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A COMPARISON OF DEGRADATION AND FAILURE-TIME ANALYSIS METHODS FOR ESTIMATING A TIME-TO-FAILURE DISTRIBUTION

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Abstract: Degradation analysis can be used to assess reliability when few or even no failures are expected in a life test. In this paper, we use a simple but useful degradation model to compare degradation analysis and traditional failure-time analysis in terms of asymptotic efficiency. The comparisons consider a range of practical testing situations and provide insight into the trade-offs between these two methods of estimating the quantiles of the time-to-failure distribution. We investigate the effect that the number of inspections, the amount of measurement error, and the quantile of interest have on the asymptotic variances of the quantile estimators. Although measurement error can induce some loss of precision in degradation analysis, our comparisons show that, except in extreme cases, degradation analysis provides more precision than traditional failure-time analysis.

Key words and phrases: First crossing time, life data analysis, measurement error, relative efficiency.

1. Introduction

Traditional failure-time analysis (FTA) methods for estimating component reliability record only the time-to-failure (for units that fail) or the running time (for units that do not fail). In life tests for high reliability components there will be few or no failures, making reliability assessment difficult. Degradation analysis (DA) is an alternate approach that uses a sequence of degradation measures to assess reliability. Lu and Meeker (1993) discuss a particular approach to degradation analysis. Nelson (1990, chapter 11) discusses other methods for analyzing degradation data, particularly with acceleration. In this paper we use a simple, but physically reasonable, degradation model to compare DA and FTA. This degradation model implies a lognormal distribution for the corresponding time-to-failure distribution. We use the ratio of the asymptotic variances of estimators of a quantile of the time-to-failure distribution to compare DA and FTA.

Suzuki, Maki and Yokogawa (1993) (SMY) also compare degradation and failure-time analysis, but use different assumptions about how data become available, focus on different quantities of interest, and use a somewhat different model. In particular,

- SMY do not let the ability to observe degradation depend on the level of degradation (in some applications, units must be removed from service due to failure or for safety reasons).
- SMY evaluate the ability to estimate mean time to failure, while our evaluations are for selected percentiles of the time-to-failure distribution.
- SMY deal with an accelerated degradation model. We are concerned with tests and inferences under a specified set of conditions.

2. Models

2.1. Degradation model

We assume a degradation model in which the degradation level is proportional to time with a degradation rate that is random from unit to unit. More specifically, we assume that measured degradation = $\exp(\Theta)\times\text{time}\times\exp(\epsilon)$ which implies the following simple path model: $Y = \Theta + x + \epsilon$, $\Theta \sim \mathrm{N}\left(\mu_\Theta, \sigma^2_\Theta\right)$, $\epsilon \sim \mathrm{N}\left(0, \sigma^2_\epsilon\right)$, where $Y = \log(\text{measured degradation})$, $\Theta = \log(\text{degradation rate})$, $x = \log(\text{time})$, ϵ is the measurement error, and Θ and ϵ are independent. We assume that "failure" occurs when *actual* degradation $(\Theta + x)$ reaches a specified critical level D_c, that is, when $\Theta + x = D_c$.

2.2. Time-to-failure model

From the degradation model in Section 2.1 the random log time-to-failure can be expressed as $X = D_c - \Theta \sim \mathrm{N}\left(D_c - \mu_\Theta, \sigma^2_\Theta\right)$. Therefore, the time-to-failure distribution is lognormal with parameters $\mu_X = D_c - \mu_\Theta$ and $\sigma_X = \sigma_\Theta$.

3. Data, Estimation and Asymptotic Variances

In this section we define the inspection data used in the comparison, we give estimators of the model parameters, and we provide expressions for the asymptotic variances of these estimators. For both the DA and FTA methods, we assume fixed log inspection times $x_1, \ldots, x_m$. For DA we observe a degradation level (plus noise) at each inspection. At each inspection in FTA we observe the number of units that have failed up to the current inspection.

3.1. Degradation analysis

3.1.1. Degradation inspection data

In some situations degradation will result in a sudden catastrophic failure. Typically, however, there is not an *exact* relationship between the level of degradation and the catastrophic failure. In other situations, performance degrades more gracefully. Then, service life of a unit would end around the time that degradation has caused system performance to reach a specified level. In such situations failure can be defined as some observed level of system performance or in terms of actual degradation on a component that can be measured, with some degree of measurement error. In either case, it may be necessary to replace a degrading unit after degradation has reached a specified level, either because of loss of adequate functionality or for safety reasons. In some reliability studies, this will restrict the time of observation.

In our comparison, we assume that there are m planned inspection times at log(times) $x_1, \ldots, x_m$. The actual number m_{Θ_i} of inspections on a unit is random and depends on the unit's log(degradation rate) $= \Theta_i$ as follows: $m_{\Theta_i} = j$ ($j = 2, \ldots, m-1$) if the unit "fails" between inspection times x_j and x_{j+1} (i.e., if $\Theta_i + x_j < D_c \leq \Theta_i + x_{j+1}$), $m_{\Theta_i} = m$ when the unit survives beyond x_m (i.e., $\Theta_i + x_m < D_c$). Finally, when $\Theta_i + x_2 > D_c$, we assume that measurements are available on the unit at x_1 and x_2. This would be realistic if we get a signal at the time that $\Theta + x$ exceeds D_c (e.g., we detect loss of performance), even though there is not a catastrophic failure and we cannot observe $\Theta + x$ directly. Having two inspections on each unit insures that the parameters Θ and σ_ϵ can be estimated for each unit.

Thus, have data $Y_{ij} = \Theta_i + x_j + \epsilon_{ij}$, $j = 1, \ldots, m_{\Theta_i}$, $2 \leq m_{\Theta_i} \leq m$, where ϵ_{ij} is the measurement error for the jth inspection on the ith unit.

In situations with large measurement error, our criterion for stopping observations will not provide a description of the actual stop-observation rule. The description is, however, useful for comparing DA and FTA for situations when observation is limited by high levels of degradation.

3.1.2. Two-stage estimation of the degradation parameters

We use the two-stage estimation procedure described in Lu and Meeker (1993) to estimate the model parameters for DA. In the first stage, least squares estimation gives

$$\begin{aligned}
\hat{\Theta}_i &= \frac{1}{m_{\Theta_i}} \sum_{j=1}^{m_{\Theta_i}} (Y_{ij} - x_j), \\
\hat{\sigma}^2_{\epsilon i} &= \frac{1}{m_{\Theta_i} - 1} \sum_{j=1}^{m_{\Theta_i}} \left[Y_{ij} - (\hat{\Theta}_i + x_j) \right]^2.
\end{aligned}$$

By linear model normal theory, the conditional distributions of $\hat{\Theta}_i$ and $\hat{\sigma}^2_{\epsilon i}$, given Θ_i, are $\hat{\Theta}_i \sim \mathrm{N}\left(\Theta_i, \sigma^2_\epsilon / m_{\Theta_i}\right)$ and $(m_{\Theta_i} - 1)\hat{\sigma}^2_{\epsilon i}/\sigma^2_\epsilon \sim \chi^2_{m_{\Theta_i}-1}$, where Θ_i and m_{Θ_i} are realizations of Θ and m_Θ, respectively.

