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POSITIVE FALSE DISCOVERY PROPORTIONS: INTRINSIC BOUNDS AND ADAPTIVE CONTROL

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Supplementary Material

This note contains proofs for Theorems 4.2, 4.3, 5.1 and 5.2. The proofs for the first two theorems are similar to those for the other two. Since the latter ones are more of interest to applications, they will be demonstrated in detail. The proofs of Theorems 4.2 and 4.3 will be outlined afterwards.

S1. Notation

For random variables X_n and Y_n, $X_n \ge_p Y_n$, $X_n \le_p Y_n$ and $X_n \sim_p Y_n$ denote $P(X_n \ge Y_n) \to 1$, $P(X_n \le Y_n) \to 1$ and $X_n/Y_n \xrightarrow{\text{P}} 1$, respectively. The notation $X_n = o_p(Y_n)$ means "$|X_n| \le_p \epsilon |Y_n|$ for any $\epsilon > 0$", whereas $X_n = O_p(Y_n)$ means "for any $\epsilon > 0$, there are $M > 0$ and $n_0 > 0$, such that $P(|X_n| \ge M|Y_n|) < \epsilon$ for all $n \ge n_0$". When it is necessary to indicate the number n of tested hypotheses, we use a superscript. For example, denote by $R^{(n)}$ the number of rejections when there are n null hypotheses.

It will be easier to work with continuous time to prove the theorems. For procedure (5.1), given p-values $\xi_1^{(n)}, \ldots, \xi_n^{(n)}$, the $R_{\tau_n}^{(n)}$ smallest ones are rejected, where

$$\tau_n = \sup_{t\in[0,1]} \left\{ \text{qbin}\left(\Gamma_*(t); R_t^{(n)}, \frac{n(t \vee \xi_{n:k_n})}{R_t^{(n)} \vee k_n} \wedge 1 \right) \le \alpha(R_t^{(n)} \vee 1) \right\}. \qquad \text{(S1.1)}$$

Denote $q_n(t;z) := \text{qbin}(z;\, R_t^{(n)},\, (nt/R_t^{(n)}) \wedge 1)$. For brevity, write $\tau = \tau_n$, $R_t = R_t^{(n)}$ and $V_t = V_t^{(n)}$. By definition, $R = R_\tau$ and $V = V_\tau$. The same relationship holds for the BH procedure (4.4), except that

$$\tau = \tau_n = \sup\{t \in [0,1] : nt \le R_t\}. \qquad \text{(S1.2)}$$

S2. Subcritical case with increasingly sparse false nulls

Define $\eta_t = \eta^{(n)} = nt/R_t$, $\theta_t = \theta_t^{(n)} = t/F_n(t)$ and $\rho = 1/\alpha - 1 > 0$. Then $q_n(z;t) = \text{qbin}(z; R_t,\ \eta_t \wedge 1)$ and by $u_n = \alpha F_n(u_n)$,

$$\theta_{u_n} = \alpha, \quad (\pi_n + \rho)u_n = \pi_n G_n(u_n). \qquad \text{(S2.1)}$$

Under the subcritical condition (5.4), the following lemmas hold.

Lemma S2.1. *For procedure* (5.1) *and the BH procedure* (4.4), $\tau/u_n \xrightarrow{\mathrm{P}} 1$ *and*

$$\text{(a)}\ \frac{G_n(\tau)}{G_n(u_n)} \xrightarrow{\mathrm{P}} 1, \quad \text{(b)}\ \theta_\tau \xrightarrow{\mathrm{P}} \alpha, \quad \text{(c)}\ \eta_\tau \xrightarrow{\mathrm{P}} \alpha, \quad \text{(d)}\ \frac{R_\tau}{(\log n)^4} \xrightarrow{\mathrm{P}} \infty.$$

Lemma S2.2. *For both procedures,* $\theta_\tau R_\tau - n\tau = \alpha R_{u_n} - nu_n + o_p(\sqrt{nu_n})$.

Lemma S2.3. *For procedure* (5.1), $\alpha R_\tau = q_n(\tau; \Gamma(\tau)) + O_p(1)$ *and for the BH procedure* (4.4), $\alpha R_\tau = n\tau$.

Let $R_{t-} = R_{t-}^{(n)}$ and $V_{t-} = V_{t-}^{(n)}$ be the numbers of rejected nulls and rejected true nulls, respectively, whose p-values are strictly less than t.

Lemma S2.4. *Given* $t \in (0,1)$ *and* $k > 0$, *for procedure* (5.1), *conditioning on* $\tau = t$ *and* $R_{\tau-} = k$, $V_{\tau-} \sim \mathrm{Bin}(k, t/F_n(t))$. *The statement holds as well for the BH procedure* (4.4).

Recall that if $p_n \in (0,1)$ satisfies $np_n(1-p_n) \to \infty$, then by Lindeberg's CLT, for $S_n \sim \mathrm{Bin}(n, p_n)$ and $z \in (0,1)$,

$$\frac{S_n - np_n}{\sqrt{np_n(1-p_n)}} \xrightarrow{d} N(0,1), \qquad \frac{\mathrm{qbin}(z;\, n,\, p_n) - np_n}{\sqrt{np_n(1-p_n)}} \to \Phi^*(z). \tag{S2.2}$$

Proof of Theorem 5.1. Assume the Lemmas are true for now. We show (a)−(d) in sequel.

