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INCREMENT-BASED ESTIMATORS OF FRACTAL DIMENSION FOR TWO-DIMENSIONAL SURFACE DATA

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Abstract: In a recent paper, Kent and Wood (1997) investigated some new increment-based estimators of the fractal dimension of a stationary Gaussian process. In the present paper, we extend this work by constructing increment-based estimators based on two-dimensional sampling of surface data (as opposed to the one dimensional, or line transect, sampling previously considered). Much of our attention is focussed on two new estimators based on the "square increment". The practical performance of these estimators is examined in the study of several real datasets and via simulation. We also provide a detailed theoretical study of their properties in both Gaussian and non-Gaussian settings. Perhaps surprisingly, it turns out that there are differences in the limit theory in the Gaussian and non-Gaussian cases.

Key words and phrases: Gaussian random field, generalised least squares, self-similarity, stationary.

1. Introduction

Surface roughness is an important property of a surface in many scientific and engineering contexts, see Thomas (1982) for a general account. One popular approach has been to use fractals as models for rough surfaces, and to use fractal dimension as a measure of roughness. See, for example, Mandelbrot, Passoja and Paullay (1982), Dubuc, Zucker, Tricot, Quiniou and Wehbi (1989) and Davies and Hall (1998). For general introductions to fractals, see Mandelbrot (1982), Barnsley (1988) and Falconer (1990).

Our preference is to follow Taylor (1986), Falconer (1990) and Taylor and Taylor (1991) and define the fractal dimension of a set to be the common value when the various mathematical notions of dimension, such as box-counting, capacity, packing and Hausdorff dimension, all agree. This definition requires that the sets under consideration possess some (minimal) degree of regularity. The sets under consideration in this paper, namely the graphs of sample functions of stationary Gaussian and chi-squared fields, possess sufficient regularity (with probability one) for us to understand "fractal dimension" in the sense indicated above; see Adler (1981), Taylor (1986) and Falconer (1990) for some mathematical details. However, the reader who prefers to stick with precisely defined

mathematical entities may (correctly) take fractal dimension to be Hausdorff dimension throughout.

This paper is concerned with estimating the fractal dimension of a surface given the surface height at locations on a two-dimensional rectangular grid. Some new estimators are proposed and their properties, both numerical and theoretical, are examined. However, before describing the contents of this paper in more detail, we shall briefly review some earlier work on the (simpler) case of one-dimensional sampling.

1.1. One-dimensional sampling

The case in which the grid of sampling locations is one-dimensional (corresponding to line transect sampling of surface heights) has been studied in a number of recent papers; see, for example, Hall and Wood (1993) ("box counting" estimators), Constantine and Hall (1994) ("variogram" estimators) and Kent and Wood (1997). In each of these papers, the line transect of the surface is modelled as a random process, and the fractal dimension of the graph of the sample function is used as a measure of surface roughness. If the underlying process is assumed to be stationary and Gaussian with a covariance function, γ, which satisfies

$$\gamma(t) = \gamma(0) - c|t|^\alpha + o(|t|^\alpha) \quad \text{as} \quad t \to 0, \tag{1.1}$$

where $0 < \alpha \leq 2$ and $c > 0$, then the fractal dimension, D, of the sample paths of the process takes the following simple form:

$$D = 2 - \alpha/2; \tag{1.2}$$

see e.g. Orey (1970) and Adler (1981) for results of this kind. Thus, in the stationary Gaussian setting with covariance function γ satisfying (1.1), α in (1.1) can be thought of as a roughness parameter and, by (1.2), estimation of D is equivalent to estimation of α. Hall and Roy (1994) have shown that relation (1.2) also holds for many non-Gaussian processes.

The variance of estimators considered by Hall and Wood (1993) and Constantine and Hall (1994) (and also several other estimators) have the following theoretical rates of convergence: if $\hat{\alpha}$ is any of the above estimators, and n is the number of equally-spaced sampling locations, then $\mathrm{Var}\,(\hat{\alpha})$ is of size $O(n^{-1})$ if $0 < \alpha < 3/2$, of size $O(n^{-1} \log n)$ if $\alpha = 3/2$, and of size $O(n^{2\alpha-4})$ if $3/2 < \alpha < 2$; and so, in particular, the theory predicts that these estimators perform less well in the case of smoother surfaces (i.e. when α is close to 2). However, several simulation studies have shown that, with realistic sample sizes (such as $n = 1000$), there is no noticeable deterioration (and even sometimes an improvement) in performance when α approaches 2. Further discussion may be found in Feuerverger,

Hall and Wood (1994). Subsequent work has shown that there is a relatively clear-cut technical explanation for this phenomenon; see Kent and Wood (1997).

In the same paper, Kent and Wood constructed estimators of α with variance of size $O(n^{-1})$ for all $\alpha \in (0, 2)$, indicating improved asymptotic performance of the estimator when $3/2 < \alpha < 2$. The new idea was to base the estimator on higher-order increments (equivalently, higher-order differences). They considered both ordinary least squares (OLS) and generalised least squares (GLS) variants of this estimator. Higher-order increments have also been used in a similar fashion, in a related context, by Istas and Lang (1997).

The numerical results in Kent and Wood (1997) suggest that a potential practical benefit of using estimators based on higher-order increments (relative to the more standard estimators) is bias reduction, even though the original (theoretical) motivation for considering these estimators was variance reduction. There also seem to be modest practical benefits in using GLS rather than OLS.

1.2. Two-dimensional sampling

With two-dimension sampling, the situation under consideration here, the surface heights are measured at the vertices of a finite rectangular grid in $\mathbf{R}^2$. By analogy with the case of one-dimensional sampling, we make the working assumption that the surface is a realisation of a stationary random field. However, a new potential complication arises with two-dimensional sampling: the possibility that the (stationary) process is anisotropic. It would seem prudent to take this possibility into account in any estimation procedure. See Davies and Hall (1998) for detailed discussion of this point.

In this paper we have three main objectives: (i) to construct estimators of fractal dimension for two-dimensional surface data using higher-order increments, taking the possibility of anisotropy into account; (ii) numerical study of these estimators; and (iii) theoretical study of these estimators under both Gaussian and non-Gaussian assumptions.

Two new estimators, based on the so-called "square increment", are constructed in Sections 3 and 4. One of these estimators accounts for possible anisotropy in a generalised least squares procedure. Numerical properties of these estimators are investigated in Section 6 using both simulated and real data. Our estimators are different than those considered in Davies and Hall (1998).

Asymptotic properties of these estimators under Gaussian assumptions are presented in Theorem 3.2 and Corollary 3.3. A result similar to Theorem 3.2 was proved in the one-dimensional case in the (unpublished) research report by Kent and Wood (1995, Appendix B); here we extend this result to the multivariate case. We also establish the asymptotic properties of these estimators in a particular non-Gaussian setting; see Theorem 5.1. Perhaps surprisingly, it turns out that

there are appreciable differences in the limit theory in the Gaussian and non-Gaussian cases.

