Sharp Minimax Estimation of the Variance of Brownian Motion Corrupted with Gaussian Noise: Supplementary Material

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Supplementary Material

This note provides details of proofs and supplementary technicalities for the paper "Sharp Minimax Estimation of the Variance of Brownian Motion Corrupted with Gaussian Noise".

S1 Additional Lemmas for the Risk Estimation of $\hat{\sigma}^2$

Notation: We suppress the index n and for two sequences $(a_n)_n$ and $(b_n)_n$ we use the notation $a_n \ll b_n$ if $a_n = o(b_n)$.

Lemma S1.1. *Let* $A_n := A_n(k,m) := \sum_{i=k+1}^{m}(\sigma^2 + \tau^2\lambda_i^{-1})^{-2}$ *and* $C_n := A_n(1,n)$, *where* λ_i *is as defined in (2.2). Then for any* $\epsilon > 0$

(i)

$$\sup_{\sigma,\tau>\epsilon}\left|C_n - \frac{1}{4\tau\sigma^3}n^{1/2}\right| = o\left(n^{1/2}\right),$$

$$\sup_{\sigma,\tau>\epsilon}\frac{1}{\sigma^5\tau^5}\left|C_n^{-1} - 4\tau\sigma^3 n^{-1/2}\right| = o\left(n^{-1/2}\right), \tag{S1.1}$$

(ii) and if $k \ll n^{1/2} \ll m$ *also*

$$\sup_{\sigma,\tau>\epsilon}\frac{1}{\sigma^8\tau^8}\left|A_n^{-1} - 4\tau\sigma^3 n^{-1/2}\right| = o\left(n^{-1/2}\right),$$

and

$$\sup_{\sigma,\tau>\epsilon}\frac{1}{\sigma^4\tau^4}A_n^{-1} = O\left(n^{-1/2}\right).$$

Proof. *(i)* Let us fix the notation

$$I_n := 2n\int_0^{1/2}\frac{1}{\left(\sigma^2+\tau^2 4n\sin^2(x\pi)\right)^2}dx = \frac{32n^3\tau^4\sigma^2 + 10n^2\tau^2\sigma^4 + 32n^4\tau^6 + n\sigma^6}{\left(\sigma^2+4n\tau^2\right)^{7/2}\sigma^3}.$$

By Taylor expansion and monotonicity in σ^2, we have

$$\left(\sigma^2+4n\tau^2\right)^{7/2}-\left(4n\tau^2\right)^{7/2}\leq 7\left(\sigma^2+4n\tau^2\right)^{5/2}\sigma^2. \tag{S1.2}$$

Note that Lemma S2.2 implies for $n\geq 2$

$$\begin{aligned}
\sup_{\sigma,\tau>\epsilon}\left|C_n-\frac{1}{4\tau\sigma^3}n^{1/2}\right| &\leq O(\log n)+\sup_{\sigma,\tau>\epsilon}\left|I_n-\frac{1}{4\tau\sigma^3}n^{1/2}\right| \\
&= O(\log n)+\sup_{\sigma,\tau>\epsilon}\left|\frac{32n^3\tau^4\sigma^2+10n^2\tau^2\sigma^4+32n^4\tau^6+n\sigma^6}{\left(\sigma^2+4n\tau^2\right)^{7/2}\sigma^3}-\frac{1}{4\tau\sigma^3}n^{1/2}\right| \\
&\leq O(\log n)+\frac{1}{\epsilon\left(1+4n\right)^{1/2}}\sup_{\sigma,\tau>\epsilon}\left|\frac{32n^3\tau^4+10n^2\tau^2\sigma^2+n\sigma^4}{\left(\sigma^2+4n\tau^2\right)^3\sigma}\right| \\
&\quad+\sup_{\sigma,\tau>\epsilon}\frac{1}{\sigma^3}\left|\frac{\left(\sigma^2+4n\tau^2\right)^{7/2}-2^7\tau^7n^4}{4\tau\left(\sigma^2+4n\tau^2\right)^{7/2}}\right| \\
&= O(\log n)+\sup_{\sigma,\tau>\epsilon}\frac{7n^{1/2}}{4\sigma\tau\left(\sigma^2+4n\tau^2\right)}=O(\log n).
\end{aligned}$$