We use $\hat{\sigma}^2_{\hat{\Theta}_i} = \hat{\sigma}^2_{\epsilon i}/m_{\Theta_i}$ to estimate $\sigma^2_{\hat{\Theta}_i} = \mathrm{Var}_\epsilon(\hat{\Theta}_i|\Theta_i)$, the variance due to measurement error for a realized value of Θ_i. Taking the variability of random effects into account, the unconditional distribution of $\hat{\Theta}_i$ has mean $\mathrm{E}\,(\hat{\Theta}_i) = \mu_\Theta$ and variance $\mathrm{Var}(\hat{\Theta}_i) = \sigma^2_\Theta + \sigma^2_{\hat{\Theta}}$, where $\sigma^2_{\hat{\Theta}} = \mathrm{E}\,(\hat{\sigma}^2_{\hat{\Theta}_i}) = \mathrm{E}_{\,\Theta}[\mathrm{E}_{\,\epsilon}(\hat{\sigma}^2_{\hat{\Theta}_i}|\Theta_i)] = \mathrm{E}_{\,\Theta}(\sigma^2_\epsilon/m_{\Theta_i}) = \sigma^2_\epsilon \mathrm{E}_{\,\Theta}(1/m_{\Theta_i})$. Note that expectations without a subscript need to be taken with respect to the joint distribution of Θ and ϵ.

In Stage 2 of the DA estimation procedure, we combine $(\hat{\Theta}_i, \hat{\sigma}^2_{\hat{\Theta}_i})$, $i = 1, \ldots, n$ into $\hat{\mu}_\Theta = \sum_{i=1}^n \hat{\Theta}_i/n$ and $\hat{\sigma}^2_{\hat{\Theta}} = \sum_{i=1}^n \hat{\sigma}^2_{\hat{\Theta}_i}/n$. Then, because $\mathrm{E}\,(\hat{\Theta}_i) = \mu_\Theta$ and $\mathrm{E}\,(\hat{\sigma}^2_{\hat{\Theta}_i}) = \sigma^2_{\hat{\Theta}}$, we have $\mathrm{E}\,(\hat{\mu}_\Theta) = \mu_\Theta$ and $\mathrm{E}\,(\hat{\sigma}^2_{\hat{\Theta}}) = \sigma^2_{\hat{\Theta}}$. Let $S^2_\Theta = \sum_{i=1}^n(\hat{\Theta}_i - \hat{\mu}_\Theta)^2/(n-1)$ denote the sample variance of the Stage 1 estimates of Θ. This sample variance, $\mathrm{E}\,(S^2_\Theta) = \sigma^2_\Theta + \sigma^2_{\hat{\Theta}}$, reflects both the measurement error variance and the unit to unit random values of Θ. Therefore, we estimate σ^2_Θ by $\hat{\sigma}^2_\Theta = S^2_\Theta - \hat{\sigma}^2_{\hat{\Theta}}$. If $\hat{\sigma}^2_\Theta < 0$, setting $\hat{\sigma}^2_\Theta = 0$ is a special case of the approach used in Lu and Meeker (1993), originally suggested by Amemiya (1985).

3.1.3. Variance-covariance matrix for DA

Noting that $\mu_X = D_c - \mu_\Theta$ and $\sigma_X = \sigma_\Theta$, we take $\hat{\mu}^{[\mathrm{DA}]}_X = D_c - \hat{\mu}_\Theta$ and $\hat{\sigma}^{[\mathrm{DA}]}_X = \hat{\sigma}_\Theta$. In Appendix A, we show that the asymptotic variance-covariance matrix for the DA estimators can be expressed as

$$\mathrm{Var}\begin{pmatrix} \hat{\mu}^{[\mathrm{DA}]}_X \\ \hat{\sigma}^{[\mathrm{DA}]}_X \end{pmatrix} = \frac{\sigma^2_X}{n}\begin{pmatrix} V^{[\mathrm{DA}]}_{11} & V^{[\mathrm{DA}]}_{12} \\ V^{[\mathrm{DA}]}_{12} & V^{[\mathrm{DA}]}_{22} \end{pmatrix}.$$

Appendix B gives computational expressions for $V^{[\mathrm{DA}]}_{11}$, $V^{[\mathrm{DA}]}_{12}$, and $V^{[\mathrm{DA}]}_{22}$ and shows that they depend on the "standardized" inspection times $z_j = (x_j - \mu_X)/\sigma_X$, $j = 2, \ldots, m-1$, and the variability ratio $R_\sigma = \sigma_\epsilon/\sigma_\Theta$, the ratio of measurement error variation versus random effect variation.

3.2. Failure-time analysis

3.2.1. Failure-time data

Because DA uses inspections during the test, we also assume the use of inspections in FTA. The data consist of the number of observed failures in each interval. We let n_1 denote the number of units failed before log time x_1, let n_j denote the number of units failed between inspections at log times x_{j-1} and x_j, $j = 2, \ldots, m$, and let n_{m+1} denote the number of units that survived to the last inspection x_m. Note that $n = \sum_{j=1}^{m+1} n_j$.

3.2.2. Maximum likelihood estimation of model parameters

Detailed discussion of maximum likelihood estimation for failure-time data can be found, for example, in Lawless (1982) or Nelson (1982). For the inspection failure-time data, the maximum likelihood estimates $\hat{\mu}^{[\mathrm{FTA}]}_X$ and $\hat{\sigma}^{[\mathrm{FTA}]}_X$ are obtained by maximizing the log likelihood of n test units:

$$\mathcal{L} = n_1 \log \Phi(z_1) + \sum_{j=2}^{m} n_j \log \Big[\Phi(z_j) - \Phi(z_{j-1})\Big] + n_{m+1} \log[1 - \Phi(z_m)],$$

where $z_j, j = 1, \ldots, m$, are the standardized inspection times defined at the end of Section 3.1.3.

3.2.3. Variance-covariance matrix for FTA

The asymptotic variance-covariance matrix of the FTA maximum likelihood estimators can be expressed as

$$\mathrm{Var}\begin{pmatrix} \hat{\mu}_X^{[\mathrm{FTA}]} \\ \hat{\sigma}_X^{[\mathrm{FTA}]} \end{pmatrix} = \frac{\sigma_X^2}{n}\begin{pmatrix} V_{11}^{[\mathrm{FTA}]} & V_{12}^{[\mathrm{FTA}]} \\ V_{12}^{[\mathrm{FTA}]} & V_{22}^{[\mathrm{FTA}]} \end{pmatrix}.$$

Meeker (1986) gives expressions for computing $V_{11}^{[\mathrm{FTA}]}$, $V_{12}^{[\mathrm{FTA}]}$, and $V_{22}^{[\mathrm{FTA}]}$ and shows that these quantities depend only on the standardized inspection times z_j, $j = 1, \ldots, m$.

