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SELECTING THE LAST RECORD WITH RECALL IN A SEQUENCE OF INDEPENDENT BERNOULLI TRIALS

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Abstract: Given a sequence of N independent trials, we want to find an optimal strategy which maximizes the probability of selecting the last record with recall of length m ($m > 1$). We use the idea of pattern discussed in Bruss and Louchard (2003) (with some modification) to prove that the optimal strategy is to stop at the first appearance of the pattern (if any) after a threshold.

Key words and phrases: Optimal stopping, secretary problem with recall, last record, pattern.

1. Introduction

In the standard secretary problem, N rankable people apply for one secretary position and are interviewed sequentially in random order by a manager. It is assumed that, at each stage, the manager can rank the applicants that have so far been interviewed and he must decide immediately whether to accept or to reject the present applicant. No recall for previous applicants is permitted and the interview continues until one of the applicants is accepted. If the last applicant is presented he/she must be accepted. The strategy which maximizes the probability of selecting the best of the N applicants is of the form: reject the first $r-1$ applicants and then accept the next one who is preferable to all his/her predecessors, where

$$r = \min\{n \mid \frac{1}{n} + \cdots + \frac{1}{N-1} \leq 1\}.$$

Under this strategy, the probability of selecting the best applicant is

$$\frac{r-1}{N} \sum_{k=r}^{N} \frac{1}{k-1},$$

which tends to e^{-1} as $N \to \infty$.

Note that in the above problem, an applicant should be accepted only if relatively best among those already interviewed. An applicant relatively best so far is called a *record*, so the jth applicant is a record if and only if its relative rank is 1. For $j = 1, \ldots, N$, let

$$X_j = \begin{cases} 1 & \text{if the } j\text{th applicant is a record,} \\ 0 & \text{if the } j\text{th applicant is not a record.} \end{cases} \tag{1.1}$$

It is well known that $X_1, \ldots, X_N$ are independent random variables with $E(X_k) = 1/k$. Hence the standard secretary problem can be viewed as follows: We observe $X_1, \ldots, X_N$ sequentially and may select any one of these, but not one of the preceding X_k's. The goal is to find an optimal stopping rule which maximizes the probability of selecting the last record.

The problem of selecting the last record without recall in a sequence of independent trials was considered by Bruss (2000). Let $I_1, \ldots, I_N$ be independent Bernoulli trials with $p_j = E(I_j)$ and $r_j = p_j/(1 - p_j)$. Bruss (2000) proved that an optimal rule τ_N which maximizes the probability of selecting the last record is to select the first index (if any) k with $I_k = 1$ and $k \geq s$, where

$$s = \sup\{1, \sup\{1 \leq k \leq N \mid \sum_{j=k}^{N} r_j \geq 1\}\},$$

with $\sup \phi := -\infty$. Bruss and Paindaveine (2000) generalized this result to considering selecting a sequence of last records in independent trials.

Smith and Deely (1975) considered a version of recall to the standard secretary problem. It is assumed that at each stage the manager can select any one of the last m applicants interviewed so far, one might say the manager has "finite memory" of size m. However, only one applicant is to be selected and, once out of memory, the applicant is no longer available. The process must stop when the manager selects an applicant. The manager clearly needs to consider selecting only when the best applicant so far is about to become unavailable. They showed that the form of the strategy which maximizes the probability of selecting the best applicant is: stop at the first stage, from r^* onwards, when the relatively best applicant is about to become unavailable. They gave an algorithm for finding r^* and the maximum probability of selecting the best applicant. If $m/N = \alpha \geq 1/2$ then $r^* = m$ and the probability of selecting the best applicant using the optimal stopping rule is

$$2 - \frac{m}{N} - \sum_{j=m}^{N-1} \frac{1}{j}, \tag{1.2}$$

which tends to $2 - \alpha + \ln \alpha$ as $N \to \infty$. If m is fixed then r^*/N and the probability of selecting the best applicant using the optimal strategy both tend to e^{-1}, giving no asymptotic advantage over the standard no-recall secretary problem. If we define X_j as in (1.1), then this problem can be viewed as a problem of selecting the last record with recall of length m. Note that $X_1, \ldots, X_N$ are independent Bernoulli random variables with $E(X_j) = 1/j$ in this problem. We study the problem for general Bernoulli trials by using a technique different from Simth and Deely (1975).