2. Data Structure and Underlying Model

Consider a surface represented mathematically by the set

$$\{x(t) : t \in A\}, \tag{2.1}$$

where A is a connected subset of $\mathbf{R}^2$ with non-empty interior. In (2.1), $x(t) \in \mathbf{R}$ denotes surface height above a fixed (but arbitrary) level, and t denotes the location at which the surface height is determined. In practice, it is only possible to measure surface height at a finite set of locations $\mathcal{S} \subset A$. Throughout this paper, we shall assume that $\mathcal{S}$ consists of the vertices of a finite rectangular grid in $\mathbf{R}^2$. From a practical point of view this is a reasonable assumption as surface height datasets are often of this form; it is the case for the food wrap datasets considered in Subsection 6.2.

For the moment, it will be convenient to work in a general d-dimensional setting. Later on, we specialise to $d = 1, 2$. Any element j of $\mathbf{Z}^d$, the d-fold cartesian product of the set of integers $\mathbf{Z}$, is referred to as a multi-index of dimension d. We write $j = (j[1], \ldots, j[d])$, with squared brackets used to denote components of j. In this notation, we define the set of locations at which surface height is measured by

$$\mathcal{S}_n = \Big\{ \frac{j}{n} : 0 \le j < n \Big\} \subset [0,1]^d \tag{2.2}$$

with $d = 2$, and $\{x(t) : t \in \mathcal{S}_n\}$ is the corresponding set of observed surface heights. In the above, j and n are multi-indices, and n has strictly positive components. All the implied operations in (2.2) are performed component-wise, e.g.

$$\frac{j}{n} = \Big(\frac{j[1]}{n[1]}, \ldots, \frac{j[d]}{n[d]} \Big) \in \mathbf{R}^d,$$

and $0 \le j < n$ is equivalent to $0 \le j[\ell] < n[\ell]$ for $\ell = 1, \ldots, d$.

Observe that the set $\mathcal{S}_n$ contains $N = n[1] \ldots n[d]$ elements. Later, we consider an asymptotic framework in which the $n[\ell]$'s increase in such a way that the ratios

$$n[1]/n[\ell] \quad (\ell = 2, \ldots, d) \quad \textit{stay bounded away from} \quad 0 \quad \textit{and} \quad \infty. \tag{2.3}$$

In other words, the set $\mathcal{S}_n$ does not become "thin". In view of (2.2) and (2.3), "infill" asymptotics are operating here, in that the set $\mathcal{S}_n$ becomes increasingly dense, but is contained in a fixed set, $[0,1]^d$.

We now describe the underlying statistical model in the case $d = 2$. The basic idea is to view the surface (2.1) as a realisation of a two-dimensional stationary Gaussian random field, $\{X(t) : t \in \mathbf{R}^2\}$ say, but restricted to the set A in (2.1). The stationarity here is with respect to translations of $\mathbf{R}^2$. Since our estimators depend only on changes in surface height, and not on surface height itself, we can without loss of generality assume that the field has zero mean.

A d-dimensional zero-mean Gaussian field is characterized by its covariance function, γ, defined by $\gamma(t) = E\{X(s)X(s+t)\}$ where $s, t \in \mathbf{R}^d$. The key technical assumption we make about γ is the following:

$$\gamma(t) = \gamma(0) - ||t||^{\alpha} M(t/||t||) + O(||t||^{\alpha+\beta}) \quad \text{as} \quad ||t|| \to 0, \tag{2.4}$$

where $\alpha \in (0, 2]$ (to ensure that γ is a non-negative definite function), $\beta > 0$, $t \in \mathbf{R}^d$, $||t|| = (t^T t)^{1/2}$ is the usual Euclidean norm, and $M(.)$ is a smooth (and necessarily non-negative) function on the unit sphere in $\mathbf{R}^d$. Note that (2.4) is a natural d-dimensional extension of (1.1). If $M(.)$ in (2.4) is constant, then the process is (locally) isotropic; otherwise, it is anisotropic. The model (2.4) was considered by Davies and Hall (1998) and, briefly, by Kent and Wood (1995, Section 8).

We also need to consider a mild additional technical assumption on γ. If $r \in \mathbf{Z}^d$ is a multi-index, we write $|r| = \sum_{\ell=1}^{d} |r[\ell]|$ and we use the following shorthand for partial derivatives: for any non-negative multi-index r with $|r| = q$,

$$\gamma^{(r)} = \frac{\partial^{|r|}\gamma}{\partial t^r} = \frac{\partial^q \gamma}{\partial t[1]^{r[1]} \dots \partial t[d]^{r[d]}},$$

where $t = (t[1], \dots, t[d])^T$. Using this notation, our technical condition may be stated as follows.

Condition $\mathcal{A}_q^{(d)}$: for each non-negative multi-index r with $|r| = q$,

$$\gamma^{(r)}(t) = -\frac{\partial^{|r|}}{\partial t^r}\Big\{||t||^{\alpha} M(t/||t||)\Big\} + o(||t||^{\alpha-q}) \quad \text{as} \quad ||t|| \to 0, \tag{2.5}$$

where all partial derivatives of order $q+1$ of the function $M(\theta)$, $\theta \in \mathcal{S}^d$, are assumed continuous. Note that the derivative on the right hand side of (2.5) is of size $O(||t||^{\alpha-q})$ as we are assuming that $M(.)$ is smooth. In our setting we shall mainly be concerned with $d = 1, 2$ and $q = 2p + 2$ where $p = 0, 1$ is the order of the increment concerned (see next section).

When $d = 2$, the fractal dimension, D, of the sample function of a stationary Gaussian random field with a covariance function which satisfies (2.4) is given by

$$D = 3 - \alpha/2; \tag{2.6}$$

see e.g. Adler (1981). The relationship (2.6) also often holds if the field is non-Gaussian; see Hall and Roy (1994) for results on non-Gaussian fields in the one-dimensional case. Their arguments easily extend to higher dimensional cases.

As a consequence of relationship (2.6), estimation of the fractal dimension D is equivalent to estimating the covariance parameter α in (2.4). So, under our modelling assumptions, the basic problem may be formulated as follows: given observed data $\{x(t) : t \in \mathcal{S}_n\}$, obtain an estimate of α.

3. Increments in the Multivariate Case

We now explain how increments can be used with two-dimensional surface data. Once again, it will be convenient initially to consider the general d-dimensional setting and then specialise to $d = 2$ later. The basic ideas in the case $d = 1$ are discussed in detail by Kent and Wood (1997) and the general d-dimensional case was considered briefly by Kent and Wood (1995, Section 8). The multivariate extension of the increment-based approach is straightforward and natural, but in order to make this paper self-contained we give details here.

