Statistica Sinica **7**(1997), 577-593

EXISTENCE AND STABILITY OF WEAK SOLUTIONS TO STOCHASTIC DIFFERENTIAL EQUATIONS WITH NON-SMOOTH COEFFICIENTS

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Abstract: Weak solutions to stochastic differential equations in $\mathbb{R}^d, d \geq 2$, are continuous-time Markov processes. We show that under very general conditions such solutions possess irreducibility and continuity properties which enable criteria for Harris recurrence and transience, developed in Meyn and Tweedie (1993b), Down, Meyn and Tweedie (1995) and Stramer and Tweedie (1994), to be applied to them. All of our criteria are in terms of the second-order differential operator, and hence a unified approach to the stability classification of weak solutions is obtained which generalises that of Khas'minskii (1980); we also develop explicit forms of the stationary measure for many such processes. The results are applicable in continuous-time time series analysis (see Stramer, Tweedie and Brockwell (1996) and Stramer, Brockwell and Tweedie (1996)) and we consider a multi-dimensional threshold model as one such application.

Key words and phrases: Degenerate diffusions, exponential ergodicity, irreducible Markov processes, multi-dimensional non-linear diffusions, recurrence, stationary measures, stochastic differential equations, transience.

1. Introduction

The purpose of this paper is to investigate the stability and structure of multi-dimensional nonlinear diffusion models, in circumstances where the model coefficients fail to satisfy the smoothness conditions previously assumed in the literature (see Khas'minskii (1980), Bhattacharya (1978), Kliemann (1987) and Basak and Bhattacharya (1992)). One motivation for this is the extension of threshold models from discrete time (see Tong (1990)) to continuous time models, and we illustrate our results in Section 5 on one such example.

Our approach is to show that, under suitable but weak conditions, such diffusion models can be analyzed using general state space Markov process theory, as developed in Meyn and Tweedie (1993b). As one consequence of this, we find that conditions for recurrence or transience, as developed by say Khas'minskii (1980), give rather stronger conclusions than previously determined.

We consider processes which are solutions to stochastic differential equations of the form

$$dX(t) = b(X(t))dt + \sigma(X(t))dW(t), \tag{1}$$

or equivalently in coordinate form,

$$dX_i(t) = b_i(X(t))dt + \sum_{j=1}^{r} \sigma_{ij}(X(t))dW_j(t); \quad 1 \leq i \leq d,$$

where $W = (W_1, \ldots, W_r)$ is an r-dimensional Brownian motion ($r \geq 1$) starting from the origin, and $b : \mathbb{R}^d \to \mathbb{R}^d$ (the drift vector) and $a := \sigma\sigma^T : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ (the diffusion matrix) are locally bounded Borel measurable functions. Throughout this paper we shall assume that $d \geq 2$ (for more detailed results on the scalar case $d = 1$ see Mandl (1968)) and that conditions for non-explosion (see Narita (1982), McKean (1969), Khas'minskii (1980) and Stroock and Varadhan (1979)) are satisfied.

We will consider weak solutions to (1) as continuous time-homogeneous Markov processes evolving on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, with transition probability law $P^t(x, A) = P_x(X(t) \in A)$, $x \in \mathbb{R}^d, A \in \mathcal{B}(\mathbb{R}^d)$; here $\mathcal{B}(\mathbb{R}^d)$ denotes the Borel σ-field on $\mathbb{R}^d$. Following Meyn and Tweedie (1993a) and Meyn and Tweedie (1993b), in order to investigate stability of X, for a measurable set $A \subseteq \mathbb{R}^d$ we denote the hitting times and occupation times of X by

$$\tau_A = \inf\{t \geq 0 : X(t) \in A\}, \qquad \eta_A = \int_0^\infty \mathbb{1}\{X(t) \in A\}\, dt.$$

We write

$$L(x, A) = P_x(\tau_A < \infty); \qquad U(x, A) = E_x[\eta_A].$$

Stability properties of Markov processes are typically stronger for irreducible processes with some continuity properties, as discussed in Meyn and Tweedie (1993a). We will be interested in μ^{Leb}-irreducible chains (where μ^{Leb} is Lebesgue measure) defined as:

Definition 1.1. (i) A Markov process X is called μ^{Leb}-irreducible if

$$\mu^{\text{Leb}}(B) > 0 \Longrightarrow \int_0^\infty P^t(x, B)dt = E_x(\eta_B) > 0, \qquad \forall x \in \mathbb{R}^d.$$

(ii) The process X is called Harris recurrent with respect to μ^{Leb} if $L(x, A) \equiv 1$ whenever $\mu^{\text{Leb}}\{A\} > 0$, and transient if there exists a countable cover of $\mathbb{R}^d$ by sets A_j such that $U(x, A_j) \leq M_j < \infty$ for all j.

It is well known (see Meyn and Tweedie (1993a) and Tweedie (1994)) that if X is a time-homogeneous irreducible Markov process on $\mathbb{R}^d$ then we have a dichotomy between recurrence and transience. Moreover (see Azéma, Duflo and Revuz (1967), Getoor (1979) and Meyn and Tweedie (1993a)), if a weak solution X is Harris recurrent then an essentially unique invariant measure π exists. If the invariant measure is *finite*, then it may be normalized to a probability measure;

in this case X is called *positive Harris recurrent*. If the invariant measure is *not finite* then X is called *null Harris recurrent*. For weak solutions that are positive Harris recurrent it is further known (see Meyn and Tweedie (1993a)) that under general conditions $P^t(x,\cdot) \to \pi(\cdot)$, where π is the unique invariant measure, in some appropriate sense; and in particular it may be that such convergence occurs exponentially quickly.

Specifically, our goals here are: (a) to find conditions under which weak solutions to (1) are μ^{Leb}-irreducible with suitable continuity properties (Section 2); (b) within this context to derive criteria for the process to be Harris recurrent or transient (Section 3); (c) and for recurrent chains, to find conditions when π is finite, when convergence to π is exponentially fast, and finally to derive the structure of π under extra conditions (Section 4). We then conclude in Section 5 with an application to multi-dimensional continuous-time threshold AR models.