3.3. Asymptotic variances of $\hat{x}_p^{[\mathrm{DA}]}$ and $\hat{x}_p^{[\mathrm{FTA}]}$

The p quantile of the log time-to-failure distribution is $x_p = \mu_X + u_p\sigma_X$ where $u_p = \Phi^{-1}(p)$ is the standard normal p quantile. The DA estimator of x_p is obtained from $\hat{x}_p^{[\mathrm{DA}]} = \hat{\mu}_X^{[\mathrm{DA}]} + u_p\hat{\sigma}_X^{[\mathrm{DA}]}$. The asymptotic variances of these estimators are

$$\begin{aligned} \mathrm{Var}(\hat{x}_p^{[\mathrm{DA}]}) &= \mathrm{Var}(\hat{\mu}_X^{[\mathrm{DA}]} + u_p\hat{\sigma}_X^{[\mathrm{DA}]}) \\ &= \mathrm{Var}(\hat{\mu}_X^{[\mathrm{DA}]}) + 2u_p\mathrm{Cov}(\hat{\mu}_X^{[\mathrm{DA}]}, \hat{\sigma}_X^{[\mathrm{DA}]}) + u_p^2\mathrm{Var}(\hat{\sigma}_X^{[\mathrm{DA}]}). \end{aligned}$$

The corresponding asymptotic variance factor (VF) is

$$\mathrm{VF}^{[\mathrm{DA}]}(p) = \frac{n}{\sigma_X^2}\mathrm{Var}(\hat{x}_p^{[\mathrm{DA}]}) = V_{11}^{[\mathrm{DA}]} + 2u_pV_{12}^{[\mathrm{DA}]} + u_p^2V_{22}^{[\mathrm{DA}]}.$$

To compute $\mathrm{Var}(\hat{x}_p^{[\mathrm{FTA}]})$ and $\mathrm{VF}^{[\mathrm{FTA}]}$, we use exactly the same formulas with FTA substituted for DA.

3.4. Relative efficiency

To compare DA and FTA, we use an (asymptotic) relative efficiency (RE) computed as the ratio of the asymptotic variances of the estimated p quantile of time-to-failure distribution for the DA and FTA methods

$$\mathrm{RE} = \frac{\mathrm{Var}(\hat{x}_p^{[\mathrm{FTA}]})}{\mathrm{Var}(\hat{x}_p^{[\mathrm{DA}]})} = \frac{\mathrm{VF}^{[\mathrm{FTA}]}(p)}{\mathrm{VF}^{[\mathrm{DA}]}(p)} = \frac{V_{11}^{[\mathrm{FTA}]} + 2u_pV_{12}^{[\mathrm{FTA}]} + u_p^2V_{22}^{[\mathrm{FTA}]}}{V_{11}^{[\mathrm{DA}]} + 2u_pV_{12}^{[\mathrm{DA}]} + u_p^2V_{22}^{[\mathrm{DA}]}}.$$

4. Comparison and Discussion

4.1. Test plans used in comparisons

Meeker (1986) evaluates and compares different methods of planning inspections in life tests and suggests that "equal-probability-spacing" has good statistical properties and provides a convenient method for comparing alternative life test plans. Here, we also use the "equal-probability-spacing" inspection times to compare DA and FTA. With the "equal-probability-spacing" inspections, the expected number of units failed is the same within each inspection interval. The "equal-probability-spacing" standardized log inspection times are specified in terms of $z_j = \Phi^{-1}[(j/m)P_F]$, $j = 1, \ldots, m$, where P_F is the expected proportion of failures by the last log inspection time x_m. The actual log inspection times are related to the standardized inspection times through $x_j = \mu_X + z_j\sigma_X$, $j = 1, \ldots, m$. For purposes of comparison, the "equal-probability-spacing" inspections have the advantage of being easy to characterize and specify because they depend only on specification of m and P_F.

The variance factor $\mathrm{VF}^{[\mathrm{DA}]}(p)$ is a function of P_F, m, R_σ, and p, and $\mathrm{VF}^{[\mathrm{FTA}]}(p)$ is a function of P_F, m, and p. Thus the RE is a function of P_F, m, R_σ, and p. We compared the DA and FTA methods by computing and graphing $\mathrm{VF}^{[\mathrm{DA}]}(p)$, $\mathrm{VF}^{[\mathrm{FTA}]}(p)$, and RE for all combinations of the following factors:

- The time-to-failure distribution quantiles of interest: $p = .01, .02, \ldots, .99$.
- The expected proportion of failures (i.e., proportion exceeding D_c) before the last inspection: $P_F = .1, .2, \ldots, .9$.
- The variability ratio: $R_\sigma = .1, .5, 1, 2, 5$.
- The number of inspections: $m = 3, 5, 10, 20, 50, 100$.

4.2. Comparison figures

Figures 1 to 3 provide the results for a subset of these combinations:

- The first plot in Figure 1 shows $\mathrm{VF}^{[\mathrm{FTA}]}(p)$ versus p with $m = 10$ inspections, with separate lines for different values of P_F. This plot, for the lognormal distribution, is similar to the plot given in Meeker and Nelson (1976) of the variance factor of the estimated quantile of the Weibull time-to-failure distribution for censored data and continuous inspection. The other plots in this figure show $\mathrm{VF}^{[\mathrm{DA}]}(p)$ versus p with the same number of inspections with variability ratios $R_\sigma = .5, 2, 5$.

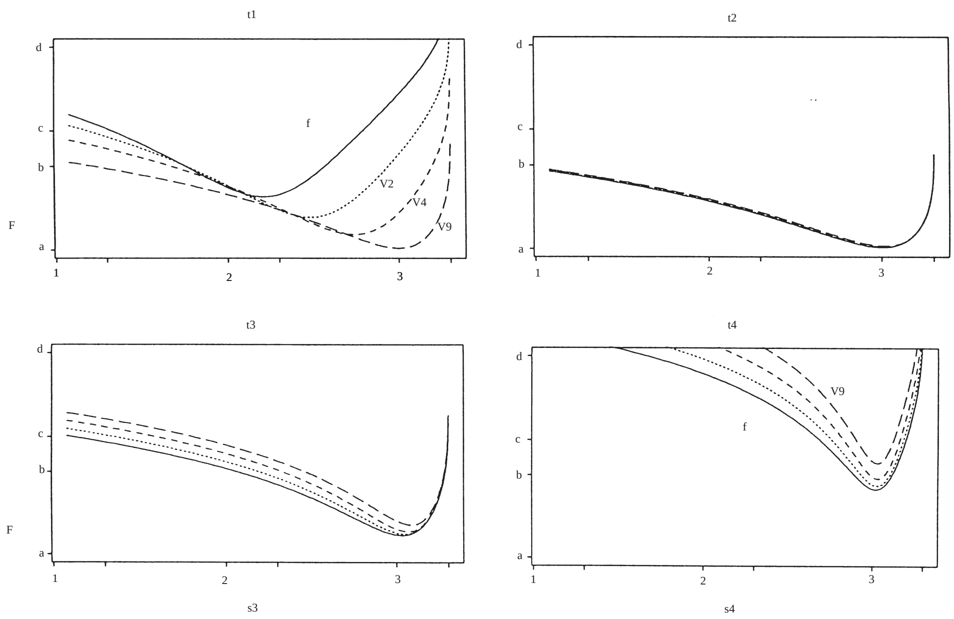

Figure 1. Variance factors for estimated quantiles.