Given multi-indices j and r we define $j^r = \prod_{\ell=1}^{d} j[\ell]^{r[\ell]}$, with the convention that $0^0 = 1$.

In the d-dimensional case, an increment of order p, where p is a non-negative integer, is a finite array $\mathbf{a} = \{a_j : -J \le j \le J\}$, where $j, J \in \mathbf{Z}^d$ and $a_j \in \mathbf{R}$, with the following properties: for all non-negative multi-indices r which satisfy $|r| \le p$,

$$\sum_{j:-J\le j\le J} j^r a_j = 0; \tag{3.1}$$

and for some non-negative multi-index r with $|r| = p + 1$,

$$\sum_{j:-J\le j\le J} j^r a_j \neq 0. \tag{3.2}$$

It is straightforward to check that when $d = 2$,

$$\mathbf{a} = \{a_{(0,0)} = -1, \quad a_{(1,0)} = 1\} \quad \text{and} \quad \mathbf{a} = \{a_{(0,0)} = -1, \quad a_{(0,1)} = 1\} \tag{3.3}$$

are both increments of order $p = 0$. Increments of order $p = 1$ include

$$\mathbf{a}\{H\} = \{a_{(-1,0)} = a_{(1,0)} = 1, \quad a_{(0,0)} = -2\}, \tag{3.4}$$

$$\mathbf{a}\{V\} = \{a_{(0,-1)} = a_{(0,1)} = 1, \quad a_{(0,0)} = -2\}, \tag{3.5}$$

$$\mathbf{a}\{\square\} = \{a_{(0,0)} = a_{(1,1)} = 1, \quad a_{(1,0)} = a_{(0,1)} = -1\}, \tag{3.6}$$

$$\mathbf{a}\{D_+\} = \{a_{(1,1)} = a_{(-1,-1)} = 1, \quad a_{(0,0)} = -2\} \tag{3.7}$$

and

$$\mathbf{a}\{D_-\} = \{a_{(-1,1)} = a_{(1,-1)} = 1, \quad a_{(0,0)} = -2\}. \tag{3.8}$$

In the above we have adopted the convention that those a_j which are not given explicitly are zero. The characters $H, V, \Box, D_+$ and D_- stand for, respectively, "horizontal", "vertical", "square", "diagonal with positive gradient" and "diagonal with negative gradient". The "square-increment" estimators described in the next section are based principally on $\mathbf{a}\{\Box\}$, but the other four increments are used to construct a suitable weight matrix for use in generalised least squares.

It may be helpful to indicate the close correspondence between increments and difference operators. Suppose $d = 2$ and let $f(j)$, $j \in \mathbf{Z}^2$, be a given function. Define the multi-indices $e_x = (1,0)$ and $e_y = (0,1)$ and let B_x and B_y be the "backwards" shift operators defined by $B_x f(j) = f(j - e_x)$ and $B_y f(j) = f(j - e_y)$. Then

$$\sum_j f(j) a_j\{H\} = f(-e_x) + f(e_x) - 2f(0) = (1 - B_x)^2 f(e_x),$$

$$\sum_j f(j) a_j\{V\} = f(-e_y) + f(e_y) - 2f(0) = (1 - B_y)^2 f(e_y),$$

and

$$\sum_j f(j) a_j\{\Box\} = f(e_x + e_y) + f(0) - f(e_x) - f(e_y) = (1 - B_x)(1 - B_y) f(e_x + e_y).$$

The diagonal increments (3.7) and (3.8) have similar definitions in terms of back-shift operators defined on the diagonals. The relationship between increments and difference operators indicated above holds generally. Our reason for working with increments rather than difference operators is purely a matter of notational convenience.

The dilation $\mathbf{a}^u = \{a_j^u : -Ju \le j \le Ju\}$ for integer $u \ge 1$ of an increment $\mathbf{a}$ is defined as follows:

$$a_j^u = \begin{cases} a_{j'} & \text{if } j = j'u, \\ & \qquad -Ju \le j \le Ju \\ 0 & \text{otherwise.} \end{cases}$$

It follows immediately from this definition that

$$\sum j^r a_j^u = u^{|r|} \sum j^r a_j \quad \text{for} \quad |r| > p, \tag{3.9}$$

while both sides vanish if $|r| \le p$.

Given an underlying stationary Gaussian random field $\{X(t) : t \in \mathbf{R}^d\}$ which is sampled at locations $t \in \mathcal{S}_n$, where $\mathcal{S}_n$ is defined in (2.2), we define the fields $Y_N^u(i)$ and $Z_N^u(i)$ by

$$Y_N^u(i) = N^{\alpha/(2d)} \sum_{-Ju \le j \le Ju} a_j^u X\Big(\frac{i+j}{n}\Big), \qquad i \in \mathcal{I}_n, \tag{3.10}$$

and

$$Z_N^u(i) = \{Y_N^u(i)\}^2, \tag{3.11}$$

where N is the cardinality of the set $\mathcal{I}_n = \{i : 0 \le i < n\}$. In order for $Y_N^u(i)$ to be well-defined when i is close to the boundary of $\mathcal{I}_n$, we assume that the process is also observed at the appropriate grid points just outside $\mathcal{I}_n$.

We now define $\bar{\mathbf{Z}}_N = (\bar{Z}_N^u, u = 1, \ldots, m)$ to be the m-dimensional column vector containing the means

$$\bar{Z}_N^u = N^{-1} \sum_{i \in \mathcal{I}_n} Z_N^u(i). \tag{3.12}$$

The reason for considering these quantities is now explained. Let $\sigma_N^{uv}(h) = \mathrm{cov}\{Y_N^u(i+h), Y_N^v(i)\}$ denote the cross-covariance function of the Y-process. Using (2.3) and (2.4) we obtain

$$\begin{aligned}
\sigma_N^{uv}(h) &= E\{Y_N^u(i) Y_N^v(i+h)\} \\
&= N^{\alpha/d} \sum_{j,k} a_j^u a_k^v \gamma\Big(\frac{h+k-j}{n}\Big) \\
&= -\sum_{j,k} a_j^u a_k^v ||h+k-j||^\alpha M\Big(\frac{h+k-j}{||h+k-j||}\Big) + o(1) \\
&= \sigma_0^{uv}(h) + o(1)
\end{aligned}$$

as $N \to \infty$, where $\sigma_0^{uv}(h)$ is defined to be the sum in the penultimate line. Using (3.9) and the fact that $E\{Z_N^u(i)\} = \mathrm{Var}\,\{Y_N^u(i)\} = \sigma_N^{uu}(0)$, we find that

$$\begin{aligned}
E\{Z_N^u(i)\} \to \sigma_0^{uu}(0) &= -\sum_{j,k} a_j^u a_k^u ||k-j||^\alpha M\Big(\frac{k-j}{||k-j||}\Big) \\
&= -u^\alpha \sum_{j,k} a_j a_k ||k-j||^\alpha M\Big(\frac{k-j}{||k-j||}\Big) \\
&= C u^\alpha.
\end{aligned} \tag{3.13}$$

Consequently, using the approximate relationship (3.13), we may estimate α by regressing $\log(\bar{Z}_N^u)$ on $\log(u)$, $u = 1, \ldots, m$. Note that (3.13) holds for any increment $\mathbf{a}$ of order $p \ge 0$.