Finally we show (S1.1). Note

$$\left|C_n^{-1}-4\tau\sigma^3n^{-1/2}\right|\leq\left|C_n^{-1}-I_n^{-1}\right|+\left|I_n^{-1}-4\tau\sigma^3n^{-1/2}\right| \tag{S1.3}$$

and by Lemma S2.2 for $n\geq 2$

$$\left|C_n^{-1}-I_n^{-1}\right|\leq 16\log n\;\sigma^{-4}I_n^{-1}C_n^{-1}. \tag{S1.4}$$

By the Cauchy-Schwarz inequality we have for all $k<m$

$$C_n^{-1}\leq A_n(k,m)^{-1}\leq\sum_{i=k+1}^{m}t_i^2\frac{\left(\sigma^2\lambda_i+\tau^2\right)^2}{\lambda_i^2},\quad\text{whenever}\quad\sum_{i=k+1}^{m}t_i=1. \tag{S1.5}$$

Hence with Lemma S2.1 it follows for k, m, $k\ll n^{1/2}\ll m$, n sufficiently large

$$\begin{aligned}
A_n(k,m)^{-1} &\leq\sum_{i=\left[n^{1/2}\right]+1}^{2\left[n^{1/2}\right]}\left[n^{1/2}\right]^{-2}\left(\sigma^2+\tau^2\lambda_i^{-1}\right)^2 \\
&\leq 2\left[n^{1/2}\right]^{-1}\left(\sigma^4+\tau^4\lambda_{2\left[n^{1/2}\right]}^{-2}\right)\leq 4n^{-1/2}\left(\sigma^4+16\pi^4\tau^4\right)
\end{aligned} \tag{S1.6}$$

and $C_n^{-1}\leq 4n^{-1/2}\left(\sigma^4+16\pi^4\tau^4\right)$. We now estimate $(\sigma\tau)^{-5}\left|C_n^{-1}-I_n^{-1}\right|$ using (S1.4) and $(a+b)^r\leq 2^r\left(a^r+b^r\right)$ for $a,b,r\geq 0$, as

$$\sup_{\sigma,\tau>\epsilon}\frac{16\log n}{\sigma^9\tau^5}I_n^{-1}C_n^{-1}\leq\sup_{\sigma,\tau>\epsilon}\frac{2^{7/2}64\log n\left(\sigma^7+2^7n^{7/2}\tau^7\right)n^{-1/2}\left(\sigma^4+16\pi^4\tau^4\right)}{\sigma^6\tau^5\left(32n^3\tau^4\sigma^2+10n^2\tau^2\sigma^4+32n^4\tau^6+n\sigma^6\right)}$$

and some elementary calculations finally yield

$$\sup_{\sigma,\tau>\epsilon} \frac{1}{\sigma^5\tau^5} \left|C_n^{-1} - I_n^{-1}\right| = O\left(n^{-1}\log n\right).$$

Note in order to bound the second term in (S1.3)

$$\begin{aligned}
\left|I_n^{-1} - 4\tau\sigma^3 n^{-1/2}\right| &\leq \left|\frac{\left(\sigma^2+4n\tau^2\right)^{7/2}\sigma^3}{32n^3\tau^4\sigma^2+10n^2\tau^2\sigma^4+32n^4\tau^6+n\sigma^6} - 4\tau\sigma^3 n^{-1/2}\right| \\
&\leq \left|\frac{\left(\left(\sigma^2+4n\tau^2\right)^{7/2} - \left(4n\tau^2\right)^{7/2}\right)\sigma^3}{32n^3\tau^4\sigma^2+10n^2\tau^2\sigma^4+32n^4\tau^6+n\sigma^6}\right| \\
&\quad + \left|\frac{\left(4n\tau^2\right)^{7/2}\sigma^3}{32n^3\tau^4\sigma^2+10n^2\tau^2\sigma^4+32n^4\tau^6+n\sigma^6} - 4\tau\sigma^3 n^{-1/2}\right|.
\end{aligned}$$