There are some other versions of the secretary problem with recall, for example, Yang (1974) considered the problem in which the manager can make an offer to any applicant already interviewed. If at stage k the manager makes an offer to applicant $k-r$, that applicant will accept the offer with a known probability $q(r)$. Yang assumes that rejections are final, that the $q(r)$ are non-increasing, and that $q(0) = 1$, which means that an applicant is sure to accept an offer immediately after his interview.

Petruccelli (1981) combined Yang's idea of recall and Smith's idea of uncertain employment by using the same approach as Yang without requiring $q(0) = 1$. In Yang's and Petrucceli's model, the manager is allowed to continue the process if an offer is rejected. Sweet (1994) considered the problem of a single offer with both the possibility of a recall of applicants and the possibility of a rejection of the offer. In Sweet's model, the process ends after any offer is made.

In this paper, we consider the general problem of selecting the last record, with recall, in a sequence of independent Bernoulli trials. Let $m \geq 1$ be fixed. Let $I_1, \ldots, I_N$ be a sequence of independent Bernoulli random variables with $E(I_j) = p_j < 1$. We say that I_k is a record if $I_k = 1$. We observe $I_1, \ldots, I_N$ sequentially and may select any one of these. It is assumed that at each stage we can select any one of the last m objects observed so far. However, only one object is to be chosen and, once out of the length of recall, the object is no longer available. We want to find an optimal stopping rule that maximizes the probability of selecting the last record.

It is clear that we will stop at either stage k, with $I_{k-m+1} = 1$ and $I_{k-m+2} = \cdots = I_k = 0$, or at stage N. Note that if we decide to stop at stage k before N, then the string $I_{k-m+1}I_{k-m+2}\cdots I_k$ forms the pattern $\underbrace{100\cdots0}_{m}$. Hence we can use the idea of patterns discussed in Bruss and Louchard (2003) to find the optimal stopping rule.

2. Form of the Optimal Stopping Rule

Because we are allowed to recall the previous m items, it is natural to consider the occurrence of the pattern $H = \underbrace{100\cdots0}_{m}$ in the whole sequence, or the pattern $H_l = \underbrace{100\cdots0}_{l}$ for $l \leq m$ in the string $I_k I_{k+1} \cdots I_N$ with $k \geq N-m+1$. We say that H occurs at stage $k < N$ if $I_{k-m+1}I_{k-m+2}\cdots I_k = H$, and at stage N if $I_{N-l+1}I_{N-l+2}\cdots I_N = H_l$ for some $1 \le l \le m$. We observe $I_1,\ldots,I_N$ sequentially. Whenever the pattern H occurs, we are allowed either to stop or to continue. We want to find the strategy which maximizes the probability of stopping on the last occurrence of the pattern H within a string of fixed length N.

Let O_n be the number of occurrences of pattern H in the string $I_{N-n+1}I_{N-n+2}\cdots I_N$. To find the optimal stopping rule, we need the following lemma.

Lemma 1. *If $P(O_{n+1}=0) > P(O_{n+1}=1)$, then $P(O_n=0) > P(O_n=1)$.*

Proof. By the definitions of H and O_n, since the random variables $I_1, I_2, \cdots$ are independent, we have

$$\begin{aligned} P(O_{n+1}=0) &= P(O_{n+1}=0, I_{N-n}=0) = (1-p_{N-n})P(O_n=0) \\ &> P(O_{n+1}=1) \qquad \text{by assumption} \\ &\ge P(I_{N-n}=0, O_{n+1}=1) = (1-p_{N-n})P(O_n=1), \end{aligned}$$

and thus $(1-p_{N-n})P(O_n=0) > (1-p_{N-n})P(O_n=1)$, which implies that $P(O_n=0) > P(O_n=1)$.

Let B be the set of states for which the decision to stop is at least as good as to continue for exactly one more period and then stop. The policy that stops at the first time the process enters a state in B is called the one-stage look-ahead stopping policy.

Theorem 2.1.(Ross (1983)) *If B is a closed set of states, the one-stage look-ahead stopping policy is optimal. Here B is a closed set means that once the process enters B, it never leaves.*

We are now ready to describe the optimal strategy.