Let $\hat{\alpha}$ be an estimator obtained using this regression procedure. A Taylor expansion argument shows that the stochastic component of the error in estimating α with $\hat{\alpha}$ is determined by the covariance matrix $\Phi_N = (\phi_N^{uv})$ of $\bar{\mathbf{Z}}_N$. Using a standard formula for the covariance of two sums of squared, correlated Gaussian variables, we find that

$$\phi_N^{uv} = \mathrm{cov}(\bar{Z}_N^u, \bar{Z}_N^v) = N^{-1} \sum_{-n<h<n} \Big\{ \prod_{\ell=1}^{d} \Big(1 - \frac{|h[\ell]|}{n[\ell]}\Big) \Big\} 2\{\sigma_N^{uv}(h)\}^2, \tag{3.14}$$

where, for each $h \in \mathbf{Z}^d$, $\sigma_N^{uv}(h) \to \sigma_0^{uv}(h)$.

In the three results below, it is assumed that the underlying random field is stationary and Gaussian with a covariance function which satisfies (2.4), and that the sampling condition (2.3) is satisfied. Proofs are given in Section 7.

Lemma 3.1. *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be increments of order* $p' \geq p$ *and* $q' \geq q$*, respectively, and define*

$$Y_{N,\mathbf{c}}^u(i) = N^{\alpha/(2d)} \sum_j c_j^u X\Big(\frac{i+j}{n}\Big), \qquad \mathbf{c} = \mathbf{a}, \mathbf{b}. \tag{3.15}$$

If condition $\mathcal{A}_{p+q+2}^{(d)}$ *in* (2.5) *holds, then*

$$\operatorname{cov}\{Y_{N,\mathbf{a}}^u(i), Y_{N,\mathbf{b}}^v(i+h)\} = O\{(1 + ||h||)^{\alpha-p-q-2}\}$$

uniformly for $||h|| < N^{1/d}$*, as* $N \to \infty$.

Remark 3.1. Lemma 3.1 is a straightforward extension of part (i), Theorem 3.1, in Kent and Wood (1997). The need to consider increments $\mathbf{a}$ and $\mathbf{b}$ with $\mathbf{a} \neq \mathbf{b}$ arises in the proof of Theorem 5.1 below. The quantities p' and q' appear in the statement of the lemma because this is usually the form in which the lemma is used below.

Theorem 3.2. *Suppose that* $d = 1, 2$*,* $(\bar{Z}_N^u : u = 1, \ldots, m)$ *in* (3.12) *is based on an increment of order* $p \geq 1$*,* (2.3) *and* (2.4) *hold, and* $\mathcal{A}_4^{(d)}$ *in* (2.5) *is satisfied. Then*

$$N\phi_N^{uv} \to \phi_0^{uv} = \sum_{h \in \mathbf{Z}^d} 2\{\sigma_0^{uv}(h)\}^2. \tag{3.16}$$

Moreover, $N^{1/2}\{\bar{\mathbf{Z}}_N - E(\bar{\mathbf{Z}}_N)\}$ *converges in distribution to the* m*-variate normal distribution with mean zero and covariance matrix* $\Phi_0 = (\phi_0^{uv})$*.*

Remark 3.2. If the increment in Theorem 3.2 is of order $p = 0$, then a rather different asymptotic picture emerges. When $p \geq 1$, Theorem 3.2 tells us that $\phi_N^{uv} = O(N^{-1})$; but when $p = 0$,

$$\phi_N^{uv} = \begin{cases} O(N^{-1}) & \text{if } 0 < \alpha < 2 - d/2, \\ O\{N^{-1}L(N)\} & \text{if } \alpha = 2 - d/2, \\ O(N^{(2\alpha-4)/d}) & \text{if } 2 - d/2 < \alpha < 2, \end{cases} \tag{3.17}$$

where $L(\cdot)$ is a function which is slowly-varying at infinity. Moreover, the limit distribution of $\{\operatorname{cov}(\bar{\mathbf{Z}}_N)\}^{-1/2}\{\bar{\mathbf{Z}}_N - E\bar{\mathbf{Z}}_N\}$ is non-normal if $2 - d/2 < \alpha < 2$, but normal if $0 < \alpha < 2 - d/2$. The case $p = 0$ and $d = 1$ is covered by Constantine and Hall (1994); see also Kent and Wood (1997). A result for general d incorporating both Theorem 3.2 and (3.17) was stated without proof in Kent and Wood (1995).

Remark 3.3. In cases where the limit distribution is non-normal (i.e. when $p = 0$ and $2 - d/2 < \alpha < 2$) the relevant limit theory can be established using results due to Taqqu (1975), Dobrushin and Major (1979) and Major (1981).

In our third result, let L_u, $u = 1, \ldots, m$, be any fixed numbers such that

$$\sum_u L_u = 0 \quad \text{and} \quad \sum_u L_u \log(u) = 1, \tag{3.18}$$

and define the estimator $\hat{\alpha}$ by

$$\hat{\alpha} = \sum_{u=1}^{m} L_u \log(\bar{Z}_N^u). \tag{3.19}$$

Note that the (unknown) factor $N^{\alpha/d}$ which appears in the definition of $\bar{Z}_N^u$ via (3.10)-(3.12) disappears from (3.19) because of the first condition in (3.18).

Corollary 3.3. *Under the conditions of Theorem* 3.2*,*

$$\hat{\alpha} - \alpha = N^{-1/2}\xi + o_p(N^{-1/2}) + O(N^{-\beta/d}) \tag{3.20}$$

where $\xi \sim N(0, \sigma_\xi^2)$*,* $\sigma_\xi^2 = \mathbf{t}^T \Phi_0 \mathbf{t}$*,* $\mathbf{t} = (L_1/E(\bar{Z}_N^1), \ldots, L_m/E(\bar{Z}_N^m))^T$*, and* $\beta > 0$ *is the quantity which appears in* (2.4)*.*

Proof. From (3.19),

$$\hat{\alpha} = \sum_u L_u \log(\bar{Z}_N^u) = \sum_u L_u \log\Big(1 + \frac{\bar{Z}_N^u - E(\bar{Z}_N^u)}{E(\bar{Z}_N^u)}\Big) + \sum_u L_u \log\{E(\bar{Z}_N^u)\}. \tag{3.21}$$

The first two terms on the right hand side of (3.20) are accounted for by Theorem 3.2, combined with Taylor expansion applied to the first term on the right hand side of (3.21). Also, using (3.12) and an argument rather similar to that leading to (3.13), but taking explicit account of the remainder term in (2.4), it is found that $E(\bar{Z}_N^u) = Cu^\alpha\{1 + O(N^{-\beta/d})\}$. Consequently, using (3.18), it is seen that $\sum_u L_u \log\{E(\bar{Z}_N^u)\} = \alpha + O(N^{-\beta/d})$, and the desired conclusion follows.