(a) By Lemma S2.1, we get $\eta_\tau \xrightarrow{\mathrm{P}} \alpha$ and $n\tau(1-\eta_\tau) = R_\tau \eta_\tau(1-\eta_\tau) \sim_p \alpha(1-\alpha)R_\tau \xrightarrow{\mathrm{P}} \infty$. Then by Lemma S2.3,

$$\begin{aligned} P(V_\tau \le \alpha R_\tau) &= P\left(V_\tau \le q_n(\tau;\Gamma(\tau)) + O_p(1)\right) + o(1) \\ &= P\left(\frac{V_\tau - n\tau}{\sqrt{n\tau(1-\eta_\tau)}} \le \frac{q_n(\tau;\Gamma(\tau)) - n\tau}{\sqrt{n\tau(1-\eta_\tau)}} + o_p(1)\right) + o(1). \end{aligned}$$

Since $q_n(\tau;\Gamma(\tau)) = \mathrm{qbin}(\Gamma(\tau); R_\tau, \eta_\tau)$ and $n\tau = R_\tau\eta_\tau$, Lindeberg's CLT yields

$$\frac{q_n(\tau;\Gamma(\tau)) - n\tau}{\sqrt{n\tau(1-\eta_\tau)}} \sim_p \Phi^*(\Gamma(\tau)) = \sqrt{\frac{1-\tau}{1-\alpha}}\Phi^*(1-\gamma)\,. \tag{S2.3}$$

Write $(V_\tau - n\tau)/\sqrt{n\tau(1-\eta_\tau)} = Z_1 Z + Z_2$, where

$$Z_1 = \frac{V_\tau - \theta_\tau R_\tau}{\sqrt{\theta_\tau(1-\theta_\tau)R_\tau}}, \quad Z = \sqrt{\frac{\theta_\tau(1-\theta_\tau)R_\tau}{n\tau(1-\eta_\tau)}}, \quad Z_2 = \frac{\theta_\tau R_\tau - n\tau}{\sqrt{n\tau(1-\eta_\tau)}}\,.$$

By Lemma S2.4, conditioning on $\tau = t$ and $R_{\tau-} = k$, $V_{\tau-} \sim \text{Bin}(k, \theta_t)$. By $R_\tau - R_{\tau-}$, $V_\tau - V_{\tau-} \in \{0,1\}$ and $\theta_\tau R_\tau \to \infty$, we get $Z_1 \xrightarrow{d} N(0,1)$. By Lemma S2.1, $Z \xrightarrow{\text{P}} 1$. By Lemma S2.2, $Z_2 = Z_2' + o_p(1)$, where $Z_2' = (\alpha R_{u_n} - nu_n)/\sqrt{n\tau(1-\eta_\tau)}$. From $F_n(u_n) = u_n/\alpha \to 0$, $R_{u_n} \sim \text{Bin}(n, F_n(u_n))$, $\tau/u_n \xrightarrow{\text{P}} 1$ and $\eta_\tau \xrightarrow{\text{P}} \alpha$, it follows that

$$Z_2' = \frac{\alpha(R_{u_n} - nF_n(u_n))}{\sqrt{nF_n(u_n)(1-F_n(u_n))}}\sqrt{\frac{u_n(1-F_n(u_n))}{\alpha\tau\,(1-\eta_\tau)}} \xrightarrow{d} \sqrt{\frac{\alpha}{1-\alpha}}N(0,1)$$

and hence $Z_2 \xrightarrow{d} \sqrt{\alpha/(1-\alpha)}N(0,1)$.

We now show $(Z_1, Z_2) \xrightarrow{d} (U_1, \sqrt{\alpha/(1-\alpha)}\,U_2)$, where U_1, U_2 are i.i.d. $\sim N(0,1)$. Let $f(x,y) = E(e^{ixZ_1+iyZ_2})$. Then by Lemma S2.4 and CLT, for any $a_n \to \infty$ and $t_n \in (0,1)$, as long as $a_n\theta_{t_n}(1-\theta_{t_n}) \to \infty$,

$$\lim_{n\to\infty} E(e^{ixZ_1} \,|\, R_\tau = a_n,\ \tau = t_n) = e^{-x^2/2}.$$

Since Z_2 is a deterministic function of τ and R_τ, by $R_\tau\theta_\tau(1-\theta_\tau) \xrightarrow{\text{P}} \infty$ and dominated convergence,

$$\begin{aligned} E(e^{ixZ_1+iyZ_2}) &= E(E(e^{ixZ_1+iyZ_2} \,|\, R_\tau, \tau)) \\ &\sim e^{-x^2/2}E(e^{iyZ_2}) \to \exp\left\{-\frac{x^2}{2} - \frac{\alpha y^2}{2(1-\alpha)}\right\}. \end{aligned}$$