A number of authors have considered related problems, under various conditions on the drift vector b and the diffusion matrix a. Conditions for recurrence, non-recurrence and infinite mean recurrence time relative to an open set O (i.e conditions for $P_x(\tau_O < \infty) \equiv 1$, $P_x(\tau_O = \infty) > 0$ and $E_x[\tau_O] = \infty$) appear to be first derived in Khas'minskii (1980). There it is shown that under the assumption that b and a are Lipschitzian and a is positive definite, such single set recurrence and non-recurrence gives "open-set" recurrence or non-recurrence for all open sets. It is further shown in Bhattacharya (1978) that the process is either "open-set" recurrent or transient under weaker assumptions on b and a (namely if b is locally bounded and Borel measurable and a is continuous and positive definite). An "open-set" transience-recurrence dichotomy based on invariant control sets is also derived in Kliemann (1987) for degenerate diffusions under the assumption that b and a are C^∞.

We will show that even under more general "non-smooth" assumptions on the drift vector b and the diffusion matrix a, we can derive not only a dichotomy for open sets, but a stronger version, namely a Harris recurrence-transience dichotomy w.r.t. μ^{Leb}.

A similar test function approach for weak (rather than our total variation norm) convergence of the transition probability $P^t(x,\cdot)$ to $\pi(\cdot)$ is derived in Basak and Bhattacharya (1992) for a broad class of diffusions under the assumption that a and b are Lipschitzian. Their models include singular diffusions and highly nonradial nonsingular diffusions, and are not in general irreducible, so the criteria of Khas'minskii (1980) can not be applied or else fail. Our results typically do not apply as given here to this class, since we focus on irreducibility conditions; but we believe some of our results could be used to remove the smoothness conditions even without irreducibility, using, for example, Theorem 4.5 of Meyn and Tweedie (1993b).

2. Irreducibility Properties

Our goal in this section is to show that a broad class of diffusions are μ^{Leb}-irreducible and have continuity properties which make them T-processes. These are defined following Meyn and Tweedie (1993a) by:

Definition 2.1. A process will be called a T-process if for some probability measure λ on $[0,\infty)$ there exists a kernel $T(x,A)$ with $T(x,\mathbb{R}^d)>0\ \forall x\in\mathbb{R}^d$ such that

(i) For $A\in\mathcal{B}(\mathbb{R}^d)$ the function $T(\,\cdot\,,A)$ is lower semi-continuous;
(ii) For all $x\in\mathbb{R}^d$ and $A\in\mathcal{B}(\mathbb{R}^d)$ the measure $T(x,\,\cdot\,)$ satisfies $\int_0^\infty P^t(x,A)\lambda(dt)\ge T(x,A)$.

We will often use the next Proposition, which follows from Theorem 6.4 of Tweedie (1994), and which gives an intuitive feeling for T-processes:

Proposition 2.2. *If $X(t)$ is μ^{Leb}-irreducible and has the weak Feller property (i.e. $E_x(f(X(t)))$ is continuous in x when f is bounded and continuous), then $X(t)$ is a T-process.*

We first find sufficient conditions for weak solutions of (1) to be μ^{Leb}-irreducible T-processes in the case when the diffusion matrix is not necessarily continuous provided the noise can drive the process to a sufficiently large set of states. If $x=[\ x_1\ \cdots\ x_d\ ]$ and $y=[\ y_1\ \cdots\ y_d\ ]$ then we use the inner product notation $(x,y)=\sum_{i=1}^d x_i y_i$, and $||x||^2=(x,x)$.

Theorem 2.3. *Suppose that $\mathbb{R}^d$ can be divided up into finitely many polyhedra $O_1,\ldots,O_k$ such that $\mathbb{R}^d=\cup_{i=1}^k\overline{O_i}$ and the O_i have pairwise disjoint interiors. Assume also that the diffusion matrix $a(x)$ satisfies a Lipschitz condition in the interior of each O_i and that for some $0<k_1<k_2$ and all $\theta\in\mathbb{R}^d$,*

$$k_1||\theta||^2\le(\theta,a(x)\theta)\le k_2||\theta||^2.$$

Then if for any initial value x there exists a unique (in law) weak solution $(\Omega,\mathcal{F},\{\hat{\mathcal{F}}_t\},\ \hat{P}_x,\hat{W},X^x)$ to (1) with $X_0^x=x$, this solution is a μ^{Leb}-irreducible T-process.

Proof. We first assume that $b(\cdot)$ and $\sigma(\cdot)$ are globally bounded. For simplicity we also assume that $k=2$ (i.e. $\mathbb{R}^d=\cup_{i=1}^2\overline{O_i}$ and hence the boundary of O_1, ∂O_1 is the same as that of O_2). The extension to $k>2$ is straightforward. By the Cameron-Martin-Girsanov formula (see e.g. Stroock and Varadhan (1979) Theorem 6.4.3) there exists a weak solution $(\Omega,\mathcal{F},\{\mathcal{F}_t\},P_x,W,Y^x)$ to (1) with diffusion matrix $a(x)$, drift vector zero and initial value $Y_0^x=x$, where the measure P_x on $(\Omega, \mathcal{F})$ satisfies $d\hat{P}_x = M^x(Y^x, t)dP_x$ and

$$M^x(Y^x,t) = \exp\Big[-\frac{1}{2}\int_0^T (b(Y^x(s)), (a^{-1}b)(Y^x(s)))ds + \int_0^T ((a^{-1}b)(Y^x(s)), dY^x(s))\Big]. \tag{2}$$