Excluding exceptional circumstances, the $O(N^{-\beta/d})$ in (3.20) will be zero only if the remainder term in (2.4) is zero.

4. The Square-increment Estimator

We now define two estimators of α, focussing exclusively on the case $d = 2$. Both estimators are based on the "square increment" (3.6) and they are calculated using linear regression to estimate α in the approximate relationship

$$\log(\bar{Z}_N^u) \simeq c + \alpha \log(u), \quad u = 1, \ldots, m, \tag{4.1}$$

where m is fixed. The heuristic justification for assuming that (4.1) holds approximately is now explained: (3.13) implies that $E(\bar{Z}_N^u) \simeq Cu^\alpha$ or, equivalently, $\log\{E(\bar{Z}_N^u)\} \simeq c + \alpha \log u$, where $c = \log(C)$; and, assuming that $\bar{Z}_N^u$ is a consistent estimator of $E(\bar{Z}_N^u)$ as $N \to \infty$, $\log(\bar{Z}_N^u)$ will be close to $\log\{E(\bar{Z}_N^u)\}$ with high probability when N is sufficiently large. The combination of these two approximations yields (4.1), since m is assumed fixed.

The difference between the two estimators of α we consider is that one is obtained using ordinary least squares (OLS) while the other is obtained using generalised least squares (GLS) with a suitable symmetric weight matrix; cf. Kent and Wood (1997).

Let $\boldsymbol{\lambda} = (\lambda_u : u = 1, \ldots, m)$ where $\lambda_u = \log u$. Given a multi-index $i \in \mathcal{I}_n$ and integers q and r, we define $i_{q,r} = (i[1] + q, i[2] + r)$ and write

$$\bar{A}_\square^u = N^{-1} \sum_{i \in \mathcal{I}_n} \Big\{ X\Big(\frac{i_{0,0}}{n}\Big) + X\Big(\frac{i_{u,u}}{n}\Big) - X\Big(\frac{i_{u,0}}{n}\Big) - X\Big(\frac{i_{0,u}}{n}\Big) \Big\}^2. \tag{4.2}$$

Note that $\bar{Z}_N^u = N^{\alpha/d} \bar{A}_\square^u$ where $\bar{A}_\square^u$ depends only on the data and $\bar{Z}_N^u$ is defined in (3.12) and is based on the square increment (3.6). In order that all terms in the sum (4.2) are well-defined, it is assumed that we actually observe the random field $X(t)$ at all $t \in \mathcal{I}_{n,m}^*$ where $\mathcal{I}_{n,m}^* = \{n^{-1} i_{q,r} : i \in \mathcal{I}_n;\ q, r = 1, \ldots, m\}$.

4.1. The OLS estimator

The OLS estimator, $\hat{\alpha}_1$, is defined as

$$\hat{\alpha}_1 = \frac{\sum_{u=1}^m \Big(\lambda_u - m^{-1} \sum_{v=1}^m \lambda_v\Big) \log \bar{A}_\square^u}{\sum_{u=1}^m \Big(\lambda_u - m^{-1} \sum_{v=1}^m \lambda_v\Big)^2}. \tag{4.3}$$

4.2. The GLS estimator

Given an $m \times m$ weight matrix W, the GLS estimator of α based on W is given by

$$\hat{\alpha}_2 = \frac{(\mathbf{1}^T W \mathbf{1})(\boldsymbol{\lambda}^T W \mathbf{y}) - (\mathbf{1}^T W \boldsymbol{\lambda})(\mathbf{1}^T W \mathbf{y})}{(\mathbf{1}^T W \mathbf{1})(\boldsymbol{\lambda}^T W \boldsymbol{\lambda}) - (\mathbf{1}^T W \boldsymbol{\lambda})^2}, \tag{4.4}$$

where $\mathbf{y} = (\log \bar{A}_\square^u : u = 1, \ldots, m)$ and $\mathbf{1}$ is the m-vector of ones. See Kent and Wood (1997, Section 4) for relevant discussion. Observe that if W is the $m \times m$ identity matrix, or $m = 2$, then $\hat{\alpha}_2 = \hat{\alpha}_1$. The GLS estimator we consider is based on a choice of W given below.

4.3. The weight matrix W

Our choice of weight matrix is motivated by the elliptical covariance function model (i.e. it is assumed that the contours of the covariance function are

elliptical). This model was discussed in some detail by Davies and Hall (1998). In our notation, the assumption of elliptical contours corresponds to choosing the function M in (2.4) to be of the form

$$M(t/||t||) = \Big\{\frac{e_1^2 t_1^2 + e_2^2 t_2^2 + 2e_{12}t_1 t_2}{t_1^2 + t_2^2}\Big\}^{\alpha/2}. \tag{4.5}$$

We take W to be the inverse of the matrix with element (u, v), $u, v = 1, \ldots, m$, given by

$$2N^{-1} \sum_{-n<h<n} \Big(1 - \frac{|h[1]|}{n[1]}\Big)\Big(1 - \frac{|h[2]|}{n[2]}\Big)\frac{\{\hat{\sigma}_0^{uv}(h)\}^2}{\hat{\sigma}_0^{uu}(0)\hat{\sigma}_0^{vv}(0)}, \tag{4.6}$$

where

$$\hat{\sigma}_0^{uv}(h) = -\sum_{j,k} a_j^u a_k^v ||h + k - j||^{\hat{\alpha}_1} \hat{M}\Big(\frac{h + k - j}{||h + k - j||}\Big) \tag{4.7}$$

and $\hat{M}$ is M as given by (4.5), but with e_1, e_2, e_{12} replaced by estimators $\hat{e}_1, \hat{e}_2, \hat{e}_{12}$ which are defined explicitly in Appendix A, and α estimated by $\hat{\alpha}_1$, the estimator defined in (4.3). If the elliptical model (4.5) is correct, then (4.6) is a consistent estimate of the covariance matrix of $\log(\bar{Z}_u)$, which may be obtained using (3.14) and Taylor expansion. If the elliptic assumption is not exactly correct, but anisotropy is still present, it is still reasonable to hope that use of the weight matrix W will at least partially take account of any anisotropy.

4.4. Standard errors

Under the conditions of Theorem 3.2, the asymptotic behaviour of $\hat{\alpha}_1$ and $\hat{\alpha}_2$ is described by Corollary 3.3. The standard errors of $\hat{\alpha}_1$ and $\hat{\alpha}_2$ can be estimated by $N^{-1/2}\hat{\sigma}_\xi$ where $\hat{\sigma}_\xi$ is a suitable estimator of σ_ξ in (3.20) in each case.