Theorem 2.2. *Let*

$$r = \min\{\ n \mid P(O_{n+1}=0) < P(O_{n+1}=1),\ 0 \le n \le N-m-1\ \}$$

with the convention that $\min\emptyset = N-m$. *The strategy that maximizes the probability of stopping on the last pattern H in $I_1,\ldots,I_N$ stops at the first pattern H (if any) within the string $I_{N-r-m+1}\cdots I_N$. Then the probability of selecting the last record is $V = P(O_{r+m}=1)$.*

Proof. We first note that the strategy in question success if and only if there is exactly one pattern H in the string $I_{N-r-m+1}\cdots I_N$.

Let $\mathcal{F}_k$ be the σ-field generated by $\{I_1,\ldots,I_k\}$ and let C be the set of all stopping rules τ such that $\tau \le N$, adapted to $\{\mathcal{F}_k\}_{k=1}^N$. To find the stopping rule which maximizes the probability of stopping on the last pattern H in $I_1, \ldots, I_N$, we need to define a suitable reward function. For each $k = 1, \ldots, N$, let $X_k = E(L_k|\mathcal{F}_k)$, where L_k is 1 if the last time of completion of pattern H occurs at stage k, and is 0 otherwise. Here we note that X_k is $\mathcal{F}_k$-measurable and the problem is equivalent to solving the stopping problem $\{X_k, \mathcal{F}_k\}_{k=1,\ldots,N}$, that is, to find a stopping rule $\tau_N \in C$ such that $E(X_{\tau_N}) = \sup_{\tau \in C} E(X_\tau)$.

Note that we may stop at stage k only if $I_{k-m+1} \cdots I_k = H$ or $k = N$. Therefore, the well-known one-stage look-ahead stopping policy can be described as follows: if we are at stage N, then we stop; if we are at stage $k \leq N-1$ and $I_{k-m+1} \cdots I_k = H$, then we stop if and only if the conditional expected reward of stopping at the next j with $I_{j-m+1} \cdots I_j = H$ or $j = N$ is not greater than the present reward $P(O_{N-k} = 0)$.

More precisely, let v_n be the optimal value (success probability) when we have just observed the pattern H, but still have a string of length n, that is, $I_{N-n-m+1} I_{N-n-m+2} \cdots I_{N-n} = H$. Note that $n \leq N - m$. If we stop immediately, the reward is the probability that no further pattern appears in the future, i.e., $P(O_n = 0)$. If we continue observing, then the optimal value is $\sum_{j=1}^{n} P_{n,j} v_{n-j}$, where $P_{n,j}$ denotes the probability that the first pattern H within the last n observations is completed at time $N - (n - j)$. Thus the principle of optimality yields

$$v_n = \max\{P(O_n = 0), \sum_{j=1}^{n} P_{n,j} v_{n-j}\}.$$

Let B be the stopping region determined by the one-stage look-ahead policy. Then

$$\begin{aligned} B &= \{\ n \mid P(O_n = 0) \geq \sum_{j=1}^{n} P_{n,j} P(O_{n-j} = 0),\ 1 \leq n \leq N - m\ \} \cup \{0\} \\ &= \{\ n \mid P(O_n = 0) \geq P(O_n = 1),\ 1 \leq n \leq N - m\ \} \cup \{0\}, \end{aligned}$$

with the convention that $P(O_0 = 0) = 1$ and $P(O_0 = 1) = 0$. In the following, we prove that B is the same as $\tilde{B} = \{\ n \mid 0 \leq n \leq r\ \}$, where $r = \min\{\ n \mid P(O_{n+1} = 0) < P(O_{n+1} = 1),\ 0 \leq n \leq N - m - 1\ \}$ as in the statement of the theorem, with the convention that $\min \emptyset = N - m$. If $B = \tilde{B}$, then B is closed in the sense that once the process enters B, it never leaves. Then, by the theory of optimal stopping, the strategy that maximizes the probability of stopping at the last pattern H in $I_1, \ldots, I_N$ is to stop at the first pattern H (if any) within the string $I_{N-r-m+1} I_{N-r-m+2} \cdots I_N$. What remains to be shown is that $B = \tilde{B}$.