Furthermore, with $0 \le t_1 < t_2 < \cdots < t_n$

$$\hat{P}_x[(X_{t_1}, \ldots, X_{t_n}) \in \Gamma] = E_x[\mathbb{1}_{\{(Y^x_{t_1},\ldots,Y^x_{t_n})\in\Gamma\}} M^x(Y^x, t_n)]; \quad \Gamma \in \mathcal{B}(\mathbb{R}^{dn}), \tag{3}$$

where E_x denotes expectation relative to P_x. Therefore since $M^x(Y^x, t)$ is always positive it suffices to prove that Y^x is irreducible. We first show that X^x and hence also Y^x has the weak Feller property. Let $d(x) : \mathbb{R}^d \to \mathbb{R}$ be the Euclidean distance from x to ∂O_1 and define for any $\epsilon > 0$ $f^\epsilon(x) : \mathbb{R}^d \to \mathbb{R}$ to be 1 if $d(x) < \epsilon$, 0 if $d(x) > 2\epsilon$ and such that $f^\epsilon(\cdot)$ is continuous and bounded by 1. To show that X^{x_n} converges weakly to X^x as $x_n \to x$ we first note that X^{x_n} is relatively weakly compact (cf. Exercise 5.3.15 of Karatzas and Shreve (1991)). Without loss of generality and using the Skorokhod representation we assume that X^{x_n} converges with probability one to $\hat{X}$ as $x_n \to x$. To complete the proof we now show that $\hat{X}$ is a weak solution to (1) with $a(\cdot)$, $b(\cdot)$ as defined in Theorem 2.3 and initial point $X^{x_n} = x$. From continuity of $b(\cdot)(1 - f^\epsilon(\cdot))$ and $\sigma(\cdot)(1 - f^\epsilon(\cdot))$ for any $\epsilon > 0$ we have that as $x_n \to x$,

$$\int_0^T b(X^{x_n}(s))(1 - f^\epsilon(X^{x_n}(s)))ds \to \int_0^T b(\hat{X}(s))(1 - f^\epsilon(\hat{X}(s)))ds; \tag{4}$$

$$\int_0^T \sigma(X^{x_n}(s))(1 - f^\epsilon(X^{x_n}(s)))dW(s) \to \int_0^T \sigma(\hat{X}(s))(1 - f^\epsilon(\hat{X}(s)))dW(s). \tag{5}$$

Using Exercise 7.3.2 of Stroock and Varadhan (1979) we have that for all initial points x there exists some constant $C > 0$ such that

$$\Big|E_x(\int_0^T f^\epsilon(X(t))dt)\Big| \le C\epsilon, \tag{6}$$

where C depends on k_1, k_2 and T. From (4), (5) and (6) we now have that $\hat{X}$ is a weak solution to (1) and the Feller property follows from uniqueness of solutions to (1). The Feller property in the case that the drift vector $b(\cdot)$ and the diffusion matrix $a(\cdot)$ are unbounded follows directly from Lemma 11.1.1 of Stroock and Varadhan (1979). Using (6) and Theorem 3.2.1 of Stroock and Varadhan (1979) we have that for Y^x, $L(y, A) > 0$ for any $y \in \mathbb{R}^d$ and any set $A \subset \mathbb{R}^d$ with positive Lebesgue measure and hence by Proposition 2.1 of Meyn and Tweedie (1993a) Y^x is φ-irreducible with $\varphi = \mu^{\text{Leb}}R$, where $R(x, \cdot) = \int_0^\infty P^t(x, \cdot)e^{-t}dt$ is the resolvent kernel. We then infer from Theorem 3.2.1 of Stroock and Varadhan (1979) that if $\mu^{\text{Leb}}(A \cap O_i) > 0$ for some $1 \le i \le k$ then $\int_0^\infty P^t(x, A)dt > 0$ for all $x \in O_i$. The μ^{Leb}-irreducibility follows now directly from $R(x, A) > 0$, $x \in O_i$ which implies that $\varphi(A) > 0$ whenever $\mu^{\text{Leb}}(A) > 0$. The proof that X^x is a T-process now follows from μ^{Leb}-irreducibility and Proposition 2.2.

Remark 2.4. From Exercise 7.3.2 of Stroock and Varadhan (1979) we have that under the assumptions of Theorem 2.3 existence of a weak solution to (1) always holds and this solution is unique (in law) if $d = 2$ (see Exercise 7.3.4 of Stroock and Varadhan (1979) for details). In the case when $d > 2$ we have from Bass and Pardoux (1987) that uniqueness holds if $a(x)$ is constant in the interior of each O_i. For the more general case when the diffusion matrix is not necessarily constant in the interior of each O_i the question of uniqueness is still open.

As a second major class of diffusions satisfying (1), which fall into the μ^{Leb}-irreducible T-process framework, we next consider stochastic systems of the form

$$\begin{pmatrix} dX(t) \\ dY(t) \end{pmatrix} = \begin{pmatrix} f(Y(t)) \\ b(X(t), Y(t)) \end{pmatrix} dt + \begin{pmatrix} 0 \\ A \end{pmatrix} dW(t), \tag{7}$$

where A is a non-singular matrix. These processes are studied in Kliemann (1987) under the assumption that f and b are C^∞, and we relax this continuity assumption below. In particular this class includes continuous-time threshold ARMA processes with constant white noise variance (see Brockwell and Stramer (1995)), which can therefore be studied in this context. For the case $d = 2$ we will give more results than in the case when $d > 2$.

In order to study (7), we need the following notation and assumptions. Let B be a $d-p$ dimensional Brownian motion with $B(0) = 0$ defined on the probability space $(C[0,\infty)^{(d-p)}, \mathcal{B}[0,\infty)^{(d-p)}, P)$ and let $\mathcal{F}_t = \sigma\{B(s), s \le t\} \vee \mathcal{N}$, where $\mathcal{N}$ is the σ-algebra of P-null sets of $\mathcal{B}[0,\infty)^{(d-p)}$ and P denotes the law of B; we use E to denote expectation relative to P. For any $x \in \mathbb{R}^{(p)}$ and $y \in \mathbb{R}^{(d-p)}$ let $Z_t(x, y) = (x + \int_0^t f(B_s + y)ds, B_t + y)$, where f is an R^p-valued function on R^{d-p} which satisfies global Lipschitz conditions.

We will use the following assumption:

Controllability Assumption: We assume that the transition probability $P_y(Z_t \in D)$ of $Z_t(x, y)$ is positive for any set D in $\mathbb{R}^d$ with positive Lebesgue measure.