Combining all the above results, it follows that

$$\frac{V_\tau - n\tau}{\sqrt{n\tau(1-\eta_\tau)}} \xrightarrow{d} \frac{U}{\sqrt{1-\alpha}} \quad \text{with} \quad U \sim N(0,1)\,,$$

which, together with (S2.3) and $\tau \xrightarrow{\text{P}} 0$, implies

$$P(V_\tau \le \alpha R_\tau) \sim P\left(U \le \sqrt{1-\alpha}\Phi^*(\Gamma(\tau))\right) \to 1-\gamma.$$

This completes the proof of part (a).
(b) This directly follows from Lemma S2.1(d).
(c) For the BH procedure (4.4), from (S1.2), $0 \le \alpha R_\tau - n\tau \le \alpha$. By Lemma S2.2, $P\left(V_\tau \le \alpha R_\tau\right) = P\left(V_\tau \le n\tau + O(1)\right) = P\left(Z_1 \le Z_2 + o(1)\right)$, with

$$\begin{aligned} Z_1 &= \frac{V_\tau - \theta_\tau R_\tau}{\sqrt{\theta_\tau(1-\theta_\tau)R_\tau}}, \\ Z_2 &= \frac{n\tau - \theta_\tau R_\tau}{\sqrt{\theta_\tau(1-\theta_\tau)R_\tau}}. \end{aligned}$$

Following the argument for part (a), $(Z_1, Z_2) \xrightarrow{d} (U_1, \sqrt{\alpha/(1-\alpha)}\, U_2)$, where U_1 and U_2 are i.i.d. $\sim N(0,1)$. As a result, $P(V_\tau \le \alpha R_\tau) \to 1/2$.
(d) From (5.4), $n\pi_n \to \infty$. Since $n - N_0 \sim \mathrm{Bin}(n, \pi_n)$, by the law of large numbers (LLN), $(n - N_0)/(n\pi_n) \xrightarrow{\mathrm{P}} 1$. By Lemmas S2.1, S2.2 and S2.4,

$$\frac{R_\tau}{nF_n(u_n)} \xrightarrow{\mathrm{P}} 1, \quad \frac{R_\tau - V_\tau}{R_\tau} \xrightarrow{\mathrm{P}} 1 - \alpha.$$

Therefore, by (S2.1) and $\pi_n \to 0$,

$$\frac{\psi_n}{G_n(u_n)} = \frac{R_\tau - V_\tau}{G_n(u_n)(n - N_0)} \sim_p \frac{(1-\alpha)nF_n(u_n)}{G_n(u_n)n\pi_n} = \frac{\rho u_n}{\pi_n G_n(u_n)} \xrightarrow{\mathrm{P}} 1.$$

The proof for the BH procedure (4.4) is similar and hence is omitted.

To show the lemmas, the following representation of the p-values $\xi_1^{(n)}, \ldots, \xi_n^{(n)}$ will be used. Let $\zeta_k^{(n)} = F_n(\xi_k^{(n)})$. Then $\zeta_1^{(n)}, \ldots, \zeta_n^{(n)}$ are i.i.d. $\sim U(0,1)$. Let

$$W_t := W_t^{(n)} = \#\left\{k \ge 1 : \zeta_k^{(n)} \le t\right\}, \tag{S2.4}$$

so $R_t = W_{F_n(t)}$. Recall the following result Shorack and Wellner (1986, p.600). Let $b_n = \sqrt{2\log\log n}$, $c_n = 2\log\log n + \log\sqrt{\log\log n} - \log\sqrt{4\pi}$ and $Z_t = (W_t - nt)/\sqrt{nt(1-t)}$. Then for any $x \in (-\infty, \infty)$, as $n \to \infty$

$$P\left(b_n \sup_{t\in[0,1]} |Z_t| \ge c_n + x\right) \to e^{-4e^x}. \tag{S2.5}$$

From (5.4), it can be seen that as $n \to \infty$, $nu_n/(\log n)^4 \to \infty$ and

$$\frac{\sqrt{n}}{(\log n)^2} \frac{\pi_n^2\left[\lambda G_n(u_n) - G_n(\lambda u_n)\right]}{\sqrt{u_n}} \to \begin{cases} \infty, & \text{if } \lambda > 1 \\ -\infty, & \text{if } 0 < \lambda < 1. \end{cases} \tag{S2.6}$$

Indeed, because G_n is strictly concave, if $\lambda > 1$, (5.4) implies that

$$\frac{\sqrt{nu_n}}{(\log n)^2} \pi_n \left(\lambda - \frac{G_n(\lambda u_n)}{G_n(u_n)}\right) \to \infty.$$

Then the first limit in (S2.6) follows by (S2.1). The second limit similarly holds.

Proof of Lemma S2.1. We will only show the Lemma for procedure (5.1). The proof for the BH procedure (4.4) is similar.

