	0.297	0.298	0.110	0.111
				250	3.003	-0.502	0.186	0.187	0.069	0.069
				500	3.000	-0.500	0.131	0.132	0.048	0.049
2	0.5	3	-0.5	50	3.017	-0.509	0.379	0.377	0.135	0.133
				100	3.007	-0.504	0.261	0.261	0.091	0.091
				250	3.001	-0.501	0.162	0.163	0.056	0.056
				500	3.001	-0.501	0.115	0.115	0.040	0.040
2	1	1	0.5	50	1.100	0.446	0.410	0.415	0.157	0.163
				100	1.052	0.473	0.284	0.286	0.110	0.112
				250	1.025	0.489	0.179	0.178	0.069	0.069
				500	1.009	0.495	0.125	0.125	0.049	0.049
2	0.5	1	0.5	50	1.086	0.449	0.322	0.317	0.147	0.147
				100	1.040	0.475	0.219	0.217	0.100	0.101
				250	1.017	0.490	0.135	0.135	0.062	0.063
				500	1.008	0.495	0.094	0.095	0.044	0.044
5	1	7.5	-0.5	50	7.474	-0.494	0.744	0.751	0.121	0.122
				100	7.483	-0.497	0.516	0.519	0.083	0.083
				250	7.500	-0.500	0.324	0.324	0.052	0.052
				500	7.500	-0.500	0.229	0.228	0.036	0.036
5	0.5	7.5	-0.5	50	7.479	-0.495	0.723	0.726	0.114	0.114
				100	7.490	-0.498	0.499	0.500	0.078	0.077
				250	7.495	-0.499	0.313	0.311	0.048	0.048
				500	7.495	-0.499	0.217	0.218	0.033	0.033
5	1	2.5	0.5	50	2.733	0.450	0.692	0.697	0.133	0.137
				100	2.618	0.474	0.478	0.477	0.093	0.094
				250	2.551	0.489	0.298	0.296	0.059	0.058
				500	2.525	0.495	0.208	0.208	0.041	0.041
5	0.5	2.5	0.5	50	2.736	0.450	0.667	0.677	0.131	0.134
				100	2.616	0.475	0.461	0.461	0.091	0.092
				250	2.547	0.490	0.285	0.285	0.057	0.057
				500	2.520	0.495	0.200	0.200	0.040	0.040
15	1	22.5	-0.5	50	22.371	-0.491	1.950	1.994	0.119	0.122
				100	22.431	-0.495	1.365	1.379	0.083	0.084
				250	22.476	-0.498	0.864	0.862	0.053	0.053
				500	22.488	-0.499	0.605	0.606	0.037	0.037
15	0.5	22.5	-0.5	50	22.382	-0.492	1.946	1.993	0.119	0.122
				100	22.421	-0.495	1.372	1.379	0.084	0.084
				250	22.452	-0.497	0.860	0.862	0.052	0.053
				500	22.491	-0.499	0.610	0.606	0.037	0.037
15	1	7.5	0.5	50	8.192	0.452	1.954	1.977	0.129	0.132
				100	7.896	0.473	1.356	1.356	0.090	0.090
				250	7.657	0.489	0.841	0.838	0.056	0.056
				500	7.567	0.495	0.595	0.589	0.039	0.039
15	0.5	7.5	0.5	50	8.272	0.447	1.972	1.985	0.131	0.132
				100	7.876	0.475	1.349	1.355	0.090	0.090
				250	7.645	0.490	0.842	0.838	0.056	0.056
				500	7.578	0.495	0.589	0.589	0.039	0.039

Table S5: ML estimation for simulated softplus INGARCH(1, 1) processes: mean of estimates, and s. e. of estimates compared to mean of approximate s. e.

#	α_0	α_1	β_1	n	Mean of $\hat{\alpha}_0$	Mean of $\hat{\alpha}_1$	Mean of $\hat{\beta}_1$	Simulated s. e. and mean approx. s. e. for $\hat{\alpha}_0$		$\hat{\alpha}_1$		$\hat{\beta}_1$	
$\mu = 2$, $c = 1$													
1	0.6	0.25	0.45	100	0.964	0.238	0.287	0.665	0.540	0.126	0.107	0.349	0.273
				250	0.767	0.251	0.369	0.399	0.345	0.071	0.069	0.210	0.186
				500	0.684	0.250	0.410	0.260	0.236	0.050	0.049	0.141	0.132
2	2.4	0.25	-0.45	100	2.328	0.250	-0.421	0.715	0.660	0.114	0.113	0.323	0.283
				250	2.371	0.251	-0.439	0.540	0.493	0.072	0.072	0.237	0.210
				500	2.383	0.250	-0.443	0.397	0.374	0.051	0.051	0.173	0.159
3	1.6	-0.25	0.45	100	1.694	-0.270	0.423	0.701	0.583	0.113	0.110	0.347	0.279
				250	1.631	-0.258	0.443	0.539	0.457	0.072	0.070	0.258	0.211
				500	1.618	-0.256	0.447	0.376	0.347	0.050	0.050	0.177	0.159
4	3.4	-0.25	-0.45	100	3.139	-0.263	-0.307	0.668	0.548	0.122	0.106	0.348	0.264
				250	3.278	-0.258	-0.380	0.410	0.361	0.071	0.068	0.214	0.185
				500	3.343	-0.254	-0.417	0.271	0.249	0.049	0.048	0.143	0.131
5	0.6	0.45	0.25	100	0.785	0.436	0.175	0.455	0.425	0.123	0.118	0.236	0.218
				250	0.677	0.445	0.218	0.258	0.250	0.076	0.074	0.139	0.135
				500	0.635	0.447	0.236	0.172	0.169	0.053	0.052	0.095	0.094
6	1.6	0.45	-0.25	100	1.653	0.435	-0.265	0.592	0.576	0.118	0.122	0.253	0.249
				250	1.635	0.444	-0.263	0.381	0.380	0.075	0.076	0.166	0.164
				500	1.615	0.447	-0.255	0.266	0.265	0.054	0.053	0.116	0.114
7	2.4	-0.45	0.25	100	2.391	-0.457	0.260	0.557	0.531	0.118	0.121	0.266	0.255
				250	2.388	-0.456	0.262	0.360	0.348	0.075	0.075	0.171	0.164
				500	2.389	-0.454	0.259	0.247	0.244	0.052	0.052	0.118	0.115
8	3.4	-0.45	-0.25	100	3.292	-0.464	-0.181	0.479	0.435	0.124	0.118	0.247	0.215
				250	3.362	-0.456	-0.224	0.277	0.268	0.075	0.074	0.144	0.137
				500	3.381	-0.453	-0.239	0.186	0.185	0.053	0.052	0.099	0.096
$\mu = 2$, $c = 0.5$													
1	0.6	0.25	0.45	100	0.928	0.242	0.294	0.574	0.468	0.114	0.100	0.313	0.255
				250	0.741	0.250	0.380	0.330	0.290	0.065	0.063	0.186	0.168
				500	0.668	0.251	0.415	0.206	0.197	0.045	0.045	0.120	0.117
2	2.4	0.25	-0.45	100	2.346	0.251	-0.425	0.652	0.597	0.103	0.103	0.298	0.264
				250	2.369	0.252	-0.437	0.477	0.443	0.065	0.065	0.216	0.194
				500	2.391	0.251	-0.446	0.344	0.330	0.047	0.047	0.154	0.144
3	1.6	-0.25	0.45	100	1.674	-0.268	0.433	0.644	0.553	0.096	0.096	0.311	0.263
				250	1.619	-0.259	0.450	0.472	0.415	0.060	0.060	0.223	0.191
				500	1.608	-0.254	0.450	0.335	0.313	0.043	0.043	0.157	0.143
4	3.4	-0.25	-0.45	100	3.152	-0.263	-0.310	0.613	0.503	0.102	0.091	0.313	0.243
				250	3.289	-0.256	-0.388	0.366	0.320	0.060	0.057	0.187	0.162
				500	3.352	-0.253	-0.423	0.229	0.218	0.041	0.040	0.118	0.112
5	0.6	0.45	0.25	100	0.755	0.438	0.180	0.368	0.337	0.114	0.108	0.215	0.198
				250	0.658	0.445	0.225	0.202	0.196	0.069	0.069	0.125	0.122
				500	0.629	0.449	0.237	0.135	0.134	0.049	0.048	0.084	0.085
6	1.6	0.45	-0.25	100	1.669	0.433	-0.273	0.510	0.496	0.108	0.111	0.231	0.228
				250	1.632	0.445	-0.264	0.319	0.317	0.069	0.069	0.147	0.145
				500	1.615	0.447	-0.255	0.225	0.222	0.049	0.049	0.104	0.102
7	2.4	-0.45	0.25	100	2.380	-0.459	0.268	0.498	0.484	0.100	0.102	0.238	0.233
				250	2.387	-0.455	0.261	0.316	0.310	0.062	0.062	0.152	0.148
				500	2.395	-0.453	0.256	0.219	0.217	0.045	0.044	0.105	0.103
8	3.4	-0.45	-0.25	100	3.310	-0.465	-0.189	0.429	0.389	0.106	0.100	0.215	0.190
				250	3.365	-0.457	-0.226	0.248	0.238	0.063	0.062	0.125	0.119
				500	3.387	-0.453	-0.240	0.169	0.165	0.044	0.044	0.085	0.083