Remark 2.5. If $f(x) = Cx$, where C is a $p \times (d-p)$ matrix then it is well known (see Proposition 5.6.5 of Karatzas and Shreve (1991)) that the Controllability Assumption holds in the case when

$$\text{rank}(e, \hat{C}e, \ldots, \hat{C}^{(d-1)}e) = d,$$

where $\hat{C}$ and e are $d \times d$ matrices such that

$$\hat{c}_{ij} = \begin{cases} c_{ij}, & \text{if } 1 \le i \le p \text{ and } p+1 \le j \le d, \\ 1, & \text{if } i = j \ge p+1, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$e_{ij} = \begin{cases} 1, & \text{if } i = j \ge p+1, \\ 0, & \text{otherwise.} \end{cases}$$

We now have

Theorem 2.6. *Assume that the Controllability Assumption holds and consider stochastic differential equations of the form* (7)*; that is,*

$$dX_i(t) = f_i(Y(t))dt \qquad 1 \le i \le p$$

$$dY_i(t) = b_i(X(t), Y(t))dt + \sum_{j=p+1}^{d} \sigma_{ij} dW_j(t); \quad p+1 \le i \le d, \tag{8}$$

where $W = (W_{p+1}, \ldots, W_d)$ *is a* $(d-p)$*-dimensional Brownian motion starting from the origin,* $b : \mathbb{R}^d \to \mathbb{R}^{d-p}$ *is Borel-measurable and satisfies linear growth conditions and* $a_{(d-p)\times(d-p)}$ *is a positive definite matrix* ($a_{ik} = \sum_{j=p+1}^{d} \sigma_{ij}\sigma_{jk}$, $p+1 \le i, k \le d$)*. Then*

(i) *For any* $x \in \mathbb{R}^p$, $y \in \mathbb{R}^{d-p}$ *there exists a unique (in law) weak solution* $(X(t), Y(t))$ *to* (8) *with initial point* (x, y)*, and this solution is a* μ^{Leb}*-irreducible process.*

(ii) *If* Γ *is the set of all* $x \in \mathbb{R}^d$ *for which* $b(x)$ *is discontinuous, and for each* $\epsilon > 0$ *there exists a closed set* C_ϵ *such that* $\Gamma \subset C_\epsilon$ *and for each* $t > 0$ $\mu^{\text{Leb}}(0 < s < t : Z_s(x, y) \in C_\epsilon) \le K\epsilon$*, where* K *depends only on* t*, then* $(X(t), Y(t))$ *is also a* T*-process.*

Proof. For simplicity we assume that the matrix $\{\sigma_{ij}\}$, $p+1 \le i \le d$, $p+1 \le j \le d$ is the identity matrix I. It is obvious that $Z_t(x, y)$ as defined in the Controllability Assumption is the unique strong solution to

$$dX_i(t) = f_i(Y(t))dt \qquad 1 \le i \le p$$

$$dY_i(t) = dW_i(t) \qquad p+1 \le i \le d$$

with initial point (x, y). By the Camerom-Martin-Girsanov formula there exists a unique weak solution to (8) on $(C[0,\infty)^{d-p}, \mathcal{B}[0,\infty)^{d-p}, \hat{P}_{x,y})$. We now infer from Remark 5.3.8 of Karatzas and Shreve (1991) that $d\hat{P}_x = M^x(Y^x, t)dP_x$, where

$$\hat{P}_{x,y}[(X_t, Y_t) \in \Gamma] = E[\mathbb{1}_{\{Z_t(x,y) \in \Gamma\}} M(Z_t(x, y))]; \quad \Gamma \in \mathcal{B}(\mathbb{R}^d), \tag{9}$$

and

$$M(Z_t(x,y)) = \exp\Big[-\frac{1}{2}\sum_{i=p+1}^{d}\int_0^t (b_i(Z_s(x,y)))^2 ds + \sum_{i=p+1}^{d}\int_0^t b_i(Z_s(x,y))dB_i(s)\Big].$$

Thus (i) follows from positivity of $M(Z_t(x,y))$ and the Controllability Assumption.

The proof of (ii) will now follow directly from Proposition 2.2, provided we can show that $(X(t), Y(t))$ is weak Feller. From the assumption that the occupation time of $Z_t(x,y)$ in a small neighborhood around the boundary is small we have that if g is bounded and continuous then as $(x_n, y_n) \to (x,y)$

$$g(Z_t(x_n,y_n))M(Z_t(x_n,y_n)) \to g(Z_t(x,y))M(Z_t(x,y))$$

almost everywhere with respect to μ^{Leb}. According to Corollary 3.5.16 of Karatzas and Shreve (1991) $M(Z_t(x,y))$ is a martingale under the measure P and hence for each $x \in \mathbb{R}^p$ and $y \in \mathbb{R}^{d-p}$, $E[M(Z_t(x,y))] = 1$ and clearly $\{M(Z_t(x_n,y_n))\}$ is uniformly integrable. We now observe that $\{g(M(Z_t(x_n,y_n)))\}$ is uniformly bounded in n and hence $\{g(Z_t(x_n,y_n))M(Z_t(x_n,y_n))\}$ is uniformly integrable. The Feller property of (X,Y) follows now from Exercise 16.21 of Billingsley (1986), p.223 and this completes the proof.

Remark 2.7. Theorem 2.6 can be generalized to the case when σ_{ij} are functions of $(X(t), Y(t))$ which satisfy Lipschitz conditions. Then $Z_t(x,y)$ is the unique strong solution (for existence and uniqueness see Protter (1990)) to

$$dX_i(t) = f_i(Y(t))dt \qquad 1 \le i \le p$$

$$dY_i(t) = \sum_{j=p+1}^{d} \sigma_{ij}(X(t),Y(t))dW_j(t); \quad p+1 \le i \le d$$

and the proof is now similar to the proof of Theorem 2.6.