The main part of the proof is devoted to $\tau/u_n \xrightarrow{\mathrm{P}} 1$. Denote

$$d_n = \sqrt{n\log\log n}, \quad f_n(t) := \pi_n G_n(t) - (\pi_n + \rho)t.$$

Because G_n is strictly concave, so is f_n. By (S2.1), $f_n(u_n) = 0$. Also, $f_n(t) > (<)\ 0$ for $t < (>)\ u_n$. Given $\lambda > 1$, let $v_n = u_n/\lambda$. On the one hand, by (S2.2), $q_n(v_n; \Gamma(v_n)) \le nv_n + \sqrt{nv_n}A_n$, with $A_n \to \Phi^*(\Gamma(v_n))$. On the other, by (S2.5), for large n, $R_{v_n} = W_{F_n(v_n)} \ge_p nF_n(v_n) - 2d_n\sqrt{F_n(v_n)}$. Therefore, by $F_n(v_n) \le \alpha v_n$ and $F_n(v_n) \le F_n(u_n) = u_n/\alpha$,

$$\begin{aligned} R_{v_n} &- \frac{1}{\alpha} q_n(v_n; \Gamma(v_n)) \\ &\ge_p n\left(F_n(v_n) - (1+\rho)v_n\right) - \sqrt{\frac{nF_n(v_n)}{\alpha}}\, A_n - 2d_n\sqrt{F_n(v_n)} \\ &\ge_p nf_n(v_n) - 3d_n\sqrt{\frac{u_n}{\alpha}}. \end{aligned}$$

By $\pi_n + \rho = \pi_n G_n(u_n)/u_n$,

$$nf_n(v_n) = n\pi_n\left(G_n(v_n) - \frac{\pi_n G_n(u_n)}{u_n} v_n\right) \ge n\pi_n\left(G_n(\frac{u_n}{\lambda}) - \frac{G_n(u_n)}{\lambda}\right).$$

From (S2.6), it follows that $nf_n(v_n) - 3d_n\sqrt{u_n/\alpha} \xrightarrow{\mathrm{P}} \infty$, yielding $\alpha R_{v_n} - q_n(v_n; \Gamma(v_n)) \xrightarrow{\mathrm{P}} \infty$ and hence $P(\tau > v_n) \to 1$.

Now let $w_n = \lambda u_n$. Then for all $t \ge w_n$, $F_n(t) < t/\alpha$. Similar to the above argument, the probability that

$$R_t - \frac{1}{\alpha} q_n(t; \Gamma(t)) \le f_n(t) + 3d_n\sqrt{F_n(t)} \le f_n(t) + 3d_n\sqrt{\frac{t}{\alpha}}, \quad \text{all } t \ge w_n$$

tends to 1. Because $f_n(t)$ is concave, it is upper bounded by $f_n(w_n)t/w_n$. Note $f_n(w_n) < 0$. Therefore, the probability that

$$\begin{aligned} R_t - \frac{1}{\alpha} q_n(t; \Gamma(t)) &\le f_n(w_n)\frac{t}{w_n} + 3d_n\sqrt{\frac{t}{\alpha}} \\ &\le \sqrt{\frac{t}{w_n}}\underbrace{\left(f_n(w_n) + 3d_n\sqrt{\frac{w_n}{\alpha}}\right)}_{x_n} \end{aligned}$$

tends to 1. Similar to the above argument, $x_n \to -\infty$ by (S2.6). Then $P(\tau \le w_n) \to 1$. Together with $P(\tau > v_n) \to 1$ and $\lambda > 1$ being arbitrary, $\tau/u_n \xrightarrow{\mathrm{P}} 1$. Now we can show parts (a)−(d) in sequel.

(a) Since G_n is concave, $G_n(u_n/\lambda) > G_n(u_n)/\lambda$ and $G_n(\lambda u_n) < \lambda G_n(u_n)$. Then from $\tau/u_n \xrightarrow{\mathrm{P}} 1$, $P(G_n(u_n)/\lambda \le G_n(\tau) < \lambda G_n(u_n)) \to 1$ and hence $G_n(\tau)/G_n(u_n) \xrightarrow{\mathrm{P}} 1$.

(b) Since $F_n(t) = (1-\pi_n)t + \pi_n G_n(t)$, from the above results, it follows that $F_n(\tau)/F_n(u_n) \xrightarrow{\mathrm{P}} 1$. By $F_n(u_n) = u_n/\alpha$, $\theta_\tau = \tau/F_n(\tau) \xrightarrow{\mathrm{P}} \alpha$.
(c) Given $\lambda > 1$, define v_n and w_n as above. By $P(R_\tau > R_{v_n}) \to 1$ and $P(\tau < w_n) \to 1$, $P(\eta_\tau < n w_n/R_{v_n}) \to 1$. On the other hand, since $n v_n \to \infty$, by the CLT in (S2.2), $R_{v_n} = W_{F_n(v_n)} \sim n F_n(v_n)$ and $w_n/F_n(v_n) = \lambda^2 v_n/F_n(v_n) \le \alpha\lambda^2$. Then $P(\eta_\tau < \alpha\lambda^2) \to 1$. Similarly, $P(\eta_\tau > \alpha/\lambda^2) \to 1$. Hence $\eta_\tau \xrightarrow{\mathrm{P}} \alpha$.
(d) This follows from $R_\tau \ge R_{v_n} \sim n F_n(v_n) \ge n v_n$ and $n v_n/(\log n)^4 \to \infty$.