Using (S1.2) yields

$$\begin{aligned}
&\sup_{\sigma,\tau>\epsilon} \frac{1}{\sigma^5\tau^5} \left|\frac{\left(\left(\sigma^2+4n\tau^2\right)^{7/2} - \left(4n\tau^2\right)^{7/2}\right)\sigma^3}{32n^3\tau^4\sigma^2+10n^2\tau^2\sigma^4+32n^4\tau^6+n\sigma^6}\right| \\
&\leq 2^{5/2}7 \sup_{\sigma,\tau>\epsilon} \frac{1}{\sigma^5\tau^5} \left|\frac{\sigma^{10}+2^5 n^{5/2}\tau^5\sigma^5}{32n^4\tau^6+n\sigma^6}\right| = O\left(n^{-1}\right).
\end{aligned}$$

Finally,

$$\begin{aligned}
&\sup_{\sigma,\tau>\epsilon} \frac{1}{\sigma^5\tau^5} \left|\frac{\left(4n\tau^2\right)^{7/2}\sigma^3}{32n^3\tau^4\sigma^2+10n^2\tau^2\sigma^4+32n^4\tau^6+n\sigma^6} - 4\tau\sigma^3 n^{-1/2}\right| \\
&= \sup_{\sigma,\tau>\epsilon} \frac{4n^{-1/2}}{\sigma^2\tau^4} \left|\frac{32n^3\tau^4\sigma^2+10n^2\tau^2\sigma^4+n\sigma^6}{32n^3\tau^4\sigma^2+10n^2\tau^2\sigma^4+32n^4\tau^6+n\sigma^6}\right| = O\left(n^{-1}\right).
\end{aligned}$$

(ii) Note that since $C_n^{-1} \leq A_n^{-1}$ and due to *(i)*

$$\begin{aligned}
&\sup_{\sigma,\tau>\epsilon} \frac{1}{\sigma^8\tau^8} \left|A_n^{-1} - 4\tau\sigma^3 n^{-1/2}\right| \\
&\quad \leq \sup_{\sigma,\tau>\epsilon} \frac{1}{\sigma^8\tau^8} \left|A_n^{-1} - C_n^{-1}\right| + \sup_{\sigma,\tau>\epsilon} \frac{1}{\sigma^8\tau^8} \left|C_n^{-1} - 4\tau\sigma^3 n^{-1/2}\right| \\
&\quad \leq \sup_{\sigma,\tau>\epsilon} \frac{1}{\sigma^8\tau^8} \left(C_n - A_n\right) A_n^{-2} + o\left(n^{-1/2}\right).
\end{aligned}$$

By (S1.6) it holds further for sufficiently large n

$$C_n - A_n = \sum_{i=1}^{k} \left(\sigma^2+\tau^2\lambda_i^{-1}\right)^{-2} + \sum_{i=m+1}^{n} \left(\sigma^2+\tau^2\lambda_i^{-1}\right)^{-2} \leq \sigma^{-4}k + \tau^{-4} \sum_{i=m+1}^{n} \lambda_i^2.$$

This finally yields, applying Lemma S2.1 again,

$$\sup_{\sigma,\tau>\epsilon} \frac{1}{\sigma^8\tau^8} \left|A_n^{-1} - C_n^{-1}\right|$$
$$\leq \sup_{\sigma,\tau>\epsilon} \left(\sigma^{-4}k + \tau^{-4} \sum_{i=m+1}^{n} \lambda_i^2\right) 16n^{-1} \left(\tau^{-4} + 16\pi^4\sigma^{-4}\right)^2 = o\left(n^{-1/2}\right).$$