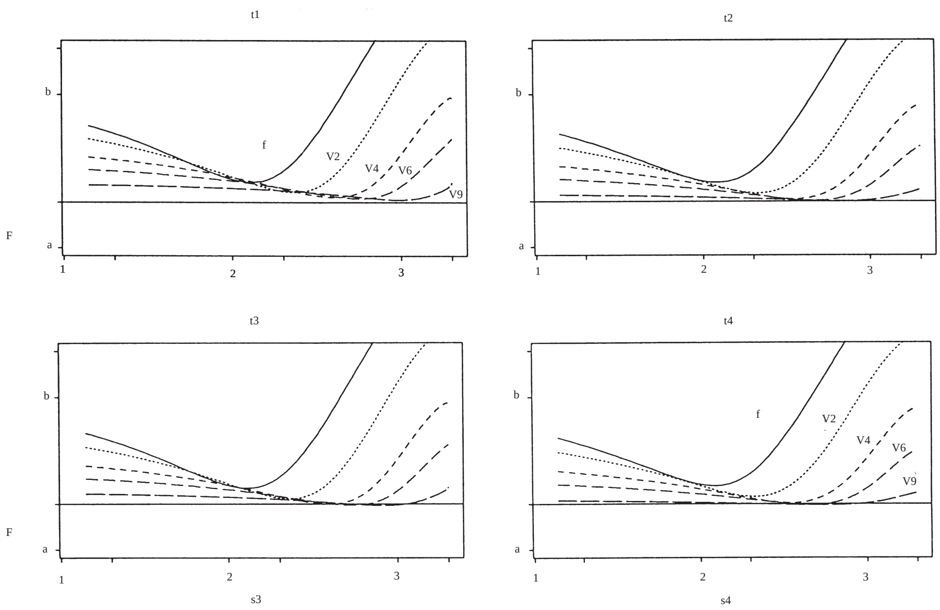

Figure 2. Degradation analysis versus failure-time analysis: $R_\sigma = .1, .5$ and $m = 10, 100$.

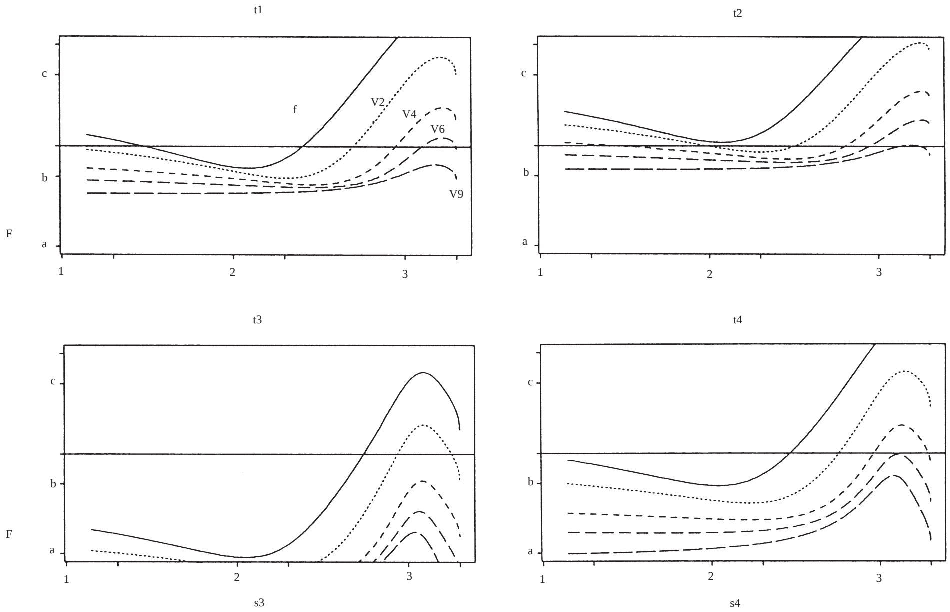

Figure 3. Degradation analysis versus failure-time analysis: $R_\sigma = 2, 5$ and $m = 10, 100$.

- Figures 2 and 3 plot RE versus p for several values of P_F for all combinations of variability ratios $R_\sigma = .1, .5, 2, 5$, and number of inspections $m = 10, 100$.

4.3. Discussion

Figure 1 shows, for FTA, that

- $\mathrm{VF}^{[\mathrm{FTA}]}(p)$ decreases and then increases as p increases. There is less precision for estimating quantiles that are remote from P_F.
- The effect of different P_F on $\mathrm{VF}^{[\mathrm{FTA}]}(p)$ is stronger when estimating larger quantiles; there is more spread on the right-hand side than on the left. This indicates that precision drops off rapidly when extrapolating into the upper tail of the time-to-failure distribution.

The following points can be seen most clearly in graphs of $\mathrm{VF}^{[\mathrm{FTA}]}(p)$ versus P_F which, to save space, are not shown here.

- With continuous inspection, $\mathrm{VF}^{[\mathrm{FTA}]}(p)$ is strictly decreasing as a function of P_F because as more test units fail, more information about the time-to-failure distribution becomes available and, hence, we can estimate the quantile of the time-to-failure distribution more precisely.
- With only interval-information on the time to failure, $\mathrm{VF}^{[\mathrm{FTA}]}(p)$ decreases as P_F increases over most of the range of P_F, but it is possible for $\mathrm{VF}^{[\mathrm{FTA}]}(p)$ to increase slightly beginning at some point after P_F exceeds p. This is a result of

the limited information from the discrete inspection data. For example, in the extreme case with only $m = 2$ inspections, as the last inspection time becomes large, the resulting data from that inspection is unimportant to estimate a small quantile. This is the reason that there is some crossing of lines in the plot of $\mathrm{VF}^{[\mathrm{FTA}]}(p)$ versus p in Figure 1.

The plots in Figure 1 also shows, for DA, that

- As with the FTA plots, the plots of $\mathrm{VF}^{[\mathrm{DA}]}(p)$ versus p also have a "U" shape.
- $\mathrm{VF}^{[\mathrm{DA}]}(p)$ increases as P_F increases. This general behavior is opposite to that for the FTA. This is because increasing P_F is equivalent to increasing the test length and, hence, with constant m, reducing the expected number of inspections before the actual degradation crosses D_c. The effect is smaller when the measurement error ratio R_σ is small.