Table S5 *(continued)*: ML estimation for simulated softplus INGARCH(1, 1) processes: mean of estimates, and s. e. of estimates compared to mean of approximate s. e.

					Mean of			Simulated s. e. and mean approx. s. e. for					
#	α_0	α_1	β_1	n	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{\alpha}_0$		$\hat{\alpha}_1$		$\hat{\beta}_1$	
$\mu = 5$, $c = 1$													
1	1.5	0.25	0.45	100	2.481	0.247	0.258	1.339	1.099	0.105	0.096	0.281	0.238
				250	1.958	0.255	0.354	0.777	0.689	0.060	0.060	0.170	0.158
				500	1.741	0.254	0.398	0.500	0.468	0.042	0.043	0.114	0.111
2	6	0.25	-0.45	100	5.755	0.254	-0.405	1.451	1.347	0.100	0.101	0.278	0.252
				250	5.878	0.254	-0.429	1.097	1.013	0.063	0.063	0.206	0.184
				500	5.930	0.251	-0.436	0.817	0.777	0.045	0.045	0.151	0.141
3	4	-0.25	0.45	100	4.239	-0.272	0.425	1.419	1.271	0.094	0.098	0.278	0.249
				250	4.126	-0.259	0.435	1.104	0.983	0.060	0.061	0.212	0.184
				500	4.046	-0.254	0.445	0.808	0.752	0.043	0.043	0.153	0.139
4	8.5	-0.25	-0.45	100	7.715	-0.267	-0.274	1.372	1.113	0.097	0.091	0.282	0.229
				250	8.115	-0.262	-0.359	0.816	0.715	0.058	0.057	0.171	0.155
				500	8.308	-0.256	-0.405	0.513	0.488	0.040	0.040	0.112	0.109
5	1.5	0.45	0.25	100	1.914	0.443	0.173	0.845	0.767	0.107	0.102	0.198	0.185
				250	1.670	0.448	0.217	0.466	0.454	0.064	0.064	0.116	0.116
				500	1.577	0.450	0.235	0.313	0.310	0.045	0.045	0.081	0.081
6	4	0.45	-0.25	100	4.165	0.433	-0.268	1.191	1.171	0.102	0.105	0.226	0.225
				250	4.086	0.444	-0.263	0.769	0.756	0.065	0.066	0.148	0.144
				500	4.052	0.447	-0.258	0.532	0.529	0.046	0.046	0.102	0.101
7	6	-0.45	0.25	100	5.926	-0.453	0.267	1.186	1.150	0.094	0.098	0.231	0.229
				250	5.957	-0.453	0.261	0.767	0.743	0.060	0.060	0.152	0.146
				500	5.977	-0.452	0.256	0.530	0.522	0.042	0.042	0.105	0.102
8	8.5	-0.45	-0.25	100	8.222	-0.460	-0.182	0.925	0.817	0.097	0.093	0.199	0.178
				250	8.390	-0.455	-0.222	0.517	0.497	0.059	0.058	0.116	0.112
				500	8.452	-0.453	-0.237	0.349	0.343	0.041	0.041	0.080	0.079
$\mu = 5$, $c = 0.5$													
1	1.5	0.25	0.45	100	2.468	0.246	0.261	1.295	1.093	0.105	0.096	0.275	0.237
				250	1.960	0.254	0.354	0.752	0.678	0.061	0.059	0.166	0.156
				500	1.743	0.255	0.397	0.476	0.459	0.042	0.042	0.110	0.109
2	6	0.25	-0.45	100	5.760	0.254	-0.405	1.416	1.329	0.099	0.100	0.272	0.249
				250	5.886	0.252	-0.428	1.093	1.006	0.063	0.063	0.205	0.184
				500	5.932	0.251	-0.437	0.805	0.767	0.045	0.045	0.149	0.139
3	4	-0.25	0.45	100	4.272	-0.273	0.420	1.423	1.258	0.093	0.096	0.279	0.246
				250	4.124	-0.259	0.435	1.098	0.966	0.060	0.060	0.211	0.181
				500	4.040	-0.255	0.447	0.790	0.741	0.042	0.043	0.148	0.137
4	8.5	-0.25	-0.45	100	7.723	-0.267	-0.273	1.349	1.099	0.095	0.089	0.276	0.226
				250	8.120	-0.261	-0.361	0.803	0.704	0.057	0.056	0.169	0.152
				500	8.295	-0.257	-0.402	0.510	0.484	0.040	0.039	0.112	0.107
5	1.5	0.45	0.25	100	1.912	0.442	0.176	0.832	0.746	0.104	0.100	0.194	0.182
				250	1.664	0.449	0.218	0.455	0.438	0.064	0.063	0.116	0.113
				500	1.585	0.449	0.234	0.305	0.302	0.045	0.045	0.080	0.080
6	4	0.45	-0.25	100	4.169	0.433	-0.269	1.202	1.153	0.101	0.104	0.230	0.223
				250	4.055	0.445	-0.257	0.755	0.744	0.064	0.065	0.146	0.143
				500	4.041	0.448	-0.256	0.526	0.521	0.046	0.045	0.102	0.100
7	6	-0.45	0.25	100	5.916	-0.452	0.269	1.187	1.137	0.091	0.094	0.233	0.227
				250	5.949	-0.453	0.262	0.758	0.731	0.058	0.058	0.150	0.144
				500	5.963	-0.452	0.259	0.520	0.513	0.041	0.040	0.103	0.101
8	8.5	-0.45	-0.25	100	8.222	-0.462	-0.181	0.877	0.801	0.091	0.087	0.190	0.174
				250	8.399	-0.455	-0.223	0.500	0.483	0.055	0.054	0.113	0.108
				500	8.451	-0.453	-0.236	0.332	0.334	0.038	0.038	0.075	0.076

Table S5 *(continued)*: ML estimation for simulated softplus INGARCH(1, 1) processes: mean of estimates, and s. e. of estimates compared to mean of approximate s. e.