The second statement follows directly from (S1.6). □

Lemma S1.2. *Let* $k = \left[n^{1/2-b}\right]$ *and* $m = \left[n^{1/2+b}\right]$, $0 < b < 1/2$. *Then for any* $\epsilon > 0$

$$\sup_{\sigma,\tau>\epsilon} (\sigma\tau)^{-8} \left|\mathrm{E}\left(\hat{A}_n^{-1}\right) - A_n^{-1}\right| = o\left((nk)^{-1/2}\right).$$

Proof. Arguing as in (S1.5) yields

$$\hat{A}_n^{-1} \leq \sum_{i=k+1}^{m} t_i^2 \frac{\left(\bar{\sigma}^2\lambda_i + \hat{\tau}^2\right)^2}{\lambda_i^2}, \quad \text{whenever} \quad \sum_{i=k+1}^{m} t_i = 1$$

and with the choice $t_i = A_n^{-1}\lambda_i^2/\left(\sigma^2\lambda_i + \tau^2\right)^2$, $i = k+1, \ldots, m$ we have

$$\hat{A}_n^{-1} \leq A_n^{-2} \sum_{i=k+1}^{m} \frac{\lambda_i^2}{\left(\sigma^2\lambda_i + \tau^2\right)^4} \left(\bar{\sigma}^2\lambda_i + \hat{\tau}^2\right)^2.$$

Hence with similar arguments as in (2.18)

$$\mathrm{E}\left(\hat{A}_n^{-1}\right) \leq A_n^{-1}\left(1 + k^{-1/2}\right)$$
$$+ 2\left(1 + k^{1/2}\right) A_n^{-2} \sum_{i=k+1}^{m} \frac{\lambda_i^2}{\left(\sigma^2\lambda_i + \tau^2\right)^4} \left(\mathrm{MSE}\left(\bar{\sigma}^2\right)\lambda_i^2 + \mathrm{MSE}\left(\hat{\tau}^2\right)\right)$$
$$\leq A_n^{-1}\left[1 + k^{-1/2} + 2\left(1 + k^{1/2}\right)\left(\frac{1}{\sigma^4}\mathrm{MSE}\left(\bar{\sigma}^2\right) + \frac{1}{\tau^4}\mathrm{MSE}\left(\hat{\tau}^2\right)\right)\right].$$

It follows from (2.12) and (2.13) that for sufficient large n,

$$\mathrm{Bias}^2\left(\bar{\sigma}^2\right) \leq \mathrm{Bias}^2\left(\hat{\tau}^2\right), \quad \mathrm{Var}\left(\bar{\sigma}^2\right) \leq \mathrm{Var}\left(\hat{\tau}^2\right) + \frac{2}{k}\left(\sigma^2 + \tau^2\right)^2$$

and hence by (2.6)

$$\sup_{\sigma,\tau>\epsilon} \frac{1}{\sigma^4\tau^4} \mathrm{MSE}\left(\bar{\sigma}^2\right) = O\left(k^{-1}\right). \tag{S1.7}$$

This yields for $\sigma, \tau > \epsilon$

$$\frac{1}{\sigma^8\tau^8}\left|\mathrm{E}\left(\hat{A}_n^{-1}\right) - A_n^{-1}\right|$$
$$\leq \left(\frac{1}{\sigma^4\tau^4}A_n^{-1}\right)\left[k^{-1/2}\epsilon^{-8} + 2\left(1 + k^{1/2}\right)\left(\frac{1}{\sigma^8\tau^4}\mathrm{MSE}\left(\bar{\sigma}^2\right) + \frac{1}{\sigma^4\tau^8}\mathrm{MSE}\left(\hat{\tau}^2\right)\right)\right]$$

and thus $\sup_{\sigma,\tau>\epsilon} (\sigma\tau)^{-8}\left|\mathrm{E}\left(\hat{A}_n^{-1}\right) - A_n^{-1}\right| = O\left(n^{-1/2}k^{-1/2}\right).$ □