Figures 2 to 3, which focus on the RE of DA versus FTA, show that

- RE usually decreases as P_F increases. This is because, as explained earlier, increasing the test length with the same number of inspections generally provides more information for FTA, but less information for DA.
- Especially when R_σ is small, RE is much larger when estimating quantiles in the upper tail of the time-to-failure distribution, particularly for smaller P_F. This shows the advantage of DA over FTA when extrapolating into the upper tail of the time-to-failure distribution.
- From Figure 2, when the variability ratio is small (e.g., $R_\sigma = .1$ or $.5$), the number of inspections has little effect on RE.
- With large R_σ, RE drops well below 1 for some combinations of P_F and p.
- In Figure 3, however, we see that the number of inspections can compensate for a large value of R_σ (e.g., $R_\sigma = 5$). With high measurement error variability, for many combinations of P_F and p, DA can provide better precision than FTA only if the number of inspections is large enough. Meeker (1986) shows that increasing m beyond 10 has little effect on the precision of FTA estimates.
- Figure 3 also shows that RE begins to decrease with p, particularly for large p. This is because, as seen in Figure 1, $\mathrm{VF}^{[\mathrm{DA}]}(p)$ increases with p more rapidly than $\mathrm{VF}^{[\mathrm{FTA}]}(p)$ for large p.

5. Concluding Remarks

In this paper we compare DA and FTA methods in terms of asymptotic variance factors and relative efficiency of the estimators of quantiles of time-to-failure distribution. From the figures and discussion, we can summarize the comparison as follows

- As the P_F increases, there are different effects for FTA and DA. For FTA the expected proportion of failures increases with P_F, increasing precision. For the DA, however, the expected number of inspections before failure decreases as P_F increases, causing a decrease in precision.

- Even with large measurement error variability, DA performs better than the FTA, provided there are enough inspections to compensate for the measurement error variance.
- The results show that DA is especially better suited than the FTA to make inference on the quantiles that are larger than P_F.

The advantages of DA do not come entirely for free. For FTA we need to specify a time-to-failure distribution. For DA we need to specify a degradation model, implying a time-to-failure distribution. In either case data can be used to assess the adequacy of the model *within the range of the data*. In terms of statistical efficiency, DA has its biggest advantages when estimating quantiles of failure probabilities *beyond the range of the data*. With either FTA or DA, there will be potential for substantial model error (or bias) in estimates that extrapolate beyond the range of the data. Because DA offers more precision in these estimates we could, outside the range of the data, expect less robustness with the use of an inadequate model. On the other hand, degradation modeling should, whenever possible, be tied closely to the physics of failure, providing more confidence for degradation models than is typical in the commonly used curve-fitting techniques of FTA (which are, of course, adequate for interpolative inferences).

Although working with different data and model assumptions, our conclusions are consistent with those of Suzuki, Maki, and Yokogawa (1993). Also, their results, based on a model with terms allowing for fixed-effect curvature in the degradation paths, suggest that similar conclusions would be obtained if our setup were extended to include terms for curvature in the degradation paths.

In this paper we have provided evaluations of asymptotic variance factors for a range of degradation testing situations, giving a general picture of the effect that the various factors have on estimation precision. To answer specific questions about the design of such experiments (i.e., length of test and sample size) it is useful to do specific computations of asymptotic variances. Also, at the expense of having to use more computer time, asymptotic evaluations can be supplemented with Monte Carlo simulation to allow evaluations without having to rely on asymptotic approximations.

In our comparisons we have not accounted for the fact that there may be different costs for obtaining time-to-failure and degradation data. In many (but not all) situations, degradation measurements result in more expense. The cost varies, depending on the situation. Often the larger part of the cost is in metrology research or needed capital equipment needed to make measurements. In many other situations it involves painstaking disassembly and measurement. In many electronic tests, however, the cost of taking measurements is extremely low because the measurement work is done by computer recording of the signals that directly input from test equipment.

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Appendix

A. Expected Values, Variances and Covariances for the Two Stage Estimators

In this appendix we derive the asymptotic expected values and variance-covariance matrices of the two–stage estimators $\hat{\mu}_X^{[\mathrm{DA}]}, \hat{\sigma}_X^{[\mathrm{DA}]}$. Recall that $\hat{\mu}_X^{[\mathrm{DA}]} = D_c - \hat{\mu}_\Theta$ and $(\hat{\sigma}_X^{[\mathrm{DA}]})^2 = \hat{\sigma}_\Theta^2 = S_\Theta^2 - \hat{\sigma}_{\hat{\Theta}}^2$, where $\hat{\mu}_\Theta = \sum_{i=1}^n \hat{\Theta}_i/n$, $S_\Theta^2 = \sum_{i=1}^n (\hat{\Theta}_i - \hat{\mu}_\Theta)^2/(n-1)$, and $\hat{\sigma}_{\hat{\Theta}}^2 = \sum_{i=1}^n \hat{\sigma}_{\hat{\Theta}_i}^2/n$. For any fixed i, $\hat{\Theta}_i$ and $\hat{\sigma}_{\hat{\Theta}_i}^2$ are independent. Also, units i and k ($i \neq k$) are independent, which implies, for example, that $\mathrm{Cov}(\Theta_i, \Theta_k) = \mathrm{Cov}(\hat{\Theta}_i, \hat{\Theta}_k) = \mathrm{Cov}(\hat{\Theta}_i, \hat{\sigma}_{\hat{\Theta}_k}^2) = 0$.

A.1. Expected values

First, we show that $\hat{\mu}_X^{[\mathrm{DA}]}, (\hat{\sigma}_X^{[\mathrm{DA}]})^2$ are unbiased estimates of μ_X, σ_X^2.

$$\mathrm{E}\,(\hat{\mu}_X^{[\mathrm{DA}]}) = D_c - \mathrm{E}\,(\hat{\Theta}_i) = D_c - \mathrm{E}_\Theta\Big[\mathrm{E}_\epsilon(\hat{\Theta}_i|\Theta_i)\Big] = D_c - \mathrm{E}_\Theta(\Theta_i) = D_c - \mu_\Theta = \mu_X,$$
$$\mathrm{E}\Big[(\hat{\sigma}_X^{[\mathrm{DA}]})^2\Big] = \mathrm{E}\,(S_\Theta^2) - \mathrm{E}\,(\hat{\sigma}_{\hat{\Theta}}^2) = \sigma_\Theta^2 + \sigma_{\hat{\Theta}}^2 - \sigma_{\hat{\Theta}}^2 = \sigma_\Theta^2 = \sigma_X^2.$$

Using a Taylor series expansion, up to terms of order n^{-1}, $\hat{\sigma}_X^{[\mathrm{DA}]} = \sigma_\Theta + (\hat{\sigma}_\Theta^2 - \sigma_\Theta^2)/(2\sigma_\Theta)$.