					Mean of			Simulated s. e. and mean approx. s. e. for					
#	α_0	α_1	β_1	n	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{\alpha}_0$		$\hat{\alpha}_1$		$\hat{\beta}_1$	
$\mu = 15$, $c = 1$													
1	4.5	0.25	0.45	100	8.173	0.255	0.201	3.445	3.043	0.103	0.096	0.233	0.213
				250	6.561	0.263	0.300	2.137	1.935	0.059	0.058	0.149	0.143
				500	5.714	0.262	0.358	1.391	1.327	0.041	0.041	0.102	0.102
2	18	0.25	-0.45	100	16.738	0.255	-0.370	3.656	3.482	0.099	0.102	0.243	0.228
				250	17.186	0.255	-0.400	2.942	2.723	0.061	0.062	0.191	0.171
				500	17.544	0.253	-0.422	2.187	2.138	0.044	0.044	0.137	0.132
3	12	-0.25	0.45	100	13.276	-0.276	0.392	3.635	3.333	0.097	0.101	0.245	0.224
				250	12.807	-0.263	0.409	2.971	2.640	0.061	0.061	0.194	0.168
				500	12.445	-0.256	0.427	2.204	2.102	0.043	0.043	0.140	0.131
4	25.5	-0.25	-0.45	100	22.328	-0.279	-0.207	3.516	2.957	0.097	0.092	0.238	0.203
				250	23.669	-0.270	-0.307	2.193	1.931	0.058	0.057	0.150	0.140
				500	24.409	-0.264	-0.363	1.426	1.347	0.040	0.040	0.103	0.101
5	4.5	0.45	0.25	100	6.090	0.449	0.145	2.358	2.139	0.101	0.097	0.174	0.167
				250	5.197	0.453	0.201	1.328	1.280	0.061	0.061	0.108	0.107
				500	4.861	0.452	0.224	0.904	0.890	0.044	0.043	0.077	0.077
6	12	0.45	-0.25	100	12.451	0.433	-0.264	3.504	3.234	0.099	0.102	0.229	0.213
				250	12.265	0.444	-0.262	2.315	2.218	0.064	0.064	0.151	0.144
				500	12.135	0.448	-0.257	1.592	1.557	0.046	0.045	0.104	0.101
7	18	-0.45	0.25	100	17.730	-0.453	0.271	3.530	3.249	0.097	0.100	0.234	0.219
				250	17.840	-0.453	0.264	2.308	2.201	0.062	0.062	0.152	0.145
				500	17.920	-0.452	0.257	1.581	1.550	0.044	0.043	0.106	0.102
8	25.5	-0.45	-0.25	100	24.351	-0.470	-0.152	2.406	2.138	0.096	0.092	0.177	0.163
				250	24.988	-0.460	-0.206	1.377	1.313	0.059	0.058	0.109	0.106
				500	25.225	-0.455	-0.226	0.933	0.920	0.041	0.041	0.076	0.076
$\mu = 15$, $c = 0.5$													
1	4.5	0.25	0.45	100	8.123	0.255	0.204	3.434	3.013	0.104	0.095	0.233	0.211
				250	6.575	0.263	0.299	2.167	1.933	0.059	0.058	0.151	0.142
				500	5.707	0.261	0.359	1.383	1.324	0.040	0.041	0.101	0.102
2	18	0.25	-0.45	100	16.695	0.256	-0.368	3.650	3.482	0.100	0.101	0.243	0.227
				250	17.198	0.254	-0.400	2.954	2.719	0.062	0.062	0.191	0.171
				500	17.570	0.253	-0.425	2.212	2.127	0.044	0.044	0.139	0.131
3	12	-0.25	0.45	100	13.259	-0.276	0.393	3.625	3.309	0.096	0.101	0.244	0.222
				250	12.765	-0.262	0.411	2.955	2.637	0.061	0.061	0.193	0.168
				500	12.400	-0.257	0.430	2.242	2.087	0.043	0.043	0.142	0.130
4	25.5	-0.25	-0.45	100	22.376	-0.280	-0.210	3.457	2.953	0.097	0.092	0.234	0.204
				250	23.676	-0.271	-0.306	2.159	1.939	0.058	0.057	0.149	0.140
				500	24.408	-0.264	-0.363	1.417	1.348	0.040	0.040	0.103	0.101
5	4.5	0.45	0.25	100	6.029	0.449	0.149	2.348	2.113	0.101	0.096	0.175	0.166
				250	5.209	0.454	0.199	1.324	1.282	0.062	0.061	0.107	0.108
				500	4.851	0.452	0.225	0.900	0.887	0.043	0.043	0.077	0.077
6	12	0.45	-0.25	100	12.434	0.433	-0.264	3.498	3.249	0.100	0.103	0.230	0.214
				250	12.256	0.444	-0.262	2.310	2.207	0.064	0.064	0.149	0.144
				500	12.131	0.447	-0.256	1.600	1.564	0.045	0.045	0.105	0.102
7	18	-0.45	0.25	100	17.703	-0.450	0.270	3.548	3.264	0.095	0.100	0.234	0.220
				250	17.782	-0.452	0.266	2.318	2.202	0.061	0.062	0.152	0.145
				500	17.890	-0.451	0.258	1.616	1.553	0.044	0.043	0.107	0.102
8	25.5	-0.45	-0.25	100	24.336	-0.469	-0.152	2.388	2.134	0.095	0.092	0.175	0.162
				250	24.968	-0.458	-0.206	1.384	1.321	0.059	0.058	0.108	0.106
				500	25.228	-0.456	-0.226	0.932	0.920	0.042	0.041	0.076	0.076

Table S6: ML estimation for simulated softplus INGARCH(1,1) processes: mean of estimates, and s. e. of estimates compared to mean of approximate s. e.

#	α_0	α_1	β_1	n	Mean of $\hat{\alpha}_0$	Mean of $\hat{\alpha}_1$	Mean of $\hat{\beta}_1$	Sim. s.e. $\hat{\alpha}_0$	Approx. s.e. $\hat{\alpha}_0$	Sim. s.e. $\hat{\alpha}_1$	Approx. s.e. $\hat{\alpha}_1$	Sim. s.e. $\hat{\beta}_1$	Approx. s.e. $\hat{\beta}_1$
$\mu=2$, $c=1$													
1	0.1	0.2	0.75	100	0.645	0.207	0.514	0.698	0.413	0.117	0.091	0.303	0.200
				250	0.269	0.218	0.662	0.246	0.163	0.057	0.054	0.121	0.096
				500	0.168	0.211	0.711	0.099	0.085	0.038	0.037	0.061	0.057
2	3.1	0.2	-0.75	100	2.712	0.219	-0.583	0.630	0.579	0.101	0.106	0.286	0.243
				250	2.901	0.213	-0.668	0.424	0.403	0.061	0.065	0.171	0.158
				500	2.989	0.209	-0.706	0.295	0.290	0.044	0.046	0.110	0.108
3	0.9	-0.2	0.75	100	1.247	-0.232	0.607	0.567	0.469	0.097	0.103	0.280	0.235
				250	1.083	-0.219	0.676	0.392	0.349	0.061	0.064	0.179	0.160
				500	0.994	-0.212	0.714	0.253	0.250	0.044	0.045	0.106	0.107
4	3.9	-0.2	-0.75	100	3.628	-0.231	-0.582	0.582	0.424	0.106	0.090	0.268	0.174
				250	3.815	-0.218	-0.690	0.269	0.232	0.056	0.054	0.115	0.091
				500	3.864	-0.210	-0.723	0.159	0.153	0.039	0.038	0.062	0.057
5	0.1	0.75	0.2	100	0.299	0.732	0.153	0.301	0.282	0.114	0.115	0.125	0.132
				250	0.176	0.744	0.181	0.147	0.154	0.072	0.072	0.076	0.078
				500	0.137	0.747	0.191	0.098	0.103	0.051	0.051	0.054	0.054
6	0.9	0.75	-0.2	100	0.955	0.698	-0.181	0.395	0.416	0.100	0.129	0.134	0.185
				250	0.917	0.726	-0.187	0.243	0.251	0.063	0.079	0.082	0.108
				500	0.909	0.737	-0.193	0.173	0.176	0.046	0.055	0.060	0.074
7	3.1	-0.75	0.2	100	3.095	-0.726	0.171	0.370	0.387	0.100	0.136	0.137	0.191
				250	3.100	-0.739	0.185	0.231	0.236	0.065	0.083	0.084	0.111
				500	3.101	-0.743	0.191	0.161	0.165	0.048	0.058	0.061	0.076
8	3.9	-0.75	-0.2	100	3.833	-0.757	-0.164	0.297	0.319	0.126	0.137	0.140	0.146
				250	3.876	-0.756	-0.183	0.188	0.197	0.082	0.086	0.088	0.089
				500	3.891	-0.754	-0.192	0.133	0.138	0.060	0.060	0.062	0.062
$\mu=2$, $c=0.5$													
1	0.1	0.2	0.75	100	0.524	0.210	0.530	0.550	0.312	0.107	0.085	0.277	0.186
				250	0.221	0.218	0.673	0.162	0.113	0.053	0.050	0.101	0.085
				500	0.153	0.212	0.713	0.071	0.061	0.035	0.035	0.054	0.052
2	3.1	0.2	-0.75	100	2.712	0.218	-0.578	0.577	0.531	0.092	0.098	0.270	0.232
				250	2.910	0.215	-0.671	0.359	0.354	0.055	0.059	0.148	0.142
				500	2.995	0.209	-0.708	0.247	0.251	0.039	0.041	0.095	0.096
3	0.9	-0.2	0.75	100	1.253	-0.230	0.606	0.521	0.442	0.085	0.092	0.253	0.221
				250	1.079	-0.218	0.680	0.334	0.315	0.052	0.056	0.151	0.143
				500	1.002	-0.211	0.711	0.225	0.221	0.038	0.039	0.095	0.095
4	3.9	-0.2	-0.75	100	3.683	-0.235	-0.606	0.502	0.364	0.090	0.076	0.221	0.143
				250	3.835	-0.220	-0.698	0.220	0.198	0.048	0.046	0.086	0.072
				500	3.872	-0.210	-0.726	0.138	0.134	0.032	0.032	0.048	0.046
5	0.1	0.75	0.2	100	0.216	0.728	0.158	0.215	0.185	0.117	0.117	0.127	0.132
				250	0.143	0.743	0.183	0.101	0.102	0.073	0.072	0.077	0.077
				500	0.120	0.746	0.192	0.068	0.068	0.051	0.051	0.053	0.054
6	0.9	0.75	-0.2	100	0.939	0.704	-0.187	0.301	0.314	0.092	0.118	0.121	0.165
				250	0.912	0.729	-0.191	0.184	0.190	0.059	0.073	0.074	0.096
				500	0.905	0.739	-0.195	0.129	0.133	0.043	0.051	0.055	0.067
7	3.1	-0.75	0.2	100	3.110	-0.732	0.172	0.331	0.350	0.087	0.121	0.120	0.174
				250	3.106	-0.740	0.185	0.208	0.214	0.057	0.073	0.077	0.101
				500	3.104	-0.744	0.191	0.149	0.150	0.043	0.051	0.056	0.070
8	3.9	-0.75	-0.2	100	3.849	-0.762	-0.168	0.266	0.293	0.115	0.127	0.125	0.130
				250	3.883	-0.757	-0.187	0.171	0.182	0.077	0.079	0.079	0.079
				500	3.894	-0.754	-0.194	0.125	0.128	0.054	0.056	0.055	0.055