5. The Non-Gaussian Case

So far we have assumed that the underlying random field is Gaussian. In this section we explore what happens when the field is a stationary but non-Gaussian random field. Many distinct types of departure from the Gaussian assumption are possible and, taken as a whole, these departures are difficult to characterize. Here, we focus on a particular type of departure which is easy to describe: the field is assumed to be a point transformation of a Gaussian field. More specifically, it is assumed that the observed field is of the form $g\{X(t)\}$ where $g : \mathbf{R} \to \mathbf{R}$ is a smooth, non-linear function and $\{X(t)\}$ is a stationary Gaussian field as before.

In Theorem 5.1 below, we focus on the particular choice $g(x) = x^2$, resulting in a stationary χ_1^2 field. Even under this relatively simple departure, there are

some appreciable differences to the Gaussian case, as will be seen below. Further work is required to determine the extent to which the special case $g(x) = x^2$ is representative of the general case of smooth, non-linear g.

An application of Theorem 3.1 in Hall and Roy (1994) shows that, when $d = 1$, the fractal dimension of the sample function of a χ_1^2 field constructed as indicated above satisfies (1.2) with probability one. Using broadly similar arguments, it can be shown that (2.7) holds for χ_1^2 fields when $d = 2$. Consequently, the approximate linear relationship (4.1) can still be used to estimate the fractal dimension of the sample function via (1.2) or (2.7). Of particular interest here is the question of whether the performance of estimators based on (4.1) differ in the Gaussian and non-Gaussian cases. Theorem 5.1 and its proof indicate that there are quite substantial theoretical differences.

Simplified notation will be used in this section. In particular dependence on $n \in \mathbf{Z}^d$ and $N \in \mathbf{Z}$ will often be suppressed. Write X_i for $X(i/n)$, $Y_{u,i}$ for $N^{\alpha/(2d)} \sum_{-Ju \le j \le Ju} a_j^u X_{i+j}$ and put $\mu_u = E(Y_{u,i}^2)$. The sum of squares of a typical dilated increment is given by

$$\bar{Z}_u = N^{-1} \sum_{i \in \mathcal{I}_n} \Big\{ N^{\alpha/(2d)} \sum_j a_j^u g(X_{i+j}) \Big\}^2. \tag{5.1}$$

We assume that $\hat{\alpha}$, the estimator of α under consideration, is of the form described in (3.18) and (3.19). Note that if X is a stationary Gaussian random field which satisfies (2.4), then the proof of Corollary 3.3 shows that

$$\alpha_N = \sum_{u=1}^{m} L_u \log(\mu_u) = \alpha + O(N^{-\beta/d}) \tag{5.2}$$

as $N \to \infty$. In (5.2), $\beta > 0$ is the quantity given in (2.4). We also define

$$\bar{G} = N^{-1} \sum_{i \in \mathcal{I}_n} \{g'(X_i)\}^2. \tag{5.3}$$

We now formulate our main result in the non-Gaussian case. In the theorem below, $\mathbf{a}$ is an increment of order $p \ge 0$; the number of dilations, m, stays fixed; the estimator $\hat{\alpha}$ is defined by (5.1), (3.18) and (3.19); X is a stationary d-dimensional Gaussian random field (with $d = 1$ or 2) which is assumed to satisfy (2.4) and $\mathcal{A}_4^{(d)}$ in (2.5); and (2.3) is assumed to hold.

Theorem 5.1. *Suppose that $g(\cdot) = (\cdot)^2$ and $d = 1$ or 2. Then*

$$\hat{\alpha} - \alpha = \bar{G}^{-1} T_N + O(N^{-\beta/d}) + o_p(T_N) \tag{5.4}$$

where $\bar{G}$, defined in (5.3)*, is bounded away from zero and infinity, and T_N is a random variable with the following properties: for any increment of order $p \ge 0$,*

$$E(T_N) = O(N^{-\alpha/d}) \tag{5.5}$$

and

$$\mathrm{Var}\,(T_N) = \begin{cases} O(N^{-1}) & \text{if} \quad 0 < \alpha < 2p + 2 - d/2, \\ O\{N^{-1}L(N)\} & \text{if} \quad \alpha = 2 - d/2 \text{ and } p = 0, \\ O(N^{-(4-2\alpha)/d}) & \text{if} \quad 2 - d/2 < \alpha < 2 \text{ and } p = 0, \end{cases}$$

where $L(N)$ is a function slowly varying at infinity. Note that when $p \geq 1$, $\mathrm{Var}\,(T_N)$ is $O(N^{-1})$ for all $0 < \alpha < 2$ (because $2p + 2 - d/2 > 2$ when $p \geq 1$ and $d \leq 2$).

The proof is given in Subsection 7.2 below.

Remark 5.1. Under the assumptions of Theorem 5.1, we have the following non-Gaussian analogue of Corollary 3.3, provided the increment used has order $p \geq 1$:

$$\hat{\alpha} - \alpha = O_p(N^{-1/2}) + O_p(N^{-\alpha/d}) + O(N^{-\beta/d}) \tag{5.6}$$

for any $\alpha \in (0, 2)$.

Remark 5.2. The early part of the proof of Theorem 5.1 makes use of a consistency argument for $\hat{\alpha}$ in the non-Gaussian case which is due to Peter Hall (personal communication). His argument works for any g belonging to a broad class of functions. The subtle point in the non-Gaussian case is that even though the variogram cannot be estimated consistently, the quantity $\bar{G}$ which is causing the trouble disappears when we calculate the estimator $\hat{\alpha}$.

Remark 5.3. One may ask whether the order statement for $E(T_N)$ in Theorem 5.1 is tight when $0 < \alpha < d/2$. Scrutiny of formula (7.20) indicates that the order statement is "typical" and "representative", even if (perhaps) it does not hold in every possible case.

Remark 5.4. In principle it should be possible to extend our method of proof without substantial change to the case of a general polynomial g but so far we have not attempted to do so.

Remark 5.5. Theorem 5.1 does not say anything about the limit distribution of $\{\mathrm{Var}\,(T_N)\}^{-1/2}\{T_n - E(T_N)\}$. Our guess is that the limit distribution is normal when $\mathrm{Var}\,(T_N) = O(N^{-1})$, and non-normal when $\mathrm{Var}\,(T_N) = O(N^{-(4-2\alpha)/d})$, where the relevant distribution theory in the non-normal case is indicated in Remark 3.3. However, the extra layer of non-linearity introduced by the transformation $g(\cdot)$ seems to make the problem difficult, and so far we have not obtained any rigorous results in either of the cases distinguished above.