A.2. Variance-covariance matrices

In this section we show that, up to terms of order n^{-1},

$$\mathrm{Cov}\begin{pmatrix} \hat{\mu}_X^{[\mathrm{DA}]} \\ \hat{\sigma}_X^{[\mathrm{DA}]} \end{pmatrix} = \frac{\sigma_\Theta^2}{n}\begin{bmatrix} V_{11}^{[\mathrm{DA}]} & V_{12}^{[\mathrm{DA}]} \\ V_{12}^{[\mathrm{DA}]} & V_{22}^{[\mathrm{DA}]} \end{bmatrix},$$

where
$$V_{11}^{[\mathrm{DA}]} = 1 + \frac{\sigma_{\hat{\Theta}}^2}{\sigma_\Theta^2} \tag{1}$$
$$V_{12}^{[\mathrm{DA}]} = -\frac{1}{\sigma_\Theta^3}\mathrm{Cov}_\Theta(\Theta_i, \sigma_{\hat{\Theta}_i}^2) \tag{2}$$
$$V_{22}^{[\mathrm{DA}]} = \frac{1}{2\sigma_\Theta^4}\Big\{(\sigma_\Theta^2 + \sigma_{\hat{\Theta}}^2)^2 + \mathrm{Var}_\Theta(\sigma_{\hat{\Theta}_i}^2) + 2\mathrm{Cov}_\Theta\Big[(\Theta_i - \mu_\Theta)^2, \sigma_{\hat{\Theta}_i}^2\Big] + \mathrm{E}_\Theta\Big(\frac{\sigma_{\hat{\Theta}_i}^4}{m_{\Theta_i} - 1}\Big)\Big\}. \tag{3}$$

Note that $\sigma^2_{\hat{\Theta}_i} = \mathrm{Var}_\epsilon(\hat{\Theta}_i|\Theta_i) = \sigma^2_\epsilon/m_{\Theta_i}$, which is a random variable because m_{Θ_i} depends on the random Θ.

The limiting expressions for $V_{11}^{[\mathrm{DA}]}, V_{12}^{[\mathrm{DA}]}$, and $V_{22}^{[\mathrm{DA}]}$, as $\sigma^2_{\hat{\Theta}} \to 0$ (which happens when $m \to \infty$ or $\sigma_\epsilon \to 0$) are $V_{11}^{[\mathrm{DA}]} = 1$, $V_{12}^{[\mathrm{DA}]} = 0$, and $V_{22}^{[\mathrm{DA}]} = 1/2$. As expected, these limiting values are equal to the values for the corresponding components $V_{11}^{[\mathrm{FTA}]}, V_{12}^{[\mathrm{FTA}]}$, and $V_{22}^{[\mathrm{FTA}]}$ in the failure-time analysis case when $P_F = 1$ (no right censoring) and $m = \infty$ (continuous inspection). To derive the $V_{ij}^{[\mathrm{DA}]}$, we made repeated use of the following results:

- Let W, R, S be given random variables that have finite second moments. Then (see, Searle, Casella and McCulloch (1992, page 461)):

$$\mathrm{Cov}(W,R) = \mathrm{Cov}[\mathrm{E}(W|S), \mathrm{E}(R|S)] + \mathrm{E}[\mathrm{Cov}(W,R|S)] \quad (4)$$
$$\mathrm{Var}(W) = \mathrm{Var}[E(W|S)] + E[\mathrm{Var}(W|S)] \quad (5)$$

- If W has mean μ_W, variance σ^2_W, and finite third moment, then $\mathrm{Cov}(W, W^2) = \mathrm{E}[(W-\mu_W)^3] + 2\sigma^2_W\mu_W$. To show this, expand the third moment on the right-hand side and simplify. Then, when W is symmetrically distributed, $\mathrm{E}[(W-\mu_W)^3] = 0$ and $\mathrm{Cov}(W, W^2) = 2\sigma^2_W\mu_W$.

Derivation of $\mathrm{Var}(\hat{\mu}_X^{[\mathrm{DA}]})$

$$\begin{aligned}
\mathrm{Var}(\hat{\mu}_X^{[\mathrm{DA}]}) &= \mathrm{Var}(D_c - \hat{\mu}_\Theta) = \mathrm{Var}(\hat{\mu}_\Theta) = \frac{1}{n}\mathrm{Var}(\hat{\Theta}_i) \\
&= \frac{1}{n}\Big\{\mathrm{Var}_\Theta[\mathrm{E}_\epsilon(\hat{\Theta}_i|\Theta_i)] + \mathrm{E}_\Theta[\mathrm{Var}_\epsilon(\hat{\Theta}_i|\Theta_i)]\Big\} \\
&= \frac{1}{n}\Big\{\mathrm{Var}_\Theta(\Theta_i) + \mathrm{E}_\Theta(\sigma^2_{\hat{\Theta}_i})\Big\} = \frac{1}{n}(\sigma^2_\Theta + \sigma^2_{\hat{\Theta}}) \\
&= \frac{\sigma^2_\Theta}{n} V_{11}^{[\mathrm{DA}]}.
\end{aligned}$$

Derivation of $\mathrm{Cov}(\hat{\mu}_X^{[\mathrm{DA}]}, \hat{\sigma}_X^{[\mathrm{DA}]})$

Using the delta method approximation, for large n, one gets

$$\begin{aligned}
\mathrm{Cov}(\hat{\mu}_X^{[\mathrm{DA}]}, \hat{\sigma}_X^{[\mathrm{DA}]}) &= \mathrm{Cov}(D_c - \hat{\mu}_\Theta, \hat{\sigma}_\Theta) = -\mathrm{Cov}(\hat{\mu}_\Theta, \hat{\sigma}_\Theta) \approx -\frac{1}{2\sigma_\Theta}\mathrm{Cov}(\hat{\mu}_\Theta, \hat{\sigma}^2_\Theta) \\
&= -\frac{1}{2\sigma_\Theta}\mathrm{Cov}(\hat{\mu}_\Theta, S^2_\Theta - \hat{\sigma}^2_{\hat{\Theta}}) \\
&= -\frac{1}{2\sigma_\Theta}\Big[\mathrm{Cov}(\hat{\mu}_\Theta, S^2_\Theta) - \mathrm{Cov}(\hat{\mu}_\Theta, \hat{\sigma}^2_{\hat{\Theta}})\Big]. \quad (6)
\end{aligned}$$

To simplify the expression on the right-hand side, observe that

$$
\begin{aligned}
\mathrm{Cov}(\hat{\mu}_\Theta, S^2_\Theta) &= \mathrm{Cov}\Big[\hat{\mu}_\Theta, \frac{1}{n-1}\Big(\sum_{i=1}^{n}\hat{\Theta}_i^2 - n\hat{\mu}^2_\Theta\Big)\Big] \\
&\approx \frac{1}{n}\Big[\sum_{i=1}^{n}\mathrm{Cov}(\hat{\mu}_\Theta, \hat{\Theta}_i^2) - n\mathrm{Cov}(\hat{\mu}_\Theta, \hat{\mu}^2_\Theta)\Big] \\
&= \frac{1}{n}\Big[\frac{1}{n}\sum_{i=1}^{n}\mathrm{Cov}(\hat{\Theta}_i, \hat{\Theta}_i^2) - n\mathrm{Cov}(\hat{\mu}_\Theta, \hat{\mu}^2_\Theta)\Big] \\
&= \frac{1}{n}\Big[\mathrm{Cov}(\hat{\Theta}_i, \hat{\Theta}_i^2) - n\mathrm{Cov}(\hat{\mu}_\Theta, \hat{\mu}^2_\Theta)\Big] \\
&\approx \frac{3}{n}\mathrm{Cov}_\Theta(\Theta_i, \sigma^2_{\hat{\Theta}_i}).
\end{aligned} \tag{7}
$$