Table S6 *(continued)*: ML estimation for simulated softplus INGARCH(1, 1) processes: mean of estimates, and s. e. of estimates compared to mean of approximate s. e.

					Mean of			Simulated s. e. and mean approx. s. e. for					
#	α_0	α_1	β_1	n	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{\alpha}_0$		$\hat{\alpha}_1$		$\hat{\beta}_1$	
$\mu = 5$, $c = 1$													
1	0.25	0.2	0.75	100	1.571	0.229	0.459	1.319	0.780	0.106	0.085	0.256	0.181
				250	0.707	0.239	0.621	0.462	0.291	0.052	0.048	0.103	0.083
				500	0.450	0.227	0.684	0.200	0.150	0.034	0.033	0.055	0.050
2	7.75	0.2	-0.75	100	6.509	0.221	-0.522	1.281	1.192	0.096	0.100	0.264	0.229
				250	7.012	0.221	-0.622	0.859	0.800	0.055	0.058	0.159	0.142
				500	7.314	0.215	-0.677	0.571	0.560	0.038	0.040	0.098	0.094
3	2.25	-0.2	0.75	100	3.466	-0.235	0.544	1.224	1.099	0.090	0.099	0.256	0.230
				250	2.972	-0.225	0.631	0.807	0.737	0.052	0.057	0.155	0.139
				500	2.687	-0.218	0.681	0.534	0.517	0.037	0.039	0.096	0.091
4	9.75	-0.2	-0.75	100	8.992	-0.260	-0.535	1.129	0.724	0.089	0.075	0.211	0.142
				250	9.477	-0.238	-0.656	0.456	0.364	0.048	0.045	0.089	0.072
				500	9.625	-0.223	-0.702	0.260	0.235	0.032	0.031	0.052	0.046
5	0.25	0.75	0.2	100	0.505	0.738	0.158	0.433	0.335	0.107	0.106	0.117	0.120
				250	0.339	0.747	0.183	0.189	0.178	0.067	0.067	0.071	0.072
				500	0.294	0.749	0.191	0.123	0.119	0.048	0.047	0.050	0.051
6	2.25	0.75	-0.2	100	2.346	0.706	-0.182	0.660	0.702	0.085	0.108	0.116	0.160
				250	2.278	0.730	-0.189	0.400	0.420	0.053	0.067	0.073	0.094
				500	2.261	0.739	-0.193	0.285	0.293	0.039	0.047	0.053	0.065
7	7.75	-0.75	0.2	100	7.772	-0.729	0.172	0.686	0.746	0.076	0.101	0.114	0.165
				250	7.768	-0.740	0.185	0.430	0.453	0.048	0.061	0.073	0.098
				500	7.763	-0.746	0.193	0.305	0.317	0.036	0.043	0.053	0.068
8	9.75	-0.75	-0.2	100	9.658	-0.765	-0.169	0.478	0.498	0.097	0.101	0.107	0.110
				250	9.726	-0.757	-0.188	0.304	0.306	0.063	0.063	0.068	0.068
				500	9.738	-0.755	-0.193	0.214	0.215	0.044	0.044	0.047	0.047
$\mu = 5$, $c = 0.5$													
1	0.25	0.2	0.75	100	1.552	0.232	0.457	1.281	0.758	0.105	0.083	0.247	0.178
				250	0.706	0.241	0.619	0.437	0.278	0.051	0.048	0.097	0.081
				500	0.456	0.229	0.681	0.188	0.143	0.033	0.032	0.052	0.049
2	7.75	0.2	-0.75	100	6.531	0.221	-0.527	1.235	1.184	0.093	0.099	0.255	0.228
				250	7.018	0.220	-0.623	0.827	0.794	0.054	0.058	0.154	0.141
				500	7.303	0.215	-0.676	0.564	0.555	0.038	0.040	0.098	0.093
3	2.25	-0.2	0.75	100	3.475	-0.234	0.541	1.206	1.084	0.089	0.097	0.256	0.226
				250	2.968	-0.224	0.632	0.790	0.723	0.052	0.056	0.152	0.137
				500	2.695	-0.217	0.678	0.533	0.511	0.037	0.039	0.097	0.091
4	9.75	-0.2	-0.75	100	8.992	-0.261	-0.531	1.111	0.718	0.086	0.072	0.204	0.138
				250	9.477	-0.240	-0.653	0.459	0.358	0.045	0.042	0.086	0.069
				500	9.626	-0.224	-0.700	0.255	0.231	0.031	0.029	0.048	0.044
5	0.25	0.75	0.2	100	0.446	0.737	0.160	0.350	0.253	0.106	0.105	0.115	0.117
				250	0.316	0.746	0.184	0.146	0.132	0.068	0.066	0.072	0.071
				500	0.282	0.749	0.192	0.091	0.088	0.047	0.047	0.049	0.050
6	2.25	0.75	-0.2	100	2.352	0.707	-0.185	0.626	0.664	0.082	0.105	0.112	0.155
				250	2.279	0.731	-0.190	0.377	0.397	0.052	0.064	0.070	0.092
				500	2.258	0.740	-0.193	0.265	0.276	0.038	0.045	0.052	0.063
7	7.75	-0.75	0.2	100	7.780	-0.731	0.173	0.653	0.722	0.070	0.093	0.110	0.161
				250	7.771	-0.741	0.186	0.408	0.437	0.044	0.056	0.070	0.095
				500	7.762	-0.746	0.193	0.294	0.306	0.033	0.039	0.052	0.066
8	9.75	-0.75	-0.2	100	9.680	-0.769	-0.170	0.463	0.483	0.091	0.094	0.099	0.101
				250	9.735	-0.759	-0.188	0.293	0.298	0.057	0.058	0.061	0.061
				500	9.744	-0.754	-0.194	0.210	0.209	0.041	0.041	0.043	0.043

Table S6 *(continued)*: ML estimation for simulated softplus INGARCH$(1,1)$ processes: mean of estimates, and s. e. of estimates compared to mean of approximate s. e.