Remark 5.6. Even if we could prove that $N^{1/2}\{T_N - E(T_N)\}$ is asymptotically normal when $\mathrm{Var}\,(T_N) = O(N^{-1})$, it would not be the case that $\hat{\alpha} - \alpha_N$ is asymptotically normal. The latter distribution would be a scale mixture of normals (where the random scale factor is $\bar{G}^{-1}$).

6. Numerical Results

In this section, we first compare the performance of the five order 1 increments introduced in Section 3; see (3.4)–(3.8). Then we concentrate on the square-increment and compare the performance of the OLS and GLS estimators defined in (4.3) and (4.4). We study the performance of these estimators in both Gaussian and χ_1^2 settings. Finally, in Subsection 6.2 we try these estimators out on real data consisting of surface height measurement on six food wrap surfaces. These data were analysed by Davies and Hall (1998).

6.1. Simulation studies

We simulated Gaussian random fields on grids of dimension 50×50, 128×128, and 500×500 with γ, the covariance function, given by $\gamma(t) = \exp\{-M(t/||t||)\ ||t||^\alpha\}$, where $M(t/||t||)$ is defined in (4.5). Gaussian and χ_1^2 fields with fractal dimension ranging from 2.05 to 2.95 are considered. These fractal dimensions correspond to α ranging from 0.1 to 1.9. The shape-and-scale parameters e_1, e_2 and e_{12} are chosen to be 1, 1, 0 for isotropic processes and 2, 1, 0 for anisotropic processes. The maximum dilation m is chosen to be between 2 and 8, inclusive. Note that when $m = 2$, the OLS and GLS estimators are identical. The fields were simulated using the circulant embedding procedure described by Wood and Chan (1994); for larger values of α it was necessary to use an approximate form of the procedure; see Section 4 of that paper for further discussion. In Tables 1, 2 and 3 some representative results are shown.

In the first study, we consider OLS estimators based on the five first-order increments defined in (3.4)-(3.8). The most striking (and unexpected) aspect of Table 1 is that the magnitude of the bias of all estimators, with the exception the square-increment estimator, increases sharply as α increases. These large biases appear to be a consequence of using the approximate simulation procedure described in Wood and Chan (1994, Section 4), and seem not to be due to the estimators themselves; in other words, the problem seems to be with the simulation procedure rather than the estimators. The approximate simulation procedure was used because when simulating smoother processes (i.e. processes with α close to 2) it was not feasible to use the exact form of the procedure. Similar dramatic increases in bias do not seem to occur in the one-dimensional case (which suggests that the approximate simulation procedure is satisfactory when $d = 1$); see e.g. the numerical results in Kent and Wood (1997). For some reason that we do not fully understand, the differencing employed by the square-increment estimator largely removes the distributional error in the realisation, whereas the differencing employed by the other estimators does not. Consequently, Table 1 suggests that the square-increment estimator may possess certain robustness properties not shared by the four other estimators.

Table 1. Comparison of the five order 1 increments.

		Bias	SD	MSE
$\alpha = 0.7$				
$m = 2$	H	0.0002	0.0070	0.00005
	V	−.00005	0.0072	0.00005
	$\square$	−.0028	0.0065	0.00005
	D_+	−.0027	0.0038	0.00002
	D_-	−.0009	0.0053	0.00003
$m = 5$	H	−.0018	0.0058	0.00004
	V	−.0020	0.0049	0.00003
	$\square$	−.0049	0.0048	0.00005
	D_+	−.0054	0.0053	0.00006
	D_-	−.0029	0.0040	0.00002
$\alpha = 1.3$				
$m = 2$	H	−.2208	0.0681	0.0534
	V	−.2051	0.0694	0.0469
	$\square$	0.0004	0.0062	0.00004
	D_+	−.1970	0.0233	0.0393
	D_-	−.1419	0.0325	0.0212
$m = 5$	H	−.2020	0.0392	0.0423
	V	−.1987	0.0427	0.0413
	$\square$	0.0015	0.0059	0.00004
	D_+	−.1882	0.0225	0.0359
	D_-	−.1331	0.0185	0.0181
$\alpha = 1.9$				
$m = 2$	H	−.8990	0.1033	0.8189
	V	−.9044	0.1041	0.8288
	$\square$	0.0822	0.0042	0.0068
	D_+	−.8941	0.0436	0.8013
	D_-	−.5737	0.0512	0.3318
$m = 5$	H	−.9026	0.0762	0.8204
	V	−.9014	0.0708	0.8175
	$\square$	0.0841	0.0061	0.0071
	D_+	−.8900	0.0386	0.7936
	D_-	−.5733	0.0375	0.3301

Notes: each cell estimate is based on 100 simulated Gaussian fields with covariance function (4.5); in each case, the number of grid points is $500 \times 500 = 250,000$; H, V, $\square$, D_+ and D_- denote the corresponding increments (see (3.4)–(3.8)); the bias is the mean of the 100 simulated estimates of α minus the true value of α; SD denotes the standard deviation of the 100 simulated estimates of α; $\text{MSE} = \text{Bias}^2 + \text{SD}^2$.

Table 2. Comparison of the estimators $\hat{\alpha}_1$ and $\hat{\alpha}_2$.

(a) Isotropic Gaussian Processes

	$\hat{\alpha}_1(4.3)$			$\hat{\alpha}_1(4.4)$		
	Bias	SD	MSE	Bias	SD	MSE
$\alpha = 0.1$						
$m = 2$	$-.0315$	0.0908	0.0092 (20)	$-.0315$	0.0908	0.0092 (20)
$m = 4$	$-.0326$	0.0447	0.0031 (7)	$-.0325$	0.0476	0.0033 (5)
$m = 6$	$-.0359$	0.0366	0.0026 (3)	$-.0359$	0.0390	0.0028 (4)
$m = 8$	$-.0352$	0.0269	0.0020 (1)	$-.0354$	0.0306	0.0022 (3)
$\alpha = 1.0$						
$m = 2$	$-.0034$	0.0903	0.0082	$-.0034$	0.0903	0.0082
$m = 4$	$-.0033$	0.0760	0.0058	$-.0049$	0.0722	0.0052
$m = 6$	$-.0129$	0.0788	0.0064	$-.0095$	0.0685	0.0048
$m = 8$	$-.0231$	0.0839	0.0076	$-.0105$	0.0668	0.0046
$\alpha = 1.9$						
$m = 2$	0.0070	0.0807	0.0066 (13)	0.0070	0.0807	0.0066 (13)
$m = 4$	0.0337	0.0907	0.0094 (22)	0.0138	0.0786	0.0064 (18)
$m = 6$	0.0654	0.1112	0.0166 (42)	0.0217	0.0793	0.0068 (19)
$m = 8$	0.1012	0.1342	0.0282 (56)	0.0267	0.0980	0.0103 (24)