If $B = \{0\}$, then $P(O_n = 0) < P(O_n = 1)$ for $n = 1, \ldots, N - m$. This implies that $r = 0$; hence $\tilde{B} = \{0\}$. Next, assume that B contains $\{0\}$ strictly.

Suppose that $0 < n \in B$. We claim that $n-1 \in B$. If $n = 1$, then $n-1 = 0 \in B$; if $n > 1$, then $P(O_n = 0) \geq P(O_n = 1)$ and it follows from Lemma 1 that $P(O_{n-1} = 0) \geq P(O_{n-1} = 1)$. Hence, $n-1 \in B$. It is now clear that $B = \tilde{B}$.

Corollary 2.1. *If $m \geq N/2$, then*

$$r = \min\{\ n \mid \prod_{i=N-n}^{N} (1-p_i) < \frac{1}{2},\ 0 \leq n \leq N-m-1\ \},$$

with the convention that $\min \emptyset = N - m$. *In particular, if $p_i = p$ for each i, then*

$$r = \min\{\lfloor -\log_{(1-p)} 2 \rfloor, N-m\},$$

where $\lfloor x \rfloor$ denotes the greatest integer not greater than x. Moreover, for $p > 1/2$, it is optimal to select the N-th object.

Proof. Suppose that we have just observed the pattern H, but still have a string of length n. Since $m \geq N/2$, we have $n \leq N - m \leq m$ and then

$$P(O_n = 1) = 1 - \prod_{i=N-n+1}^{N} (1-p_i).$$

Hence

$$\begin{aligned} r &= \min\{\ n \mid P(O_{n+1} = 0) < P(O_{n+1} = 1),\ 0 \leq n \leq N-m-1\ \} \\ &= \min\{\ n \mid \prod_{i=N-n}^{N} (1-p_i) < 1 - \prod_{i=N-n}^{N} (1-p_i),\ 0 \leq n \leq N-m-1\ \} \\ &= \min\{\ n \mid \prod_{i=N-n}^{N} (1-p_i) < \frac{1}{2},\ 0 \leq n \leq N-m-1\ \} \end{aligned} \tag{2.1}$$

with the convention that $\min \emptyset = N - m$.

If $p_i = p$ for each i, then by (2.1),

$$\begin{aligned} r &= \min\{\ n \mid P(O_{n+1} = 0) < P(O_{n+1} = 1),\ 0 \leq n \leq N-m-1\ \} \\ &= \min\{\ n \mid (1-p)^{n+1} < \frac{1}{2},\ 0 \leq n \leq N-m-1\ \} \\ &= \min\{\ n \mid n+1 > -\log_{(1-p)} 2,\ 0 \leq n \leq N-m-1\ \} \end{aligned}$$

with the convention that $\min \emptyset = N - m$. Hence

$$r = \min\{\lfloor -\log_{(1-p)} 2 \rfloor, N-m\}.$$

Moreover, if $p > 1/2$, then $1 - p < 1/2$ and hence $r = 0$. Therefore, it is optimal to select the Nth object for $p > 1/2$.

References

Bruss, F. T. (2000). Sum the odds to one and stop. *Ann. Probab.* **28**, 1384-1391.

Bruss, F. T. and Paindaveine, D. (2000). Selecting a sequence of last successes in independent trials. *J. Appl. Probab.* **37**, 389-399.

Bruss, F. T. and Louchard, G. (2003). Optimal stopping on patterns in strings generated by independent random variables. *J. Appl. Probab.* **40**, 49-72.

Petruccelli, J. D. (1981). Best-choice problems involving uncertainty of selection and recall of observation. *J. Appl. Probab.* **18**, 415-425.

Ross, S. (1983). *Introduction to Stochastic Dynamic Programming.* Academic Press, Inc., New York.

Smith, M. H. and Deely, J. J. (1975). A secretary problem with finite memory. *J. Am. Statist. Assoc.* **70**, 357-361.

Sweet, T. (1994). Optimizing a single uncertain selection with recall of observations. *J. Appl. Probab.* **31**, 660-672.

Yang, M. C. K. (1974). Recognizing the maximum of a random sequence based on relative rank with backward solicitation. *J. Appl. Probab.* **11**, 504-512.

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