By the weak version of Hungarian construction Shorack and Wellner (1986, p.494), for each n, there exist a Brownian bridge $B_t^{(n)} \overset{d}{=} Z_t - tZ_1$ and a stochastic process $r_t^{(n)}$ defined on the same probability space as $\zeta_1^{(n)}, \ldots, \zeta_n^{(n)}$, where Z_t is a standard Brownian motion, such that $\sup_{t\in[0,1]} |r_t^{(n)}| = O_p(1)$ and $W_t = nt + \sqrt{n} B_t^{(n)} + r_t^{(n)} (\log n)^2$.

Proof of Lemma S2.2. By $R_t = W_{F_n(t)}$ and the Hungarian construction,

$$\theta_t R_t - nt = \sqrt{n}\theta_t B_{F_n(t)}^{(n)} + (\log n)^2 \theta_t r_{F_n(t)}^{(n)} .$$

Note $\theta_{u_n} = \alpha$ and by Lemma S2.1, $\theta_\tau = O_p(1)$. Since $(\log n)^2/\sqrt{n u_n} \to 0$ and $r_t^{(n)}$ is bounded, to show Lemma S2.2 for τ, it suffices to demonstrate

$$\frac{1}{\sqrt{u_n}} \left[\theta_\tau B_{F_n(\tau)}^{(n)} - \alpha B_{F_n(u_n)}^{(n)} \right] \xrightarrow{\mathrm{P}} 0.$$

Write the left hand side as $I_1 + \alpha I_2$, where $I_1 = (\theta_\tau - \alpha) B_{F_n(\tau)}^{(n)}/\sqrt{u_n}$ and $I_2 = (B_{F_n(\tau)}^{(n)} - B_{F_n(u_n)}^{(n)})/\sqrt{u_n}$. Given $\lambda > 1$, since $u_n \to 0$, $P(\tau < \lambda u_n) \to 1$ and $F_n(\lambda u_n) < \lambda F_n(u_n) = \lambda u_n/\alpha$, it is seen that $|B_{F_n(\tau)}^{(n)}|$ asymptotically is dominated by $\sup_{t \le \lambda u_n/\alpha} |B_t^{(n)}|$, hence stochastically dominated by $\sup_{t\le \lambda u_n/\alpha} |Z_t| + (\lambda u_n/\alpha)|Z_1|$. Then $B_{F_n(\tau)}^{(n)}/\sqrt{u_n} = O_p(1)$. By $\theta_\tau \xrightarrow{\mathrm{P}} \alpha$, $I_1 \xrightarrow{\mathrm{P}} 0$.

Similarly, letting $D_n = \lambda u_n - u_n/\lambda$, $B_{F_n(\tau)}^{(n)} - B_{F_n(u_n)}^{(n)}$ asymptotically is stochastically dominated by $\sup_{t\in[0,D_n]} |Z_t| + D_n |Z_1|$. Therefore, I_2 asymptotically is stochastically dominated by $\sqrt{\lambda - 1/\lambda}\, \sup_{t\in[0,1]} |Z_t| + o_p(1)$. Because λ is arbitrary, $I_2 \xrightarrow{\mathrm{P}} 0$.

Recall that $\mathrm{qbin}(z; n, p)$ is increasing and left-continuous in z and p respectively; for $z, p \in (0,1)$,

$$\lim_{x\downarrow z} \mathrm{qbin}(x; n, p), \quad \lim_{x\downarrow p} \mathrm{qbin}(z; n, x) \in \{\mathrm{qbin}(z; n, p), \mathrm{qbin}(z; n, p) + 1\};$$

and $\mathrm{qbin}(z; n, p) \le \mathrm{qbin}(z; n-1, p) + 1$.

Proof of Lemma S2.3. By the definition of τ, when $\tau > 0$ and $R_\tau > 0$, for all $t > \tau$, $q_n(t;\Gamma(t)) > \alpha R_t$. If $t - \tau > 0$ is small enough, $R_t = R_\tau$. Since $\Gamma(t)$ is decreasing in t, $\text{qbin}(\Gamma(\tau); R_\tau, nt/R_\tau) \ge \alpha R_\tau$. Letting $t \downarrow \tau$ then yields $q_n(\tau, \Gamma(\tau)) \ge \alpha R_\tau - 1$.

On the other hand, there is a sequence $t_j \uparrow \tau$, such that $q_n(t_j;\Gamma(t_j)) \le \alpha R_{t_j}$. If R_t is continuous at τ, then for large j, $R_{t_j} = R_\tau$ and letting $j \to \infty$ yields $q_n(\tau, \Gamma(\tau)) \le \alpha R_\tau$. If R_t has a jump at τ, then for large j, $R_{t_j} = R_\tau - 1$ and letting $j \to \infty$ yields

$$\begin{aligned} &\text{qbin}\left(\Gamma(\tau); R_\tau - 1, \frac{n\tau}{R_\tau - 1}\right) \le \alpha R_\tau + 1 \\ \Longrightarrow\ &\text{qbin}\left(\Gamma(\tau); R_\tau, \frac{n\tau}{R_\tau - 1}\right) \le \alpha R_\tau + 2. \end{aligned}$$

Then $q_n(\tau, \Gamma(\tau)) - \alpha R_\tau \le \text{qbin}\left(\Gamma(\tau); R_\tau, n\tau/(R_\tau - 1)\right) - \alpha R_\tau \le 2$.