					Mean of			Simulated s. e. and mean approx. s. e. for					
#	α_0	α_1	β_1	n	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{\alpha}_0$		$\hat{\alpha}_1$		$\hat{\beta}_1$	
$\mu = 15$, $c = 1$													
1	0.75	0.2	0.75	100	6.081	0.254	0.341	3.759	2.363	0.111	0.086	0.218	0.170
				250	3.131	0.275	0.518	1.571	0.970	0.055	0.048	0.099	0.083
				500	2.005	0.262	0.605	0.749	0.510	0.034	0.032	0.058	0.051
2	23.25	0.2	-0.75	100	18.344	0.222	-0.444	3.134	3.164	0.104	0.105	0.238	0.219
				250	19.867	0.224	-0.548	2.215	2.168	0.058	0.061	0.154	0.141
				500	20.882	0.222	-0.613	1.505	1.531	0.039	0.041	0.098	0.094
3	6.75	-0.2	0.75	100	11.665	-0.238	0.462	3.122	3.019	0.102	0.105	0.239	0.218
				250	10.137	-0.231	0.557	2.174	2.029	0.057	0.060	0.153	0.135
				500	9.165	-0.225	0.614	1.507	1.459	0.039	0.040	0.100	0.092
4	29.25	-0.2	-0.75	100	25.594	-0.306	-0.398	3.114	1.974	0.099	0.078	0.182	0.140
				250	27.522	-0.283	-0.550	1.429	0.949	0.050	0.046	0.092	0.075
				500	28.309	-0.259	-0.628	0.772	0.572	0.033	0.031	0.057	0.049
5	0.75	0.75	0.2	100	1.487	0.751	0.146	0.980	0.690	0.098	0.094	0.104	0.106
				250	1.005	0.755	0.176	0.408	0.343	0.062	0.061	0.066	0.066
				500	0.873	0.753	0.188	0.244	0.224	0.044	0.043	0.046	0.047
6	6.75	0.75	-0.2	100	7.061	0.704	-0.178	1.917	2.012	0.081	0.103	0.118	0.158
				250	6.816	0.729	-0.184	1.135	1.197	0.052	0.064	0.073	0.094
				500	6.782	0.739	-0.192	0.799	0.834	0.038	0.045	0.053	0.065
7	23.25	-0.75	0.2	100	23.283	-0.722	0.169	1.886	2.023	0.074	0.095	0.115	0.159
				250	23.249	-0.737	0.186	1.143	1.219	0.046	0.057	0.072	0.095
				500	23.274	-0.744	0.192	0.809	0.850	0.034	0.040	0.052	0.066
8	29.25	-0.75	-0.2	100	29.041	-0.774	-0.162	0.990	0.929	0.079	0.076	0.088	0.086
				250	29.180	-0.762	-0.183	0.575	0.560	0.049	0.048	0.055	0.054
				500	29.214	-0.756	-0.192	0.399	0.391	0.035	0.034	0.039	0.038
$\mu = 15$, $c = 0.5$													
1	0.75	0.2	0.75	100	6.095	0.253	0.341	3.733	2.357	0.111	0.086	0.218	0.170
				250	3.131	0.276	0.517	1.581	0.966	0.056	0.048	0.101	0.082
				500	2.000	0.262	0.606	0.748	0.510	0.034	0.032	0.056	0.051
2	23.25	0.2	-0.75	100	18.295	0.221	-0.440	3.157	3.172	0.105	0.105	0.243	0.219
				250	19.854	0.225	-0.547	2.211	2.159	0.059	0.061	0.153	0.140
				500	20.933	0.222	-0.617	1.494	1.516	0.039	0.041	0.096	0.093
3	6.75	-0.2	0.75	100	11.686	-0.241	0.464	3.136	2.972	0.101	0.104	0.238	0.214
				250	10.149	-0.231	0.555	2.180	2.034	0.058	0.060	0.153	0.135
				500	9.139	-0.225	0.616	1.499	1.450	0.039	0.040	0.099	0.091
4	29.25	-0.2	-0.75	100	25.652	-0.306	-0.401	3.104	1.953	0.097	0.078	0.181	0.138
				250	27.474	-0.283	-0.546	1.486	0.962	0.051	0.046	0.097	0.076
				500	28.303	-0.259	-0.627	0.770	0.574	0.033	0.031	0.057	0.049
5	0.75	0.75	0.2	100	1.504	0.751	0.146	1.007	0.678	0.097	0.093	0.104	0.105
				250	1.008	0.753	0.177	0.406	0.331	0.061	0.060	0.066	0.066
				500	0.872	0.753	0.188	0.241	0.213	0.043	0.043	0.046	0.047
6	6.75	0.75	-0.2	100	7.072	0.704	-0.178	1.916	2.016	0.081	0.103	0.117	0.158
				250	6.863	0.728	-0.187	1.157	1.204	0.052	0.064	0.073	0.094
				500	6.789	0.739	-0.192	0.808	0.835	0.038	0.045	0.053	0.065
7	23.25	-0.75	0.2	100	23.253	-0.722	0.171	1.866	2.034	0.074	0.094	0.113	0.160
				250	23.296	-0.737	0.184	1.138	1.213	0.047	0.057	0.072	0.094
				500	23.273	-0.744	0.192	0.794	0.848	0.033	0.040	0.052	0.066
8	29.25	-0.75	-0.2	100	29.060	-0.776	-0.160	0.991	0.918	0.077	0.074	0.086	0.084
				250	29.188	-0.762	-0.183	0.572	0.552	0.047	0.046	0.052	0.052
				500	29.224	-0.756	-0.191	0.389	0.384	0.033	0.033	0.036	0.036

Table S7: ML estimation for simulated softplus INARCH(1) processes with mean $\mu = 5$, where DGP has tabulated parameter value for c, but $c = 1$ was assumed for estimation (so italic numbers correspond to adequate models): mean of estimates, and s. e. of estimates compared to mean of approximate s. e.

				Mean of		Simulated s. e. and mean approx. s. e. for			
c	α_0	α_1	n	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\alpha}_0$		$\hat{\alpha}_1$	
2	7.5	-0.5	50	7.403	-0.445	0.751	0.771	0.120	0.124
			100	7.419	-0.448	0.529	0.533	0.084	0.085
			250	7.416	-0.448	0.328	0.332	0.052	0.053
			500	7.417	-0.448	0.231	0.234	0.037	0.037
1	*7.5*	*-0.5*	*50*	*7.474*	*-0.494*	*0.744*	*0.751*	*0.121*	*0.122*
			100	*7.483*	*-0.497*	*0.516*	*0.519*	*0.083*	*0.083*
			250	*7.500*	*-0.500*	*0.324*	*0.324*	*0.052*	*0.052*
			500	*7.500*	*-0.500*	*0.229*	*0.228*	*0.036*	*0.036*
0.5	7.5	-0.5	50	7.543	-0.511	0.734	0.750	0.119	0.121
			100	7.542	-0.512	0.523	0.519	0.084	0.083
			250	7.555	-0.514	0.324	0.324	0.052	0.052
			500	7.556	-0.514	0.231	0.228	0.037	0.036
2	2.5	0.5	50	3.099	0.414	0.748	0.749	0.137	0.138
			100	2.959	0.441	0.509	0.512	0.095	0.094
			250	2.898	0.455	0.315	0.318	0.059	0.059
			500	2.872	0.461	0.219	0.223	0.041	0.041
1	*2.5*	*0.5*	*50*	*2.733*	*0.450*	*0.692*	*0.697*	*0.133*	*0.137*
			100	*2.618*	*0.474*	*0.478*	*0.477*	*0.093*	*0.094*
			250	*2.551*	*0.489*	*0.298*	*0.296*	*0.059*	*0.058*
			500	*2.525*	*0.495*	*0.208*	*0.208*	*0.041*	*0.041*
0.5	2.5	0.5	50	2.697	0.455	0.707	0.694	0.136	0.136
			100	2.582	0.480	0.475	0.475	0.092	0.094
			250	2.509	0.495	0.292	0.294	0.058	0.058
			500	2.485	0.500	0.206	0.206	0.041	0.041

Table S8: ML estimation for simulated softplus INGARCH$(1,1)$ processes with mean $\mu = 5$, where DGP has tabulated parameter value for c, but $c = 1$ was assumed for estimation (so italic numbers correspond to adequate models): mean of estimates, and s. e. of estimates compared to mean of approximate s. e.