Lemma S1.3. *Let* $k = \left[n^{1/2-b}\right]$ *and* $m = \left[n^{1/2+b}\right]$, $0 < b < 1/18$ *and define*

$$\gamma_n := \hat{A}_n^{-2} \sum_{i=k+1}^{m} \frac{\lambda_i^2}{(\bar{\sigma}^2\lambda_i + \hat{\tau}^2)^4} \left[(\sigma^2 - \bar{\sigma}^2)^2\lambda_i^2 + (\tau^2 - \hat{\tau}^2)^2\right].$$

Then for any $\epsilon > 0$

$$\sup_{\sigma,\tau>\epsilon} (\sigma\tau)^{-8} \operatorname{E}(\gamma_n) = O(n^{9b-1}).$$

Proof. We argue with similar techniques as in the proof of Lemma S1.2. Note that

$$\begin{aligned}
&\hat{A}_n^{-2} \sum_{i=k+1}^{m} \frac{\lambda_i^2}{(\bar{\sigma}^2\lambda_i + \hat{\tau}^2)^4} \\
&\leq \hat{A}_n^{-1} A_n^{-2} \sum_{j=k+1}^{m} \frac{\lambda_j^2}{(\sigma^2\lambda_j + \tau^2)^4} \left(\bar{\sigma}^2\lambda_j + \hat{\tau}^2\right)^2 \sum_{i=k+1}^{m} \frac{\lambda_i^2}{(\bar{\sigma}^2\lambda_i + \hat{\tau}^2)^4} \\
&\leq A_n^{-1} \max_{j=k+1,\dots,m} \frac{\left(\bar{\sigma}^2\lambda_j + \hat{\tau}^2\right)^2}{\left(\sigma^2\lambda_j + \tau^2\right)^2} \max_{i=k+1,\dots,m} \frac{1}{\left(\bar{\sigma}^2\lambda_i + \hat{\tau}^2\right)^2} \\
&\leq A_n^{-1} \frac{\lambda_{k+1}^2}{\lambda_m^2} \frac{1}{\left(\sigma^2\lambda_m + \tau^2\right)^2} \leq A_n^{-1} \frac{\lambda_{k+1}^2}{\lambda_m^2} \frac{1}{\tau^4}
\end{aligned}$$

and in the same way

$$\hat{A}_n^{-2} \sum_{i=k+1}^{m} \frac{\lambda_i^4}{\left(\bar{\sigma}^2\lambda_i + \hat{\tau}^2\right)^4} \leq \frac{1}{\sigma^4} \frac{\lambda_{k+1}^2}{\lambda_m^2} A_n^{-1}.$$

This yields with Lemma S1.1, (2.6) and (S1.7)

$$\begin{aligned}
&\sup_{\sigma,\tau>\epsilon} \frac{1}{\sigma^8\tau^8} \operatorname{E}\left(\hat{A}_n^{-2} \sum_{i=k+1}^{m} \frac{\lambda_i^2}{(\bar{\sigma}^2\lambda_i + \hat{\tau}^2)^4} \left[(\sigma^2 - \bar{\sigma}^2)^2\lambda_i^2 + (\tau^2 - \hat{\tau}^2)^2\right]\right) \\
&\leq \sup_{\sigma,\tau>\epsilon} \frac{1}{\sigma^8\tau^8} \left(A_n^{-1} \frac{\lambda_{k+1}^2}{\lambda_m^2} \left(\frac{1}{\sigma^4} \operatorname{MSE}\left(\bar{\sigma}^2\right) + \frac{1}{\tau^4} \operatorname{MSE}\left(\hat{\tau}^2\right)\right)\right) = O\left(n^{-1/2} \frac{m^4}{k^5}\right).
\end{aligned}$$