Following Brockwell and Williams (1995) and Nisio (1973) we can generalize the results of Theorem 2.6 for $d = 2$ to the case when $\sigma(\cdot)$ is not necessarily continuous. In particular this class includes continuous-time threshold AR(2) processes with white noise variance that is not necessarily constant (see Brockwell and Williams (1995)).

Theorem 2.8. *Consider the following two dimensional stochastic differential equation,*

$$\begin{pmatrix} dX_1(t) \\ dX_2(t) \end{pmatrix} = \begin{pmatrix} f(X_2(t)) \\ \sum_{i=0}^k b^i(X_1(t),X_2(t))\mathbb{1}_{\{X_1(t)\in(r_i,r_{i+1})\}} \end{pmatrix} dt + \begin{pmatrix} 0 \\ \sum_{i=0}^k \sigma^i(X_1(t))\mathbb{1}_{\{X_1(t)\in(r_i,r_{i+1})\}} \end{pmatrix} dW(t), \qquad (10)$$

where $-\infty = r_0 \le r_1 \le \cdots \le r_{k+1} = \infty$*;* $c_1 \le \sigma^i(\cdot) \le c_2$ *with positive* c_1 *and* c_2*; for each* $i = 1, \ldots, k$ b^i *is Lipschitz and* σ^i *has continuous derivatives and* f *is a one to one function with continuous and bounded derivatives of order up to two. Then for any initial value* x *there exists a unique* (*in law*) *weak solution to* (10) *and this solution is a* μ^{Leb}*-irreducible* T*-process.*

Proof. Existence and uniqueness follows directly from Nisio (1973). Let (X_1, X_2) be the unique weak solution to (10) with initial point x defined on some probability space $(\Omega, \mathcal{F}, P_x)$. Without loss of generality we assume that $k = 1$. Applying Itô's rule to (10) with $F(x_1, x_2) = (x_1, f(x_2))$ we can also assume that $f(x) = x$ for all $x \in \mathbb{R}$. Using the same argument as in Brockwell and Williams (1995) (for details see the proof of Lemma 3.1) we can show that for any $\epsilon > 0$ there exists $K > 0$ such that $E^x(\int_0^\infty \mathbb{1}(|X_1(t)| < \epsilon)dt) \le K\epsilon$, where E^x denotes expectation relative to P_x. The proof that $(X_1(t), X_2(t))$ has the Feller property is now similar to the proof of Theorem 2.3. We next show that $(X_1(t), X_2(t))$ is a μ^{Leb}-irreducible process. To do this we consider for $i = 0, 1$ the following two dimensional stochastic differential equation,

$$\begin{pmatrix} dX_1^i(t) \\ dX_2^i(t) \end{pmatrix} = \begin{pmatrix} X_2^i(t) \\ b^i(X_1^i(t), X_2^i(t)) \end{pmatrix} dt + \begin{pmatrix} 0 \\ \sigma^i(X_1^i(t)) \end{pmatrix} dW(t). \tag{11}$$

Applying Itô's rule to (11) with $F(x_1, x_2) = (x_1, \int_0^{x_2} \frac{du}{\sigma^i(u)})$ we can assume that $\sigma^i(x) \equiv 1$. From Theorem 2.6 we have that for any initial point x there exists a unique (in law) weak solution to (11) and this solution is a μ^{Leb}-irreducible process. This implies that if $O_1 = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 < r_1\}$ and $O_2 = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 > r_1\}$, then for (X_1, X_2) and each $i = 1, 2$ $L(x, A) > 0$ for all $x \in O_i$ and any set $A \subset \mathbb{R}^2$ with $\mu^{\text{Leb}}(A \cap O_i) > 0$. The proof that (X_1, X_2) is a μ^{Leb}-irreducible T-process is now similar to the proof of Theorem 2.3.

3. Transience and Recurrence

We now turn to conditions for recurrence and transience. Note that our next sequence of results apply to models in Khas'minskii (1980), Bhattacharya (1978) and Kliemann(1987); for it is trivial under the assumption that the vector drift b and the diffusion matrix a are Lipschitzian and a is positive definite (see Khas'minskii (1980)) that the diffusion process is a T-process; from Theorem 2.3, all models in Bhattacharya (1978) with diffusion matrix a that are Lipschitzian are μ^{Leb}-irreducible T-processes; and it was observed in Meyn and Tweedie (1993a) that the models in Kliemann (1987) are T-processes, and now we have from Theorem 2.6 and Remark 2.7 that under the Controllability Assumption these models are also μ^{Leb}-irreducible.

3.1. Criteria for recurrence and transience

We first establish a transience-Harris recurrence dichotomy for weak solutions to (1) which are μ^{Leb}-irreducible T-processes. Note that we get a stronger version of recurrence than the one in Bhattacharya (1978), where it was shown that $P_x\{\tau_O < \infty) \equiv 1$ for each open set O: we have that the Harris condition gives $P_x\{\eta_A = \infty\} \equiv 1$ for any set A with positive Lebesgue measure, so both the class of recurrent sets and the "strength of recurrence" are extended.

Proposition 3.1. *Let X be a μ^{Leb}-irreducible weak solution of* (1)*. Then one of the following holds:*

(i) *X is Harris recurrent and for all x, and all A with $\mu^{\mathrm{Leb}}(A) > 0$*

$$P_x\{\eta_A = \infty\} \equiv 1; \tag{12}$$

(ii) *X is transient and if X is also a T-process then every compact set K is uniformly transient* (*i.e. $E_x(\eta_K) \le M < \infty$ for all x and compact K*)*.*

Proof. The fact that the chain is either transient or Harris recurrent with respect to some μ (that is, there exists a σ-finite measure μ, such that whenever $\mu\{A\} > 0$ $L(x, A) \equiv 1$) is known (see Meyn and Tweedie (1993a)). By Theorem 2.4 of Meyn and Tweedie (1993d), we then have in the recurrent case that $P_x\{\eta_A = \infty\} \equiv 1$ whenever $\varphi(A) > 0$, where $\varphi = \mu R$ and R is the resolvent kernel. We conclude from μ^{Leb}-irreducibility that $R(x, A) > 0$ for any set A in $\mathbb{R}^d$ with positive Lebesgue measure and all $x \in \mathbb{R}^d$. Thus $\varphi(A) > 0$ whenever $\mu^{\mathrm{Leb}}(A) > 0$, which gives (12) for the required sets.