					Mean of			Simulated s. e. and mean approx. s. e. for					
c	α_0	α_1	β_1	n	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{\alpha}_0$		$\hat{\alpha}_1$		$\hat{\beta}_1$	
2	6	0.25	-0.45	100	6.002	0.237	-0.394	1.537	1.416	0.101	0.101	0.289	0.258
				250	6.134	0.236	-0.417	1.181	1.093	0.063	0.063	0.217	0.194
				500	6.212	0.236	-0.433	0.875	0.836	0.044	0.045	0.156	0.147
1	*6*	*0.25*	*-0.45*	*100*	*5.755*	*0.254*	*-0.405*	*1.451*	*1.347*	*0.100*	*0.101*	*0.278*	*0.252*
				250	*5.878*	*0.254*	*-0.429*	*1.097*	*1.013*	*0.063*	*0.063*	*0.206*	*0.184*
				500	*5.930*	*0.251*	*-0.436*	*0.817*	*0.777*	*0.045*	*0.045*	*0.151*	*0.141*
0.5	6	0.25	-0.45	100	5.741	0.253	-0.404	1.444	1.335	0.100	0.101	0.278	0.250
				250	5.870	0.254	-0.430	1.085	1.006	0.063	0.063	0.203	0.183
				500	5.918	0.253	-0.439	0.811	0.771	0.045	0.045	0.150	0.140
2	4	-0.25	0.45	100	4.274	-0.254	0.419	1.488	1.314	0.094	0.098	0.289	0.253
				250	4.138	-0.239	0.430	1.206	1.035	0.061	0.061	0.228	0.191
				500	4.075	-0.235	0.438	0.889	0.809	0.043	0.043	0.166	0.147
1	*4*	*-0.25*	*0.45*	*100*	*4.239*	*-0.272*	*0.425*	*1.419*	*1.271*	*0.094*	*0.098*	*0.278*	*0.249*
				250	*4.126*	*-0.259*	*0.435*	*1.104*	*0.983*	*0.060*	*0.061*	*0.212*	*0.184*
				500	*4.046*	*-0.254*	*0.445*	*0.808*	*0.752*	*0.043*	*0.043*	*0.153*	*0.139*
0.5	4	-0.25	0.45	100	4.260	-0.274	0.422	1.443	1.262	0.094	0.097	0.283	0.247
				250	4.154	-0.261	0.430	1.112	0.979	0.061	0.061	0.215	0.183
				500	4.079	-0.258	0.441	0.794	0.747	0.043	0.043	0.150	0.138
2	4	0.45	-0.25	100	4.495	0.404	-0.262	1.337	1.263	0.104	0.105	0.249	0.232
				250	4.457	0.412	-0.261	0.876	0.847	0.066	0.065	0.161	0.154
				500	4.412	0.416	-0.257	0.602	0.595	0.046	0.046	0.111	0.109
1	*4*	*0.45*	*-0.25*	*100*	*4.165*	*0.433*	*-0.268*	*1.191*	*1.171*	*0.102*	*0.105*	*0.226*	*0.225*
				250	*4.086*	*0.444*	*-0.263*	*0.769*	*0.756*	*0.065*	*0.066*	*0.148*	*0.144*
				500	*4.052*	*0.447*	*-0.258*	*0.532*	*0.529*	*0.046*	*0.046*	*0.102*	*0.101*
0.5	4	0.45	-0.25	100	4.122	0.437	-0.266	1.178	1.168	0.102	0.106	0.226	0.225
				250	4.027	0.448	-0.258	0.755	0.746	0.065	0.066	0.146	0.143
				500	4.010	0.452	-0.257	0.527	0.522	0.046	0.046	0.103	0.100
2	6	-0.45	0.25	100	5.818	-0.414	0.276	1.326	1.227	0.096	0.098	0.257	0.238
				250	5.837	-0.410	0.268	0.864	0.825	0.060	0.060	0.165	0.157
				500	5.878	-0.409	0.259	0.601	0.582	0.042	0.042	0.117	0.111
1	*6*	*-0.45*	*0.25*	*100*	*5.926*	*-0.453*	*0.267*	*1.186*	*1.150*	*0.094*	*0.098*	*0.231*	*0.229*
				250	*5.957*	*-0.453*	*0.261*	*0.767*	*0.743*	*0.060*	*0.060*	*0.152*	*0.146*
				500	*5.977*	*-0.452*	*0.256*	*0.530*	*0.522*	*0.042*	*0.042*	*0.105*	*0.102*
0.5	6	-0.45	0.25	100	5.947	-0.462	0.271	1.180	1.146	0.096	0.098	0.229	0.229
				250	5.982	-0.462	0.264	0.749	0.732	0.062	0.061	0.147	0.144
				500	6.005	-0.461	0.258	0.526	0.513	0.044	0.042	0.104	0.101

S3. Real-Data Examples

In what follows, we provide additional results for the real-data examples discussed in Section 5 of the main manuscript.

S3.1 Acceptance Envelopes for PACFs

The data examples discussed in Sections 5.1 (strike count data) and 5.2 (chemical process data) of the main manuscript exhibit an AR(1)-like sample PACF, that is, only the lag-1 value is significantly different from zero. Accordingly, it turned out that these data are well described by a softplus-INARCH(1) model. However, because this type of model has an only approximately linear conditional mean, one may ask if the fitted softplus-INARCH(1) models are really able to mimic this AR(1)-like PACF behavior (recall the discussion in Section 3.2). For this reason, we computed acceptance envelopes in the sense of Tsay (1992) for the sample PACF of both data examples. That is, we did a parametric bootstrap based on the fitted softplus-INARCH(1) models (with 1,000 replications of lengths $n = 108$ and $n = 70$, respectively, using a pre-run of 250 for burn-in) and computed the sample PACF for each replication. At each lag k, we took the 2.5%- and 97.5%-quantile from the bootstrapped PACF values for defining the respective acceptance envelop. These envelops are shown as vertical lines in Figure S1. It can be seen that the actual sample PACF values are contained in their envelopes without exception, confirming the adequacy of the fitted softplus-INARCH(1) models.

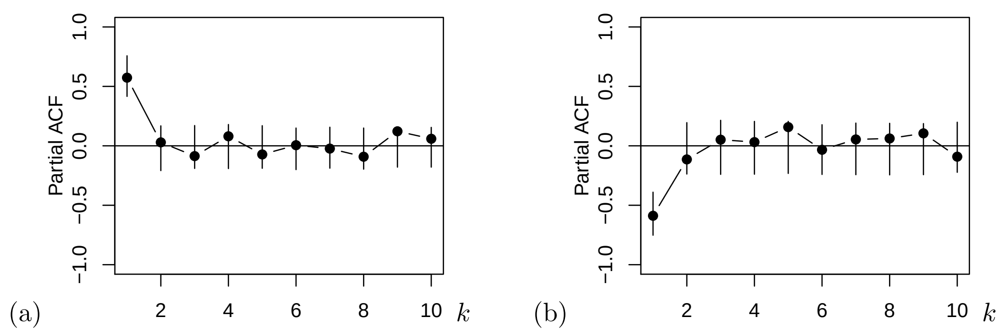

Figure S1: Sample PACF $\hat{\rho}_{\mathrm{p}}(k)$ (dots) and corresponding acceptance envelopes (vertical lines) for of (a) strike count data and (b) chemical process data, computed based on the respective fitted softplus-INARCH(1) model.

S3.2 Further Analysis Results for Crash Count Data

In what follows, we present additional results from our analysis of the crash count data. In the works by Zhu & Wang (2015); Zhu et al. (2015), several types of (log-linear) INGARCH models were fitted to the data, all having a conditional Poisson distribution. Therefore, it is reasonable to compare these models with their softplus Poisson INGARCH counterparts.

Following Zhu & Wang (2015), we first carried out ML estimation for purely INGARCH(1, 1)-type models, see the summary in Table S9. Note

Table S9: ML estimation for crash count data: estimates and approximate standard errors, maximized log-likelihood.