Equation (7) follows from

$$
\begin{aligned}
\mathrm{Cov}(\hat{\Theta}_i, \hat{\Theta}_i^2) &= \mathrm{E}\left[(\hat{\Theta}_i - \mu_\Theta)^3\right] + 2\mu_\Theta(\sigma^2_\Theta + \sigma^2_{\hat{\Theta}}) \\
&= 3\mathrm{Cov}_\Theta(\Theta_i, \sigma^2_{\hat{\Theta}_i}) + 2\mu_\Theta(\sigma^2_\Theta + \sigma^2_{\hat{\Theta}})
\end{aligned}
$$

and the approximation (obtained by ignoring terms of order n^{-2})

$$
\begin{aligned}
\mathrm{Cov}(\hat{\mu}_\Theta, \hat{\mu}^2_\Theta) &= \mathrm{E}\left[(\hat{\mu}_\Theta - \mu_\Theta)^3\right] + \frac{2}{n}(\sigma^2_\Theta + \sigma^2_{\hat{\Theta}})\mu_\Theta \\
&= \frac{1}{n^2}\mathrm{E}\left[(\hat{\Theta}_i - \mu_\Theta)^3\right] + \frac{2}{n}(\sigma^2_\Theta + \sigma^2_{\hat{\Theta}})\mu_\Theta \\
&\approx \frac{2}{n}(\sigma^2_\Theta + \sigma^2_{\hat{\Theta}})\mu_\Theta.
\end{aligned}
$$

Now,

$$
\begin{aligned}
\mathrm{Cov}(\hat{\mu}_\Theta, \hat{\sigma}^2_{\hat{\Theta}}) &= \mathrm{Cov}\Big(\frac{1}{n}\sum_{i=1}^{n}\hat{\Theta}_i, \frac{1}{n}\sum_{i=1}^{n}\hat{\sigma}^2_{\hat{\Theta}_i}\Big) = \frac{1}{n^2}\sum_{i=1}^{n}\mathrm{Cov}(\hat{\Theta}_i, \hat{\sigma}^2_{\Theta_i}) \\
&= \frac{1}{n}\mathrm{Cov}_\Theta(\Theta_i, \sigma^2_{\hat{\Theta}_i})
\end{aligned} \tag{8}
$$

which holds because $\mathrm{Cov}(\hat{\Theta}_i, \hat{\sigma}^2_{\hat{\Theta}_i}) = \mathrm{Cov}_\Theta[\mathrm{E}_\epsilon(\hat{\Theta}_i|\Theta_i), \mathrm{E}_\epsilon(\hat{\sigma}^2_{\hat{\Theta}_i}|\Theta_i)] + \mathrm{E}_\Theta[\mathrm{Cov}_\epsilon(\hat{\Theta}_i, \hat{\sigma}^2_{\hat{\Theta}_i}|\Theta_i)] = \mathrm{Cov}_\Theta(\Theta_i, \sigma^2_{\hat{\Theta}_i})$. Substituting (7) and (8) into (6) and simplifying gives

$$
\mathrm{Cov}(\hat{\mu}_X^{[\mathrm{DA}]}, \hat{\sigma}_X^{[\mathrm{DA}]}) = \frac{\sigma^2_\Theta}{n}\Big[-\frac{1}{\sigma^3_\Theta}\mathrm{Cov}_\Theta(\Theta_i, \sigma^2_{\hat{\Theta}_i})\Big] = \frac{\sigma^2_\Theta}{n}V_{12}^{[\mathrm{DA}]}.
$$

Derivation of $\mathrm{Var}(\hat{\sigma}_X^{[\mathrm{DA}]})$

$$\begin{aligned}\mathrm{Var}(\hat{\sigma}_X^{[\mathrm{DA}]}) &= \mathrm{Var}(\hat{\sigma}_\Theta) \approx \frac{1}{4\sigma_\Theta^2}\mathrm{Var}(\hat{\sigma}_\Theta^2) = \frac{1}{4\sigma_\Theta^2}\mathrm{Var}(S_\Theta^2 - \hat{\sigma}_{\hat{\Theta}}^2) \\ &= \frac{1}{4\sigma_\Theta^2}\Big[\mathrm{Var}(S_\Theta^2) + \mathrm{Var}(\hat{\sigma}_{\hat{\Theta}}^2) - 2\mathrm{Cov}(S_\Theta^2, \hat{\sigma}_{\hat{\Theta}}^2)\Big]. \end{aligned} \tag{9}$$

Now,

$$\mathrm{Var}\left(S_\Theta^2\right) \approx \frac{1}{n}\Big\{\mathrm{E}\left[(\hat{\Theta}_i - \mu_\Theta)^4\right] - (\sigma_\Theta^2 + \sigma_{\hat{\Theta}}^2)^2\Big\}, \tag{10}$$

where

$$\begin{aligned}&\mathrm{E}\left[(\hat{\Theta}_i - \mu_\Theta)^4\right] \\ &= 3(\sigma_\Theta^2 + \sigma_{\hat{\Theta}}^2)^2 + 3\mathrm{Var}_\Theta(\sigma_{\hat{\Theta}_i}^2) + 6\mathrm{Cov}_\Theta(\Theta_i^2, \sigma_{\hat{\Theta}_i}^2) - 12\mu_\Theta\mathrm{Cov}_\Theta(\Theta_i, \sigma_{\hat{\Theta}_i}^2).\end{aligned}$$

The approximation in (10) ignores terms of order lower than n^{-1} and it follows from the variance of the moment statistics (see Kendall and Stuart (1987, page 322, equation 10.9)).

Using (5), it is easy to see that

$$\mathrm{Var}(\hat{\sigma}_{\hat{\Theta}}^2) = \frac{1}{n}\Big[\mathrm{Var}_\Theta(\sigma_{\hat{\Theta}_i}^2) + 2\mathrm{E}_\Theta(\frac{\sigma_{\hat{\Theta}_i}^4}{m_{\Theta_i} - 1})\Big]. \tag{11}$$

By using the delta method we obtain, for large n, $\mathrm{Cov}(\hat{\mu}_\Theta^2, \hat{\sigma}_{\hat{\Theta}}^2) \approx \mathrm{Cov}(2\mu_\Theta\hat{\mu}_\Theta, \hat{\sigma}_{\hat{\Theta}}^2) = 2\mu_\Theta\mathrm{Cov}(\hat{\mu}_\Theta, \hat{\sigma}_{\hat{\Theta}}^2) = (2/n)\mu_\Theta\mathrm{Cov}_\Theta(\Theta_i, \sigma_{\hat{\Theta}_i}^2)$. Also, we can compute

$$\begin{aligned}\mathrm{Cov}(\hat{\Theta}_i^2, \hat{\sigma}_{\hat{\Theta}_i}^2) &= \mathrm{Cov}_\Theta\Big[\mathrm{E}_\epsilon(\hat{\Theta}_i^2 \mid \Theta_i), \mathrm{E}_\epsilon(\hat{\sigma}_i^2 \mid \Theta_i)\Big] + \mathrm{E}_\Theta\Big[\mathrm{Cov}_\epsilon(\hat{\Theta}_i^2, \hat{\sigma}_i \mid \Theta_i)\Big] \\ &= \mathrm{Cov}_\Theta(\Theta_i^2 + \sigma_{\hat{\Theta}_i}^2, \sigma_{\hat{\Theta}_i}^2) = \mathrm{Cov}_\Theta(\Theta_i^2, \sigma_{\hat{\Theta}_i}^2) + \mathrm{Var}_\Theta(\sigma_{\hat{\Theta}_i}^2).\end{aligned}$$