In the transient case, each compact set K is "small" since X is a T-process. From Theorem 8.3.5 of Meyn and Tweedie (1993c) we thus have that every compact set is uniformly transient for the resolvent, and (ii) follows now immediately from the identity (see Tuominen and Tweedie (1979))

$$U(x, A) = \sum_{n>0} R^n(x, A) = U_R(x, A),$$

where R is the resolvent chain, η_A^R is the (discrete) occupation time of A for the resolvent and $U_R(x, A) = E_x[\eta_A^R]$.

Under various continuity conditions on b and a, it is known that "open set" recurrence of the process $X(t)$ follows if $L(x, O) \equiv 1$ for some O open and bounded (see Khas'minskii (1980)), or if $L(x, K) \equiv 1$ for some K compact (see Bhattacharya (1978)). Our next result extends this.

Proposition 3.2. *Let X be a μ^{Leb}-irreducible weak solution of* (1)*.*

(i) *If X is a T-process and there exists a compact set K such that $L(x, K) \equiv 1$ then X is Harris recurrent.*

(ii) *If there exists a set* U *with positive Lebesgue measure such that for some one* x $L(x, U) < 1$ *then* X *is transient.*

Proof. (i) follows directly from Theorem 3.2 of Meyn and Tweedie (1993a) and the T-property. To see (ii), note from the dichotomy between recurrence and transience, and μ^{Leb}-irreducibility that if X is not transient then X is Harris recurrent and hence by Proposition 3.1(i) $L(x, A) \equiv 1$ for all A with positive Lebesgue measure. This obviously contradicts the assumption $L(x, U) < 1$ and so the chain is transient.

Remark 3.3. Note that in Bhattacharya (1978) it was shown that in the recurrent case

$$P_x(X(t_i) \in U \text{ for a sequence } \{t_i\} \text{ with } t_i \to \infty) = 1$$

for all $x \in \mathbb{R}^d$ and all nonempty open sets U. Clearly (12) is a much stronger condition.

3.2. Test functions for recurrence and transience

Test functions for the condition $L(x, K) \equiv 1$ are given in Khas'minskii (1980) under the condition that the process almost surely exits from each bounded set in a finite time, and from Proposition 3.2 this will give us a criterion for Harris recurrence. We now see that in fact this extra condition can be omitted for μ^{Leb}-irreducible T-processes because of the strength of the Harris recurrence/transience dichotomy.

For suppose that U is a bounded set. Under the conditions of Proposition 3.1, either the chain is transient in which case $E_x(\eta_U)$ is bounded and the chain must reach U^c almost surely; or the chain is Harris recurrent and since $\mu^{\mathrm{Leb}}(U^c) > 0$, we also have that $L(x, U^c) \equiv 1$. So the condition of Khas'minskii is always satisfied for μ^{Leb}-irreducible T-processes.

Proposition 3.4. *Assume that* X*, the weak solution to* (1), *is a* μ^{Leb}*-irreducible* T*-process. If there exists a positive definite symmetric matrix* B *such that for some compact set* C,

$$2(Bx, b(x)) + \mathrm{tr}(a(x)B) \leq \frac{2(Ba(x)Bx, x)}{(Bx, x)} \qquad x \in C^c,$$

then $L(x, C) \equiv 1$ *and the process is Harris recurrent.*

Proof. Apply Theorem 3.1 of Meyn and Tweedie (1993b) with the test function $V(x) = \log(Bx, x) + k$ (as used in Khas'minskii (1980)) to see that $L(x, C) \equiv 1$. The result follows from Proposition 3.2(i).

Proposition 3.5. *Assume that* X*, the weak solution to* (1), *is a* μ^{Leb}*-irreducible process. Then* X *is transient if there exists a semi positive definite symmetric* *matrix B with at least one of its eigenvalues nonzero and such that for some compact set C, and a constant $\alpha > 0$*

$$2(Bx, b(x)) + \mathrm{tr}(a(x)B) \geq \frac{2(1+\alpha)(Ba(x)Bx, x)}{(Bx, x)}, \qquad x \in C^c.$$

Proof. Apply Theorem 3.3 of Stramer and Tweedie (1994) with the test function $V(x) = 1 - k(Bx, x)^{-\alpha}$ (again as used in Khas'minskii (1980)) to find $L(x, C) < 1$. Now the result follows from Proposition 3.2(ii).

Remark 3.6. A condition for the stronger "transience" result $|X^x(t) - X^y(t)| \to \infty$ a.s. exponentially fast as $t \to \infty$ for all $x \neq y$ is given in Remark 2.5 of Basak and Bhattacharya (1992) for the more general case that X is not necessarily irreducible, but under the assumption that a and b are Lipschitzian. Their condition is stronger than that in Proposition 3.4.

4. Null Recurrence, Positive Recurrence and Exponential Ergodicity

For positive Harris recurrent processes it is known (see Meyn and Tweedie (1993a)) that under very general conditions $P^t(x, \cdot) \to \pi(\cdot)$. Specifically, if the chain is aperiodic in the sense that some skeleton is μ^{Leb}-irreducible (see Meyn and Tweedie (1993a) Section 5) then the transition probabilities converge in total variation to π.

Moreover, as shown in Meyn and Tweedie (1993a), often the process may converge exponentially quickly, in which case it does so in the strong sense of V-uniform ergodicity: that is, for some function $V \geq 1$ and constants $\beta < 1$ and $R < \infty$,

$$\|P^t(x, \cdot) - \pi\|_V \leq V(x)\, R\, \beta^t \qquad t \in \mathbb{R}_+,\ x \in \mathbb{R}^d, \tag{13}$$

where for any signed measure μ, $\|\mu\|_V = \sup_{|g| \leq V} |\mu(g)|$. As in Section 3.2, we can use test functions to find conditions for null recurrence, positive recurrence and V-uniform ergodicity of weak solutions to (1).