Model	c	$\hat{\alpha}_0$ or $\hat{a}_0$	s. e.	$\hat{\alpha}_1$ or $\hat{a}_1$	s. e.	$\hat{\beta}_1$ or $\hat{b}_1$	s. e.	max. L
softplus INGARCH(1, 1)	1	15.535	(1.782)	0.249	(0.041)	-0.462	(0.126)	-1090.7
log-linear INGARCH(1, 1)	—	2.771	(0.382)	0.262	(0.041)	-0.352	(0.146)	-1088.8
INGARCH(1, 1)	—	9.748	(—)	0.238	(—)	0.001	(—)	-1092.4

that for the ordinary INGARCH(1, 1) model, approximate standard errors cannot be computed because of the estimate $\hat{b}_1$ being on the bound of the bounding box. In contrast, the softplus INGARCH(1, 1) model has a significantly negative estimate for β_1. Because of the large mean, further decreasing the value of c below the default choice $c = 1$ is without effect on the estimates. However, neither the log-linear nor the softplus Poisson INGARCH(1, 1) models are appropriate for the data, because none of them is able to capture the large extent of overdispersion (dispersion ratio $\approx$ 1.712). This can be seen if analyzing the standardized Pearson residuals for these models, because their sample variance takes the value ≈ 1.614 (log-linear) and ≈ 1.623 (softplus), respectively, both being much larger than one. Therefore, in the main manuscript, we concentrate on the analyses of the log-linear and softplus INGARCH(1, 1) models with a conditional negative binomial distribution.

Instead of modeling the slowly decaying autocorrelation structure by in-

Table S10: ML estimation for crash count data, with daily temperature and weekdays indicator as covariates: estimates and approximate standard errors, maximized log-likelihood.

Model	c	$\hat{\alpha}_0$ or $\hat{a}_0$	s. e.	$\hat{\alpha}_1$ or $\hat{a}_1$	s. e.	$\hat{\gamma}_1$	s. e.	$\hat{\gamma}_2$	s. e.	max. L
softplus INARCH(1)	1	8.159	(0.618)	0.178	(0.042)	-1.202	(0.194)	3.343	(0.389)	-1044.9
log-linear INARCH(1)	—	1.908	(0.107)	0.168	(0.039)	-0.090	(0.015)	0.277	(0.035)	-1045.9

cluding the feedback term M_{t-1} in the INGARCH-type models, we followed the strategy outlined in Zhu et al. (2015) and used covariates for explaining the actual dependence structure. Therefore, we also fitted the log-linear and softplus INARCH(1) models to the data, where we used the (standardized) daily temperature and an indicator for weekdays as the covariates (we also tried an additional feedback term, but the corresponding estimate was not significant). The corresponding linear coefficients are denoted as γ_1 and γ_2, respectively. The results are summarized in Table S10. It can be seen that compared to Table S9, the maximized log-likelihood values have improved considerably, with an advantage for the softplus INARCH(1) model. However, neither model is able to fully capture the observed dispersion structure. The standardized Pearson residuals again have a sample variance being much larger than one, namely ≈ 1.376 (log-linear) and ≈ 1.373 (softplus), respectively. Therefore, in the main manuscript, we present the results for the same model structure but having a conditional negative binomial distribution.

S4. Proofs

Proof of Theorem 1. If we can verify conditions (A1)–(A3) in Doukhan & Neumann (2019), then Theorem 1 follows in terms of Corollary 2.1, Theorems 2.1 and 2.2 in Doukhan & Neumann (2019).

The softplus function $s_c(x)$ is strictly monotone increasing with $s_c(0) = c\ln 2$ and $s_c'(x) < 1$ (also see Figure 1(b)). It can thus be bounded from above by $s_c(x) \le c\ln 2 + \max\{0, x\}$. Actually, we have $\mathrm{ReLU}(x) < s_c(x) \le c\ln 2 + \mathrm{ReLU}(x)$. Together with the model recursion (3.7), and defining the constants $\bar{a}_0 = c\ln 2 + \alpha_0$, $\bar{a}_i = \max\{0, \alpha_i\}$ and $\bar{b}_j = \max\{0, \beta_j\}$, the drift condition (A1′) in Doukhan & Neumann (2019) is satisfied under the assumption that $\sum_{i=1}^{p} \max\{0, \alpha_i\} + \sum_{j=1}^{q} \max\{0, \beta_j\} < 1$. Then (A1) follows using Remark 2.1 in Doukhan & Neumann (2019).

Note that $s_c(x_2) - s_c(x_1) = \dfrac{\exp(\xi/c)}{1 + \exp(\xi/c)}(x_2 - x_1)$, where ξ is between x_1 and x_2. Thus, $s_c(x)$ is Lipschitz continuous with Lipschitz constant 1. For all $y_1, \dots, y_p \in \mathbb{R}$, $z_1, \dots, z_q$, $z_1', \dots, z_q' \ge 0$, we have

$$\Big| s_c\big(\alpha_0 + \textstyle\sum_{i=1}^{p} \alpha_i y_i + \sum_{j=1}^{q} \beta_j z_j\big) - s_c\big(\alpha_0 + \sum_{i=1}^{p} \alpha_i y_i \ + \sum_{j=1}^{q} \beta_j z_j'\big) \Big|$$

$$\le \ \Big| \textstyle\sum_{j=1}^{q} \beta_j z_j \ - \ \sum_{j=1}^{q} \beta_j z_j' \Big| \ \le \ \sum_{j=1}^{q} |\beta_j| \, |z_j - z_j'|.$$

Thus, (A2) holds. From Remark 2.3 in Doukhan & Neumann (2019), we know that (A3) holds for the Poisson distribution, so the proof is complete.

Proof of Theorem 2. Because $X_t|\mathcal{F}_{t-1}$ is a Poisson variable with mean M_t, the mth moment is $E(X_t^m|\mathcal{F}_{t-1}) = \sum_{i=0}^{m} \left\{ {m \atop i} \right\} M_t^i$, where $\left\{ {m \atop i} \right\}$ are the Stirling numbers of the second kind, see p. 933 in Ferland et al. (2006). Let $\lambda_t = \alpha_0 + \alpha_1\, X_{t-1} + \beta_1\, M_{t-1}$ and note that $M_t \leq c\, \ln 2 + \lambda_t\, \mathbb{1}_{\{\lambda_t>0\}}$. Then,

$$\begin{aligned} M_t^i &\leq (c\ln 2 + \lambda_t \mathbb{1}_{\{\lambda_t>0\}})^i \\ &\leq \sum_{l=0}^{i} \binom{i}{l} (c\ln 2 + |\alpha_0|)^{i-l} \sum_{j=0}^{l} \binom{l}{j} |\alpha_1|^j |\beta_1|^{l-j} X_{t-1}^j M_{t-1}^{l-j}. \end{aligned}$$

Note that $E\big(X_{t-1}^j M_{t-1}^{l-j}|\mathcal{F}_{t-2}\big) = \sum_{k=0}^{j} \left\{ {j \atop k} \right\} M_{t-1}^{k+l-j}$, so

$$E(M_t^i|\mathcal{F}_{t-2}) \leq \sum_{l=0}^{i} \binom{i}{l} (c\ln 2 + |\alpha_0|)^{i-l} \sum_{j=0}^{l} \sum_{k=0}^{j} \left\{ {j \atop k} \right\} \binom{l}{j} |\alpha_1|^j |\beta_1|^{l-j} M_{t-1}^{k+l-j},$$

which corresponds to equation (12) in Ferland et al. (2006) except that the equal sign therein is replaced by "$\leq$". Using arguments similar to those in Proposition 6 (p. 934) in Ferland et al. (2006), we know that the theorem holds.