From these results it follows that

$$\begin{aligned}\mathrm{Cov}(S_\Theta^2, \hat{\sigma}_{\hat{\Theta}}^2) &= \mathrm{Cov}\Big[\frac{1}{n-1}(\sum_{i=1}^n \hat{\Theta}_i^2 - n\hat{\mu}_\Theta^2), \hat{\sigma}_{\hat{\Theta}}^2\Big] \\ &\approx \frac{1}{n}\Big[\mathrm{Cov}(\sum_{i=1}^n \hat{\Theta}_i^2, \hat{\sigma}_{\hat{\Theta}}^2) - \mathrm{Cov}(n\hat{\mu}_\Theta^2, \hat{\sigma}_{\hat{\Theta}}^2)\Big] \\ &= \frac{1}{n}\Big[\mathrm{Cov}(\hat{\Theta}_i^2, \hat{\sigma}_{\hat{\Theta}_i}^2) - n\mathrm{Cov}(\hat{\mu}_\Theta^2, \hat{\sigma}_{\hat{\Theta}}^2)\Big] \\ &\approx \frac{1}{n}\Big[\mathrm{Cov}_\Theta(\Theta_i^2, \sigma_{\hat{\Theta}_i}^2) + \mathrm{Var}_\Theta(\sigma_{\hat{\Theta}_i}^2) - 2\mu_\Theta\mathrm{Cov}_\Theta(\Theta_i, \sigma_{\hat{\Theta}_i}^2)\Big]. \end{aligned} \tag{12}$$

After substituting (10)-(12) into (9) and simplifying, we obtain $\mathrm{Var}(\hat{\sigma}_X^{[\mathrm{DA}]}) = (\sigma_\Theta^2/n)V_{22}^{[\mathrm{DA}]}$.

B. Computational Formulas for the Factors $V_{ij}^{[\mathrm{DA}]}$

Observe that (1)-(3) can be expressed as follows:

$$V_{11}^{[\mathrm{DA}]} = 1 + \frac{R_\sigma^2}{m}\mathrm{E}_\Theta(r_{\Theta_i}), \tag{13}$$

$$V_{12}^{[\mathrm{DA}]} = \frac{R_\sigma^2}{m}\mathrm{Cov}_\Theta(Z_i, r_{\Theta_i}), \tag{14}$$

$$\begin{aligned} V_{22}^{[\mathrm{DA}]} = & \frac{1}{2}\Big[\Big(1+\frac{R_\sigma^2}{m}\Big)^2 + \Big(\frac{R_\sigma^2}{m}\Big)^2\mathrm{Var}_\Theta(r_{\Theta_i}) \\ & + 2\frac{R_\sigma^2}{m}\mathrm{Cov}_\Theta(Z_i^2, r_{\Theta_i}) + \Big(\frac{R_\sigma^2}{m}\Big)^2\mathrm{E}_\Theta\Big(\frac{r_{\Theta_i}^2}{m_{\Theta_i}-1}\Big)\Big], \end{aligned} \tag{15}$$

where $R_\sigma = \sigma_\epsilon/\sigma_\Theta$, $r_{\Theta_i} = m/m_{\Theta_i}$, and $Z_i = -(\Theta_i - \mu_\Theta)/\sigma_\Theta = (X_i - \mu_X)/\sigma_X$ is the standardized log time-to-failure of the ith unit. For example, using $\sigma_{\hat{\Theta}_i}^2 = \sigma_\epsilon^2/m_{\Theta_i}$, one gets

$$V_{12}^{[\mathrm{DA}]} = -\frac{1}{\sigma_\Theta^3}\mathrm{Cov}_\Theta(\Theta_i, \sigma_{\hat{\Theta}_i}^2) = \frac{\sigma_\epsilon^2}{m\sigma_\Theta^2}\mathrm{Cov}_\Theta\Big(-\frac{\Theta_i - \mu_\Theta}{\sigma_\Theta}, \frac{m}{m_{\Theta_i}}\Big) = \frac{R_\sigma^2}{m}\mathrm{Cov}_\Theta(Z_i, r_{\Theta_i}).$$

All expected values, covariances, and variances on the right-hand side of (13)-(15) can be obtained from $\mathrm{E}_\Theta[r_{\Theta_i}^2/(m_{\Theta_i}-1)]$ and expectations of the form $\mathrm{E}_\Theta(Z_i^l r_{\Theta_i}^k)$, where $l = 0,1,2$ and $k = 0,1,2$. For example, $\mathrm{Cov}_\Theta(Z_i, r_{\Theta_i}) = \mathrm{E}_\Theta(Z_i r_{\Theta_i}) - \mathrm{E}_\Theta(Z_i)\mathrm{E}_\Theta(r_{\Theta_i})$ and

$$\mathrm{E}_\Theta\Big(\frac{r_{\Theta_i}^2}{m_{\Theta_i}-1}\Big) = \sum_{j=2}^{m}\Big(\frac{m}{j}\Big)^2\frac{1}{j-1}\int_{A_j}\phi(z)dz \tag{16}$$

$$\mathrm{E}_\Theta(Z_i^l r_{\Theta_i}^k) = \sum_{j=2}^{m}\Big(\frac{m}{j}\Big)^k\int_{A_j} z^l\phi(z)dz, \tag{17}$$

where $\phi(\cdot)$ is the standard normal pdf, $A_2 = \{z \le z_2\}$, $A_j = \{z_{j-1} < z \le z_j\}$ for $j = 3,\dots,m-1$, $A_m = \{z > z_{m-1}\}$, and $z_2,\dots,z_{m-1}$ are standardized log inspection times defined by $z_j = (x_j - \mu_X)/\sigma_X$. All of the integrals needed for (16) and (17) can be computed easily from the formulas in the following table. Columns 3 and 4 of this table were obtained by substituting the standard normal density and integrating by parts.

j	$\int_{A_j}\phi(z)dz$	$\int_{A_j}z\phi(z)dz$	$\int_{A_j}z^2\phi(z)dz$
2	$\Phi(z_2)$	$-\phi(z_2)$	$-z_2\phi(z_2) + \Phi(z_2)$
$3,\dots,m-1$	$\Phi(z_j) - \Phi(z_{j-1})$	$-[\phi(z_j) - \phi(z_{j-1})]$	$-[z_j\phi(z_j) - z_{j-1}\phi(z_{j-1})]$ $+[\Phi(z_j) - \Phi(z_{j-1})]$
m	$1 - \Phi(z_{m-1})$	$\phi(z_{m-1})$	$z_{m-1}\phi(z_{m-1}) + [1 - \Phi(z_{m-1})]$

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