Proposition 4.1. *Assume that X, the weak solution to* (1), *is μ^{Leb}-irreducible and Harris recurrent. Let C be any compact set in $\mathbb{R}^d$ such that $\mu^{\mathrm{Leb}}(C) > 0$. If there are positive definite matrices B_1, B_2 and numbers $\varepsilon > 0$, $M < \infty$, such that for $x \in C^c$*

$$2(B_1 x, b(x)) + \mathrm{tr}(a(x)B_1) < M,$$

$$2(B_2 x, b(x)) + \mathrm{tr}(a(x)B_2) > \varepsilon \frac{(B_2 a(x) B_2 x, x)}{(B_2 x, x)}$$

then X is null recurrent.

Proof. Apply Theorem 4.3 of Stramer and Tweedie (1994) with $V = (B_1 x, x)$, $W = (B_2 x, x)^\alpha - k$ (as used in Khas'minskii (1980)) where $0 < \alpha < 1$ and the constant k is sufficiently large.

Proposition 4.2. *Assume that X, the weak solution to* (1), *is a μ^{Leb}-irreducible T-process.*

(i) *If there exists a semi-positive definite matrix B such that for some $c > 0$, $f \geq 1$ and $R > 0$*

$$2(Bx, b(x)) + \mathrm{tr}(a(x)B) \leq -cf(x) \quad for \quad ||x|| > R, \tag{14}$$

then X is positive recurrent;

(ii) *If X is aperiodic, and there exists a positive definite matrix B such that for some $c > 0$ and $R > 0$*

$$2(Bx, b(x)) + \mathrm{tr}(a(x)B) \leq -c(Bx, x) \quad for \quad ||x|| > R, \tag{15}$$

then the process is exponentially ergodic and for $V = (Bx, x) + 1$ we have that (13) *holds.*

Proof. Apply Theorem 4.2 of Meyn and Tweedie (1993b) with $V(x) = (Bx, x) + 1$ to get positive recurrence in (i); the same function gives V-uniform ergodicity from Theorem 6.1 of Meyn and Tweedie (1993b) also.

Remark 4.3. It is often easy to verify aperiodicity, and hence convergence in total variation as a consequence of (14). It is not hard to show that the class of processes given in Theorem 2.3 is also aperiodic under the assumption that a satisfies a Lipschitz condition in $\mathbb{R}^d$. In this case we have from (3) and Theorem 3.2.1 of Stroock and Varadhan (1979) that $P^t(x, A) > 0$ for any set $A \subseteq \mathbb{R}^d$ with positive Lebesgue measure. We conjecture that this class of processes is aperiodic without the extra assumption on a but have not yet shown this. For the class of processes defined in Theorem 2.6, aperiodicity follows from (9) and the Controllability Assumption.

Remark 4.4. From Theorem 4.5 of Meyn and Tweedie (1993b) we have that even without irreducibility, condition (14) can be used to give the existence of a stationary probability for a T-process. A similar test function to (14) can be found in Remark 2.6 of Basak and Bhattacharya (1992) under the assumption that a and b are Lipschitzian.

Finally we find an explicit expression for the stationary density in the positive recurrent case, under suitable conditions.

Theorem 4.5. *Suppose that $\mathbb{R}^d$ can be divided up into finitely many polyhedra $O_1, \ldots, O_k$ such that $\mathbb{R}^d = \cup_{i=1}^k \overline{O_i}$ and the O_i have pairwise disjoint interiors.*

Assume that in the interior of each O_i $b_0 : \mathbb{R}^d \to \mathbb{R}$ is of class C^2 with bounded second partial and bounded gradient ∇b_0. Assume also that there exist a sequence $b_n : \mathbb{R}^d \to \mathbb{R}$ such that

(i) *b_n is of class C^2 with bounded second partial and bounded gradient ∇b_n.*

(ii) *On the interior of each O_i, $1 \le i \le k$, $b_n(x) \to b_0(x)$ and $\nabla b_n(x) \to \nabla b_0(x)$ as $n \to \infty$*

(iii) $\exp^{2b_n(x)}$ *are uniformly integrable.*

Let

$$\hat{\nabla} b_0(y) = \begin{cases} \nabla b_0(y), & \text{if } y \in int(O_i), \\ arbitrary, & otherwise. \end{cases}$$

Then the weak solution X_0 to the stochastic differential equation

$$dX(t) = \hat{\nabla} b_0(X(t))dt + dW(t) \qquad 0 \le t \le \infty, \tag{16}$$

where W is a standard $\mathbb{R}^d$-valued Brownian motion, has a stationary density

$$\pi(dx) = \exp^{2b_0(x)} dx.$$

Proof. Let $(\Omega^n, \mathcal{F}^n, \{\mathcal{F}_t^n\}, P_x^n, W^n, X^n)$ be the weak solution to (16) with drift vector ∇b_n, $n \ge 0$. From (2) and (3) we have that, with $0 \le t_1 < t_2 < \cdots < t_n$

$$P_x^n[(X_n(t_1), \ldots, X_n(t_n)) \in \Gamma] = E_x[\mathbb{1}_{\{(B_{t_1}^x, \ldots, B_{t_n}^x) \in \Gamma\}} M_n^x(B^x, t_n)]; \quad \Gamma \in \mathcal{B}(\mathbb{R}^{dn}),$$

where B^x is a d-dimensional Brownian motion (with $B^x(0) = x$) and

$$M_n^x(B^x, t) = \exp\left[-\frac{1}{2}\int_0^t ||\nabla b_n(B(s))||^2 ds + \int_0^t (\nabla b_n(B(s)), dB(s))\right].$$

We first show that the finite-dimensional distributions of the process X_n converge as $n \to \infty$ to the finite dimensional distribution of X_0. By Scheffé's theorem and Exercise 21.21 of Billingsley (1986), p.289, it suffices to show that $M_n^x(B^x, t)$ converges in probability to $M_0^X(B, t)$ as $n \to \infty$. This follows since both $\int_0^t ||\nabla b_n(B(s))||^2 ds$ and $\int_0^t (\nabla b_n(B(s)), dB(s))$ converge in probability as $n \to \infty$ to the corresponding integrals with b_0 for each fixed t.