Proof of Theorem 3. We prove this theorem by verifying the conditions in Theorems 2.1 and 2.2 in Ahmad & Francq (2016). First, all moments exist according to Theorem 2, so there is no moment problem. Next, we show that the choice of the initial value is asymptotically negligible. Let $\lambda_t = c_{t-1} + \beta_1 M_{t-1}$, $\widetilde{\lambda}_t = \widetilde{c}_{t-1} + \beta_1 \widetilde{M}_{t-1}$, and $a_t = \sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} |M_t - \widetilde{M}_t|$. Using the fact that $s_c(x)$ is Lipschitz continuous with Lipschitz constant 1, we have

$$\begin{aligned}
|M_t - \widetilde{M}_t| \leq |\lambda_t - \widetilde{\lambda}_t| &\leq |c_{t-1} - \widetilde{c}_{t-1}| + |\beta_1|\,|M_{t-1} - \widetilde{M}_{t-1}| \\
&\leq |c_{t-1} - \widetilde{c}_{t-1}| + |\beta_1|\,|c_{t-2} - \widetilde{c}_{t-2}| + \beta_1^2\,|M_{t-2} - \widetilde{M}_{t-2}| \\
&\leq \ \ldots \ \leq \sum_{k=0}^{t-1} |\beta_1|^k\,|c_{t-1-k} - \widetilde{c}_{t-1-k}| \ + \ |\beta_1|^t\,|M_0 - \widetilde{M}_0|.
\end{aligned}$$

Using arguments similar to those in Francq et al. (2011), it then follows that almost surely, $a_t \leq K\rho^t, \forall t$, where $K > 0$ and $0 < \rho < 1$ are generic constants whose values can vary from line to line, then $\lim_{t\to\infty} a_t = 0$. Using the Borel-Cantelli lemma and the inequality $P(\rho^t X_t \geq \epsilon) \leq \rho^t\, E(X_t)/\epsilon$, for $\epsilon > 0$, we know that $\lim_{t\to\infty} X_t\, a_t = 0$.

Third, we show that the model is identifiable. Consider the following identification problem: $M_t(\boldsymbol{\theta}) = M_t(\boldsymbol{\theta}^0)$ almost surely if and only if $\boldsymbol{\theta} = \boldsymbol{\theta}^0$. Note that $M_t(\boldsymbol{\theta})$ can be expressed as $g(\boldsymbol{\theta}, X_{t-1}, X_{t-2}, \ldots) = s_c(\alpha_0 + \alpha_1 X_{t-1} + \beta_1 s_c(\alpha_0 + \alpha_1 X_{t-2} + \beta_1 s_c(\alpha_0 + \alpha_1 X_{t-3} + \beta_1 s_c(\cdots))))$ for some function g, then the claim holds because of the randomness and the definition of g. In fact, this conclusion can also be proved in a different way. Since $s_c(x)$ is strictly monotone increasing, if $M_t(\boldsymbol{\theta}) = M_t(\boldsymbol{\theta}^0)$, then $\lambda_t(\boldsymbol{\theta}) = \lambda_t(\boldsymbol{\theta}^0)$. Note that the conditional distribution of X_t is not degenerate and that α_{10} and β_{10} are not zero simultaneously. So the identifiability holds in terms of Remark 2.1 in Ahmad & Francq (2016).

Now all the conditions of Theorem 2.1 in Ahmad & Francq (2016) are verified, thus the strong consistency holds.

Following (1.1), let

$$a_t = \frac{\exp(\lambda_t(\boldsymbol{\theta})/c)}{1+\exp(\lambda_t(\boldsymbol{\theta})/c)}, \quad b_t = \frac{\exp(\lambda_t(\boldsymbol{\theta})/c)}{c(1+\exp(\lambda_t(\boldsymbol{\theta})/c))^2}.$$

The (i,j)th element of $\dfrac{\partial^2 M_t(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}^\top}$ is

$$\begin{aligned}
\frac{\partial^2 M_t(\boldsymbol{\theta})}{\partial\alpha_0^2} &= a_t\beta_1\frac{\partial^2 M_{t-1}(\boldsymbol{\theta})}{\partial\alpha_0^2} + b_t\left(1+\beta_1\frac{\partial M_{t-1}(\boldsymbol{\theta})}{\partial\alpha_0}\right)^2,\\
\frac{\partial^2 M_t(\boldsymbol{\theta})}{\partial\alpha_0\partial\alpha_1} &= a_t\beta_1\frac{\partial^2 M_{t-1}(\boldsymbol{\theta})}{\partial\alpha_0\partial\alpha_1} + b_t\left(1+\beta_1\frac{\partial M_{t-1}(\boldsymbol{\theta})}{\partial\alpha_0}\right)\left(X_{t-1}+\beta_1\frac{\partial M_{t-1}(\boldsymbol{\theta})}{\partial\alpha_1}\right),\\
\frac{\partial^2 M_t(\boldsymbol{\theta})}{\partial\alpha_0\partial\beta_1} &= a_t\left(\frac{M_{t-1}(\boldsymbol{\theta})}{\partial\alpha_0}+\beta_1\frac{\partial^2 M_{t-1}(\boldsymbol{\theta})}{\partial\alpha_0\partial\beta_1}\right)\\
&\quad + b_t\left(1+\beta_1\frac{\partial M_{t-1}(\boldsymbol{\theta})}{\partial\alpha_0}\right)\left(M_{t-1}(\boldsymbol{\theta})+\beta_1\frac{\partial M_{t-1}(\boldsymbol{\theta})}{\partial\beta_1}\right),\\
\frac{\partial^2 M_t(\boldsymbol{\theta})}{\partial\alpha_1^2} &= a_t\beta_1\frac{\partial^2 M_{t-1}(\boldsymbol{\theta})}{\partial\alpha_1^2} + b_t\left(X_{t-1}+\beta_1\frac{\partial M_{t-1}(\boldsymbol{\theta})}{\partial\alpha_1}\right)^2,\\
\frac{\partial^2 M_t(\boldsymbol{\theta})}{\partial\alpha_1\partial\beta_1} &= a_t\left(\frac{\partial M_{t-1}(\boldsymbol{\theta})}{\partial\alpha_1}+\beta_1\frac{\partial^2 M_{t-1}(\boldsymbol{\theta})}{\partial\alpha_1\partial\beta_1}\right)\\
&\quad + b_t\left(X_{t-1}+\beta_1\frac{\partial M_{t-1}(\boldsymbol{\theta})}{\partial\alpha_1}\right)\left(M_{t-1}(\boldsymbol{\theta})+\beta_1\frac{\partial M_{t-1}(\boldsymbol{\theta})}{\partial\beta_1}\right),
\end{aligned}$$

$$\frac{\partial^2 M_t(\boldsymbol{\theta})}{\partial \beta_1^2} = a_t \left(2\frac{\partial M_{t-1}(\boldsymbol{\theta})}{\partial \beta_1} + \beta_1 \frac{\partial^2 M_{t-1}(\boldsymbol{\theta})}{\partial \beta_1^2}\right) + b_t \left(M_{t-1}(\boldsymbol{\theta}) + \beta_1 \frac{\partial M_{t-1}(\boldsymbol{\theta})}{\partial \beta_1}\right)^2.$$

Note that $|a_t| \leq 1, |b_t| \leq 1/c$, then

$$\left|\frac{\partial M_t(\boldsymbol{\theta})}{\partial \alpha_0}\right| \leq 1 + |\beta_1| \left|\frac{\partial M_{t-1}(\boldsymbol{\theta})}{\partial \alpha_0}\right|,$$

$$\left|\frac{\partial^2 M_t(\boldsymbol{\theta})}{\partial \alpha_0 \partial \alpha_1}\right| \leq |\beta_1| \left|\frac{\partial^2 M_{t-1}(\boldsymbol{\theta})}{\partial \alpha_0 \partial \alpha_1}\right| + \frac{1}{c}\left(1 + |\beta_1| \left|\frac{\partial M_{t-1}(\boldsymbol{\theta})}{\partial \alpha_0}\right|\right)\left(X_{t-1} + |\beta_1| \left|\frac{\partial M_{t-1}(\boldsymbol{\theta})}{\partial \alpha_1}\right|\right),$$

and similar results for other derivatives hold. By the fact that all moments exist, the compactness of $\boldsymbol{\Theta}$, and the condition $\sup_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} |\beta_1| < 1$, using arguments similar to those in Fokianos et al. (2009), we know that the variables $M_t(\boldsymbol{\theta})$, $\frac{\partial M_t(\boldsymbol{\theta})}{\partial \theta_i}$ and $\frac{\partial^2 M_t(\boldsymbol{\theta})}{\partial \theta_i \partial \theta_j}$ admit moments of any order uniformly in $\boldsymbol{\theta} \in \boldsymbol{\Theta}$. Using arguments similar to those in Remark 2.3 in Ahmad & Francq (2016), we then know that all the conditions in Theorem 2.2 in Ahmad & Francq (2016) are verified, thus the asymptotic normality holds. Thus, the proof is complete.

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