□

S2 Further Technicalities

Lemma S2.1. *Let* λ_i *as defined in (2.2). Then it holds for all* $n \geq 1$ *and* $i = 1, \dots, n$

$$\pi^{-2} \frac{n}{i^2} \leq \lambda_i \leq 4 \frac{n}{i^2}.$$

Proof. It holds $x\pi/2 \leq \sin(x\pi) \leq x\pi$ whenever $x \in [0, 1/2]$. Set $x_i := (2i-1)/(4n+2)$. Hence

$$\frac{i^2}{4n} \leq \frac{ni^2\pi^2}{(4n+2)^2} \leq nx_i^2\pi^2 \leq \frac{1}{\lambda_i} \leq 4nx_i^2\pi^2 \leq \frac{i^2\pi^2}{n}.$$

□

Lemma S2.2. *Let* $g(x) := 1/\left(\sigma^2 + 4n\tau^2 \sin^2(x\pi)\right)^2$. *Define* $x_i := (2i-1)/(4n+2)$ *and let* $\xi_i \in [(i-1)/(2n), i/(2n)]$. *Then it holds for* $n \geq 2$

$$\sum_{i=1}^{n} |g(x_i) - g(\xi_i)| \leq \frac{16}{\sigma^4} \log n.$$

Proof. Obviously $|g(x_1) - g(\xi_1)| \leq |g(x_1)| + |g(\xi_1)| \leq 2/\sigma^4$. Because $\xi_i \in [(i-1)/(2n), i/(2n)]$ for $i = 1, \ldots, n$, we have by Taylor expansion for a suitable $\eta_i \in [(i-1)/(2n), i/(2n)]$,

$$|g(x_i) - g(\xi_i)| \leq |g'(\eta_i)| \left(\frac{i}{2n} - \frac{i-1}{2n}\right) = 4\tau^2\pi \sin(2\eta_i\pi)\, g(\eta_i)^{3/2}.$$

If $x \in [0, 1/2]$ then $x\pi/2 \leq \sin(x\pi)$. Hence for sufficiently large n

$$\begin{aligned}\sum_{i=2}^{n} |g(x_i) - g(\xi_i)| &\leq \sum_{i=2}^{n} \frac{4\tau^2\pi \sin(2\eta_i\pi)}{3\sigma^4 4n\tau^2 \sin^2(\eta_i\pi)} \\ &\leq \sum_{i=2}^{n} \frac{8}{3\sigma^4 n\eta_i} \leq \frac{16}{3\sigma^4} \sum_{i=1}^{n} \frac{1}{i} \leq \frac{16}{3\sigma^4}(1 + \log n).\end{aligned}$$

□

S3 A Central Limit Theorem

Theorem S3.1. *Let* $\{Z_{mk} : 1 \leq k \leq m\}$ *be a triangular array of i.i.d. random variables with mean* 0 *and variance* σ^2 *and let* c_{mk} *be some regression coefficients that satisfy the Noether condition*

(i)

$$\max_{k=1,\ldots,m} |c_{mk}| \to 0.$$

(ii)

$$\sum_{k=1}^{m} c_{mk}^2 \;\to\; C\,, \tag{S3.1}$$

where C *is a non-zero constant.*

Then it holds that

$$S_m = \sum_{k=1}^{m} c_{mk} Z_{mk} \xrightarrow{\mathcal{D}} N\left(0, C\sigma^2\right).$$

The Noether condition implies Lindeberg's condition and hence the Theorem follows by applying the Lindeberg CLT (Theorem 11.1.1 in Athreya and Lahiri (2006)).

S4 References

Athreya, K. B. and Lahiri, S. N. (2006). *Measure Theory and Probability Theory.* Springer, New York.

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