Let $\pi_n(dx) = \exp^{2b_n(x)} dx$ on $\mathcal{B}(\mathbb{R}^d)$ for all $n \ge 0$. We now have from $P_x^n(X_n^x(t) \in A) \to P_x^0(X_0^x(t) \in A)$ as $n \to \infty$ for all $A \in \mathcal{B}(\mathbb{R}^d)$ and Theorem 16.13 of Billingsley (1986) that $\int P_x^n(X_n^x(t) \in A)\pi_n(dx) \to \int P_x^0(X_0^x(t) \in A)\pi(dx)$ as $n \to \infty$ for all $A \in \mathcal{B}(\mathbb{R}^d)$. We also have from Theorem 16.13 of Billingsley (1986) that $\pi_n(\mathbb{R}^d) \to \pi(\mathbb{R}^d)$ as $n \to \infty$ and hence by Scheffé's theorem $\pi_n(A) \to \pi(A)$ as $n \to \infty$ for all $A \in \mathcal{B}(\mathbb{R}^d)$. The proof follows now directly from Exercise 5.6.18 of Karatzas and Shreve (1991).

5. A Threshold Model in Continuous Time

Example 5.1. As an application of the criteria above, we consider one special case of multivariate continuous time threshold AR(1) processes. These processes are extensions of continuous time threshold AR(1) processes (see Stramer, Brockwell and Tweedie (1996)). Here X is the weak solution to (1) with b and σ defined as follows. Let $\epsilon^1, \epsilon^2, \ldots, \epsilon^N$ $(N = 2^d)$ be all the possible vectors in $\{0,1\}^d$. For each $j = 1, \ldots, N$ define

$$A_j = \{\mathbf{x} \in \mathbb{R}^d : \ \ x_i < 0 \text{ if } \epsilon_i^j = 0 \text{ and } x_i > 0 \text{ if } \epsilon_i^j = 1\}.$$

Note that $\mathbb{R}^d = \cup_{j=1}^N \overline{A_j}$. Thus, if for example $d = 2$, then

$$A_1 = \{\mathbf{x} \in \mathbb{R}^2 : \ \ x_1 < 0 \ \text{ and } x_2 > 0\}; \quad A_2 = \{\mathbf{x} \in \mathbb{R}^2 : \ \ x_1 > 0 \ \text{ and } x_2 < 0\};$$

$$A_3 = \{\mathbf{x} \in \mathbb{R}^2 : \ \ x_1 < 0 \ \text{ and } x_2 < 0\}; \quad A_4 = \{\mathbf{x} \in \mathbb{R}^2 : \ \ x_1 > 0 \ \text{ and } x_2 > 0\}.$$

Assume that $b(\mathbf{x})$ is linear and $\sigma(\mathbf{x})$ is constant on each A_j, $j = 1, \ldots, N$, i.e. for all $\mathbf{x} \in A_j$ $b(\mathbf{x}) = b_j\mathbf{x}$ and $\sigma(\mathbf{x}) = \sigma_j I$, where σ_j are positive constants, I is the $d \times d$ identity matrix and b_j are $d \times d$ matrices. On the boundaries we define $b(\mathbf{x})$ and $\sigma(\mathbf{x})$ to be $b_j\mathbf{x}_j$ and $\sigma_j I$ for some arbitrary j such that $\mathbf{x} \in \overline{A_j}$.

Proposition 5.2. *Let X be the multivariate continuous time threshold $AR(1)$ process defined above. Then*
- (i) *X is positive recurrent if $-b_j$ is positive definite for each $j = 1, \ldots, N$.*
- (ii) *X is null recurrent if $b(\mathbf{x}) \equiv 0$ and $d = 2$;*
- (iii) *X is transient if b_j is positive definite for each $j = 1, \ldots, N$ or $b(\mathbf{x}) \equiv 0$ and $d > 2$.*

Proof. By Theorem 2.3 X is a μ^{Leb}-irreducible T-process.

Suppose first that $b(\mathbf{x}) \equiv 0$. If $d = 2$ we apply Proposition 3.4 with $B = I$ to show that X is Harris recurrent and Proposition 4.1 with $B_1 = B_2 = I$ to show that X is null recurrent. If $d > 2$ we apply Proposition 3.5. with $B = I$ to show that X is transient.

Now assume that $b(\mathbf{x}) \not\equiv 0$. If condition (i) holds then condition (14) is satisfied with $c = \max\{\lambda_j : \ 1 \le j \le N\}$, where λ_j is the largest eigenvalue of b_j for each $1 \le j \le N$, $B = I$ and $f(\mathbf{x}) = ||\mathbf{x}||^2 + 1$. Note that $\lambda_j < 0$ for each j. The proof of (i) follows now directly from Proposition 4.2. If condition (iii) holds then

$$(I\mathbf{x}, b_j\mathbf{x}) \ge \lambda_j ||\mathbf{x}||^2,$$

where $\lambda_j > 0$ is the smallest eigenvalue of b_j for each $1 \le j \le N$. The proof of (iii) follows now directly from Proposition 3.5.

Remark 5.3. We can refine the results of Proposition 5.2 if we assume that σ is constant. For simplicity let $\sigma = I$. Then we infer from Theorem 4.5 and Proposition 5.2 that if condition (i) of Proposition 5.2 holds then X is positive Harris recurrent and the stationary distribution has probability density $\pi(\mathbf{x}) \propto \exp(\sum_{j=1}^{N} 2b_j \mathbf{x} \mathbb{1}(\mathbf{x} \in \overline{A_j}))$. Moreover we have from Proposition 4.2 that since X is aperiodic and condition (15) is satisfied then X is V-exponentially ergodic with $V = ||\mathbf{x}||^2 + 1$.

Acknowledgement

This work was supported in part by NSF Grants DMS-9504798, DMS-9100392 and DMS-9205687.

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(Received August 1995; accepted June 1996)