The proof that for the BH procedure (4.4) is standard so is omitted.

Proof of Lemma S2.4. Let $\mathcal{F}_t = \sigma(\mathbf{1}\{\xi_i \le s\}, s \in [t,1], i = 1, \ldots, n)$. Then for t running backward from 1 to 0, $\mathcal{F}_t$ consist a filtration and for both procedure (5.1) and the BH procedure (4.4), τ is a stopping time with respect to the filtration. In particular, $\{\tau \ge t\} \cap \{R_{\tau-} = k\} \in \mathcal{F}_t$. Let $i_1, \ldots, i_{R_{t-}}$ be the random indices of those ξ_i that are strictly less than t. By the independence of $(\xi_1, H_1), \ldots, (\xi_n, H_n)$ and $V_{t-} = H_{i_1} + \ldots + H_{R_{t-}}$, it is not difficult to see that for any t, $A \in \mathcal{F}_t$, $k \ge 0$, and $n_1 < \ldots < n_k$, conditioning on $E = \{R_{t-} = k, i_1 = n_1, \ldots i_k = n_k\}$, V_{t-} and A are independent, i.e. $P(\{V_{t-} = v\} \cap A \,|\, E) = P(\{V_{t-} = v\} \,|\, E) \times P(A \,|\, E)$. Consequently, $P(V_{t-} = v \,|\, \tau = t, E) = P(V_{t-} = v \,|\, E)$. By Proposition 2.1, the right end is $P(S = v)$, with $S \sim \text{Bin}(k, nt/G_n(t))$. Since the conditional probability does not involve $n_1, \ldots, n_k$, then $P(V_{t-} = v \,|\, \tau = t,\, R_{t-} = k) = P(S = v)$.

S3. Supercritical case with increasing sparsity of false nulls

Let $\zeta_k^{(n)}$ be defined as in (S2.4). We need two lemmas in order to prove Theorem 5.2.

Lemma S3.1. *Given $p_0 \in (0,1)$, for any $\epsilon > 0$,*

$$\lim_{n\to\infty} \sup_{p\in[p_0,1]} P\left(|X_{1,p} + \cdots + X_{n,p} - np| \ge \epsilon n\right) = 0.$$

where for each p, $X_{1,p}, X_{2,p}, \ldots$ are i.i.d. $\sim$ Bernoulli(p).

Lemma S3.2. *If $k_n \le n$ satisfies $k_n \to \infty$, then*

$$\sup_{k_n \le k \le n} \left| \frac{\zeta_{n:k}}{k/n} - 1 \right| \xrightarrow{\text{P}} 0.$$

Proof of Theorem 5.2. Assume the lemmas are true for now. Fix $\epsilon > 0$ such that $(1-\epsilon)^2 \underline{\lim}_{n\to\infty} \alpha_*^{(n)} > \alpha$. Then by condition (5.5), for n large enough, $(1-\epsilon)^2/F_n'(0) = (1-\epsilon)^2\alpha_*^{(n)}/(1-\pi_n) > \alpha$.

First, we show that for some $K_0 > 0$, $P(R_\tau < K_0) \to 1$. Let $m_n(t) = R_t \vee k_n$. Then

$$\frac{n(t \vee \xi_{n:k_n})}{R_t \vee k_n} \geq \frac{nF_n^*\left(\zeta_{m_n(t)}\right)}{m_n(t)}. \tag{S3.1}$$

By the selection of k_n, $m_n(t) \to \infty$. Then by the convexity of $F_n^*(x)$ (because F_n is concave) and Lemma S3.2,

$$\begin{aligned}\frac{nF_n^*\left(\zeta_{m_n(t)}\right)}{m_n(t)} &\geq_p \frac{F_n^*\left((1-\epsilon)m_n(t)/n\right)}{m_n(t)/n} \\ &\geq (1-\epsilon)(F_n^*)'(0) = \frac{1-\epsilon}{F_n'(0)} \geq \frac{\alpha}{1-\epsilon}\end{aligned} \tag{S3.2}$$

and hence by Lemma S3.1, there is $K_0 > 0$, such that for all $K \geq K_0$,

$$\operatorname{qbin}\left(1-\gamma;\, K,\, \frac{1-\epsilon}{F_n'(0)}\right) > \frac{(1-\epsilon)^2 K}{F_n'(0)} > \alpha K.$$

Combined with (S3.1) and (S3.2), this implies

$$P\left(\bigcap_{t: R_t \geq K_0} \left\{\operatorname{qbin}\left(1-\gamma;\, R_t,\, \frac{n(t \vee \xi_{n:k_n})}{R_t \vee k_n}\right) > \alpha R_t\right\}\right) \to 1.$$

As a result, $P\left(R_\tau < K_0\right) \to 1$.

Now suppose condition (5.6) is satisfied. We show parts (a)−(d) in sequel.

(a) Note that

$$R_\tau = \max\left\{k \geq 1 : \operatorname{qbin}\left(1-\gamma; k, \frac{n\xi_{n:(k\vee k_n)}}{k \vee k_n} \wedge 1\right) \leq \alpha k\right\}.$$

Because $P(R_\tau < K_0) \to 1$ and $k_n \to \infty$, $P(R_\tau = R_n') \to 1$, where

$$R_n' = \max\left\{k \geq 1 : \operatorname{qbin}\left(1-\gamma; k,\, p_n \wedge 1\right) \leq \alpha k\right\}$$

with $p_n = n\xi_{n:k_n}/k_n$. Since $p_n \xrightarrow{\text{P}} \alpha_*$, by the properties of qbin as listed before the proof of Lemma S2.3, part (a) then follows from

$$P\left(\operatorname{qbin}(1-\gamma; k, p_n) \in \{\operatorname{qbin}(1-\gamma; k, \alpha_*),\, \operatorname{qbin}(1-\gamma; k, \alpha_*) + 1\}\right) \to 1.$$

(b) Since F_n is concave, given $R_\tau = \ell > 0$ and $\tau = t$, V_τ is stochastically dominated by Bin(ℓ, $(1-\pi_n)t/F_n(t)$), but stochastically dominates Bin(ℓ, $(1-\pi_n)/F_n'(0)$). Because $\tau \xrightarrow{\text{P}} 0$, $(1-\pi_n)\tau/F_n(\tau) \xrightarrow{\text{P}} \alpha_*$. Part (b) therefore follows.

(c) Let $Z_1, Z_2, \ldots$ be i.i.d. $\sim$ Bernoulli(α_*). If $\gamma < \gamma_*$, then $\ell_0 = 0$, or else there were $k > 0$ such that $P(Z_1+\cdots+Z_k \le \alpha k) \ge 1-\gamma$. Then $\gamma \ge 1-P(Z_1+\cdots+Z_k \le \alpha_k) \ge \gamma_*$, which is a contradiction. It is clear that $\ell_1 = 0$. Therefore, by part (a), $R_\tau \xrightarrow{\text{P}} 0$. On the other hand, if $\gamma > \gamma_*$, then $\ell_0 > 0$. By part (b), $P(V_\tau \le \alpha R_\tau \,|\, R_\tau = \ell_0) \to P(Z_1 + \cdots + Z_{\ell_0} \le \alpha \ell_0) \ge 1-\gamma$, and hence $\overline{\lim}_n P(V_\tau/R_\tau > \alpha \,|\, R_\tau = \ell_0) \le \gamma$. The case $R_\tau = \ell_1$ can be similarly shown as long as $\ell_1 > 0$. This completes the proof of (c).

(d) For both procedure (5.1) and the BH procedure (4.4), in order to show that their respective powers tend to 0, by $P(R \le K_0) \to 1$, it is enough to show $n - N_0 \xrightarrow{\text{P}} \infty$. Denote $s_n = F_n^*(k_n/n)$. Since $M_n := \#\{i \le n : \xi_i^{(n)} \le s_n,\ H_i^{(n)} = 1\} \sim$ Bernoulli$(n, \pi_n G_n(s_n))$ and $M_n \le n - N_0$, it is enough to show $n\pi_n G_n(s_n) \to \infty$. Since $k_n/n = F_n(s_n) = (1-\pi_n)s_n + \pi_n G_n(s_n)$ and $s_n/F_n(s_n) \to \alpha_* < 1$,

$$\frac{\pi_n G_n(s_n)}{k_n/n} = 1 - \frac{(1-\pi_n)s_n}{F_n(s_n)} \to 1 - \alpha_* > 0,$$

yielding $n\pi_n G_n(s_n) \sim (1-\alpha_*)k_n \to \infty$.

Proof of Lemma S3.1. It is enough to show

$$\lim_{n\to\infty} \sup_{p\in[p_0,1]} P\left(X_{1,p} + \cdots + X_{n,p} > (p+\epsilon)n\right) = 0, \quad \text{and}$$
$$\lim_{n\to\infty} \sup_{p\in[p_0,1]} P\left(X_{1,p} + \cdots + X_{n,p} < (p-\epsilon)n\right) = 0.$$

We will only show the first limit. The second one can be shown similarly.

Clearly, when $p \ge 1-\epsilon$, $P(X_{1,p} + \cdots + X_{n,p} > (p+\epsilon)) = 0$. If $p < 1-\epsilon$, then by Chernoff's inequality, $P\left(X_{1,p} + \cdots + X_{n,p} > (p+\epsilon)n\right) \le e^{-nI(p)}$, where $I(p) = \sup_{t>0}\left((p+\epsilon)t - \Lambda_p(t)\right)$, with $\Lambda_p(t) = \log(1-p+pe^t)$. Since $\Lambda_p(t)$ is convex and $\Lambda_p'(0) = p$, $I(p) > 0$. It can be verified that $I(p)$ is continuous on $[0, 1-\epsilon)$. Letting $I(p) = \infty$ for $p \ge 1-\epsilon$, it follows that $\inf_{p\ge p_0} I(p) > 0$, which implies the limit.

Proof of Lemma S3.2. $\xi_{n:1}, \ldots, \xi_{n:n}$ have the same joint distribution as

$$\left(\frac{S_1}{S_{n+1}}, \ldots, \frac{S_n}{S_{n+1}}\right)$$

where $S_k = U_1 + \cdots + U_k$ and $U_1, U_2, \ldots$ are i.i.d. $\sim$ Exp(1). By the LLN, $S_{n+1}/n \xrightarrow{\text{P}} 1$. Therefore, it is enough to show $\sup_{k \ge k_n} |S_k/k - 1| \xrightarrow{\text{P}} 0$, which follows from the strong law of large numbers (SLLN).

S4. Nonsparse case

Proof of Theorem 4.1. The proof of part (a) is omitted because it follows closely Genovese and Wasserman (2002). For part (b), let R' be the number of projections in (4.3). Then by Proposition 4.1, $P(R' > k_n) \to 0$. Since $P(R > 0) \le P(R' > k_n)$, part (b) follows.

To prove Theorem 4.2, we need the following standard result for empirical processes.

Lemma S4.1. *Suppose τ_n is a sequence of random variables taking values in $[0,1]$, such that for some $u \in (0,1)$, $\tau_n \xrightarrow{d} u$ as $n \to \infty$. Then, letting $\pi_0 = 1 - \pi$,*

$$\frac{V_{\tau_n} - n\pi_0\tau_n}{\sqrt{n\pi_0 u(1-\pi_0 u)}} \xrightarrow{d} N(0,1).$$

Following the proof for the sparse case, for procedure (4.7), define

$$\tau_n = \sup\left\{t \in [0,1] : \text{qbin}\left(\Gamma_*(t);\, R_t,\, \frac{\pi_0 n(t \vee \xi_{n:k_n})}{R_t \vee k_n} \wedge 1\right) \le \alpha(R_t \vee 1)\right\}$$

and for the BH procedure (4.3), define $\tau_n = \sup\{t \in [0,1] : \pi_0 nt \le R_t\}$. Then following the same notations, $R = R_\tau$ and $V = V_\tau$.

The proof of Theorem 4.2 follows closely that of Theorem 5.1, so we only give its sketch.

Proof of Theorem 4.2. (a) Following Genovese and Wasserman (2002), $\tau \xrightarrow{\text{P}} u^*$, with $u^* \in (0,1)$ the only positive solution to $\pi_0 u = \alpha F(u)$. By the definition of R and $P(\xi_i \le u^*,\ H_i = 1) = F(u^*)$, from LLN, it follows that $R/n \xrightarrow{\text{P}} F(u^*) > 0$ and hence pFDR $\sim$ FDR $= \alpha$. Furthermore, $\Gamma_*(t)$ can be replaced with $\Gamma(t)$ and by Lemma S2.3,

$$P(V_\tau \le \alpha R_\tau) = P\left(V_\tau \le \text{qbin}\left(\Gamma(\tau); R_\tau, \frac{\pi_0 n\tau}{R_\tau} \wedge 1\right) + O_p(1)\right).$$

Denote the binomial quantile on the right hand side by K. Applying the CLT to the binomial distributions, from $\tau \xrightarrow{d} u^*$, it follows that

$$\frac{K - \pi_0 n\tau}{\sqrt{n\pi_0 u^*(1-\alpha)}} \sim_p \frac{K - \pi_0 n\tau}{\sqrt{\pi_0 n\tau(1 - \pi_0 n\tau/R_\tau)}} \xrightarrow{\text{P}} \Phi^*(\Gamma(u^*)).$$

Combining this with Lemma S4.1 yields

$$P(V_\tau \le \alpha R_\tau) \to P\left(\sqrt{\frac{1-\pi_0 u^*}{1-\alpha}} Z \le \Phi^*(\Gamma(u^*))\right) = 1-\gamma.$$

(b) Since $k_n/n \to 0$ whereas $R/n \xrightarrow{d} F(u^*) > 0$, part (b) easily follows.

Proof of Proposition 4.1. (a) Following the proof of Theorem 4.2 (a),

$$P(V_\tau \le \alpha R_\tau) \to P\left(\sqrt{\frac{1-\pi_0 u^*}{1-\alpha}} Z \le 0\right) = \frac{1}{2},$$

where, for the BH procedure (4.3),

$$\tau = \tau_n = \sup\left\{t \in [0,1] : \pi(1-\pi)nt \le R_t\right\}.$$

(b) See Chi (2007).

The proof of Theorem 4.3 is almost identical to that of Theorem 5.2 and so is omitted.

Proof of Proposition 4.2. When $\alpha \in (\alpha_*, 1-\pi)$, then $\tau \xrightarrow{\mathrm{P}} u^*$. Because $R-V = \#\left\{k \le n : H_k = 1,\, \xi_k^{(n)} \le \tau\right\}$ and $n-N_0 = \#\left\{k \le n : H_k = 1\right\}$, by the LLN, $\psi_n \xrightarrow{\mathrm{P}} P(\xi \le u^* \,|\, H = 1) = G(u^*)$. On the other hand, when $\alpha < \alpha_*$, then for procedures (4.6) and (4.7), by Theorem 4.1 and 4.2, it is apparent that $\psi_n = O_p(1/n)$, and for the BH procedure (4.3), from Chi (2007), $\psi_n \xrightarrow{\mathrm{P}} 0$ as well.