(b) Anisotropic Gaussian Processes

	$\hat{\alpha}_1(4.3)$			$\hat{\alpha}_1(4.4)$		
	Bias	SD	MSE	Bias	SD	MSE
$\alpha = 0.1$						
$m = 2$	$-.0309$	0.0896	0.0090 (22)	$-.0309$	0.0896	0.0090 (22)
$m = 4$	$-.0282$	0.0408	0.0025 (4)	$-.0279$	0.0427	0.0026 (5)
$m = 6$	$-.0301$	0.0312	0.0019 (2)	$-.0286$	0.0329	0.0019 (2)
$m = 8$	$-.0348$	0.0271	0.0019 (1)	$-.0320$	0.0278	0.0018 (1)
$\alpha = 1.0$						
$m = 2$	$-.0218$	0.0880	0.0082	$-.0218$	0.0880	0.0082
$m = 4$	$-.0127$	0.0713	0.0052	$-.0167$	0.0687	0.0050
$m = 6$	$-.0148$	0.0708	0.0052	$-.0164$	0.0635	0.0044
$m = 8$	$-.0244$	0.0829	0.0075	$-.0230$	0.0646	0.0047
$\alpha = 1.9$						
$m = 2$	0.0621	0.0988	0.0136 (35)	0.0621	0.0988	0.0136 (35)
$m = 4$	0.1277	0.1124	0.0289 (59)	0.0843	0.0971	0.0165 (38)
$m = 6$	0.1932	0.1453	0.0584 (76)	0.0986	0.1096	0.0217 (47)
$m = 8$	0.2554	0.1775	0.0967 (82)	0.1155	0.1489	0.0355 (47)

Notes: each row is based on 100 simulated Gaussian fields with $50 \times 50 = 2500$ grid points; $\hat{\alpha}_1$ and $\hat{\alpha}_2$ are the OLS and GLS estimates based on the square increment; the number in curved brackets $(\cdot)$ is the number of realisations which give estimates of α which lie outside $(0, 2]$.

In the second study, we compare the performance of the OLS and GLS estimators for both isotropic and anisotropic random fields. Selected simulation results for isotropic fields are presented in Table 2(a). The performances of $\hat{\alpha}_1$ and $\hat{\alpha}_2$ are quite similar for rougher surfaces (see $\alpha = 0.1$). For smoother surfaces (see $\alpha = 1.0$ and especially $\alpha = 1.9$), $\hat{\alpha}_2$ performs better than $\hat{\alpha}_1$: in most cases, both the absolute bias and variance are smaller. The performance of both estimators deteriorates as m increases, but the deterioration is more marked in the case of $\hat{\alpha}_1$. Broadly similar results were obtained for the anisotropic fields; see Table 2(b).

The last study is concerned with the performance of both estimators for χ^2_1 fields. Table 3 shows that for rougher surfaces both estimators perform equally well. However for smoother surfaces the GLS estimator performs slightly better than the OLS estimator. Note that the absolute bias in Table 3 is large when α is either close to zero or close to 2. Note that large bias when α is close to zero is predicted by Theorem 5.1 (though it should be remembered that there are also other sources of bias which could be contributing).

Table 3. Simulation results for χ^2 fields.

	$\hat{\alpha}_1(4.3)$			$\hat{\alpha}_1(4.4)$		
α	Bias	SD	MSE	Bias	SD	MSE
0.2	−.0919	0.0795	0.0148 (9)	−.0906	0.0799	0.0146 (9)
0.4	−.1282	0.0847	0.0236	−.1286	0.0845	0.0237
0.6	−.0742	0.1085	0.0173	−.0728	0.1057	0.0165
0.8	−.0315	0.0982	0.0106	−.0313	0.0934	0.0097
1.0	−.0146	0.1188	0.0143	−.0178	0.1106	0.0125
1.2	0.0240	0.1218	0.0154	0.0161	0.1049	0.0113
1.4	0.0269	0.1331	0.0185	0.0159	0.1116	0.0127
1.6	0.0465	0.1464	0.0236 (2)	0.0385	0.1372	0.0203 (2)
1.8	0.1112	0.1458	0.0336 (21)	0.0737	0.1316	0.0228 (16)

Notes: each cell estimate is based on 100 simulated χ^2 fields with $50 \times 50 = 2500$ grid points; $m = 4$ in all cases; the number in curved brackets $(\cdot)$ is the number of realisations which give estimates of α outside the range $(0, 2]$.

Our simulation results suggest the following (rather tentative) conclusions.

(i) If we are prepared to work with the Gaussian assumption, then the estimator $\hat{\alpha}_2$ with $m = 4$ seems to be a good choice for an estimator of α.

(ii) Table 3 suggests that, under the non-Gaussian model described in Section 5, there will be little or no advantage in using $\hat{\alpha}_2$ rather than the computationally simpler $\hat{\alpha}_1$. Further numerical work would be required to establish whether or not the results in Table 3 are in fact broadly representative of non-Gaussian cases.

In the case of non-Gaussian data, a referee has suggested the possibility of applying a transformation such as Box-Cox to the raw data before constructing the estimate of α; this is an interesting avenue for further work.

6.2. An example: food wrap data

We now estimate the fractal dimension and other unknown covariance function parameters using a set of six food wrap datasets. The aim of the original study was to determine which manufacturing process produces the smoothest wrapping. This is of interest because the smoother the wrapping, the smaller is the tendency for micro-organisms to adhere. See Davies and Hall (1999) for more detailed description of these data.

Among these six food wraps, the fourth one is obviously different from the others. It is more regular and also smoother. Figure 1 shows surface 4 and one of the more typical surfaces, surface 5, while Figure 2(a) gives the six log-log plots. Three log-log plots for simulated surfaces are shown in Figure 2(b) for comparison.

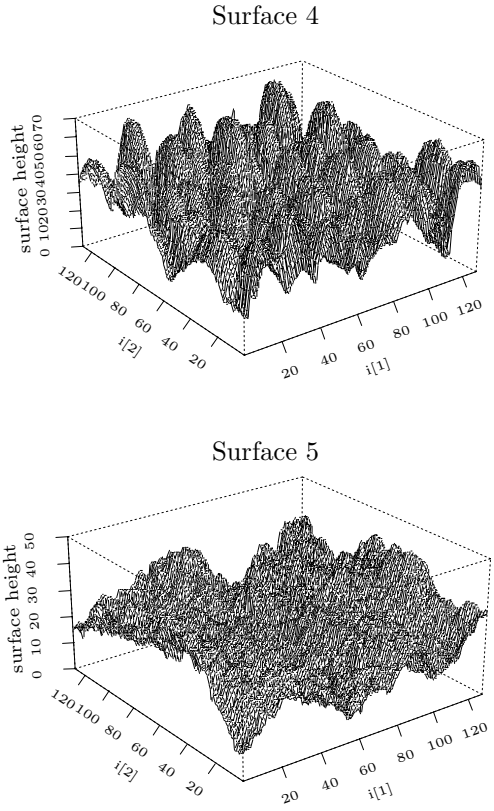
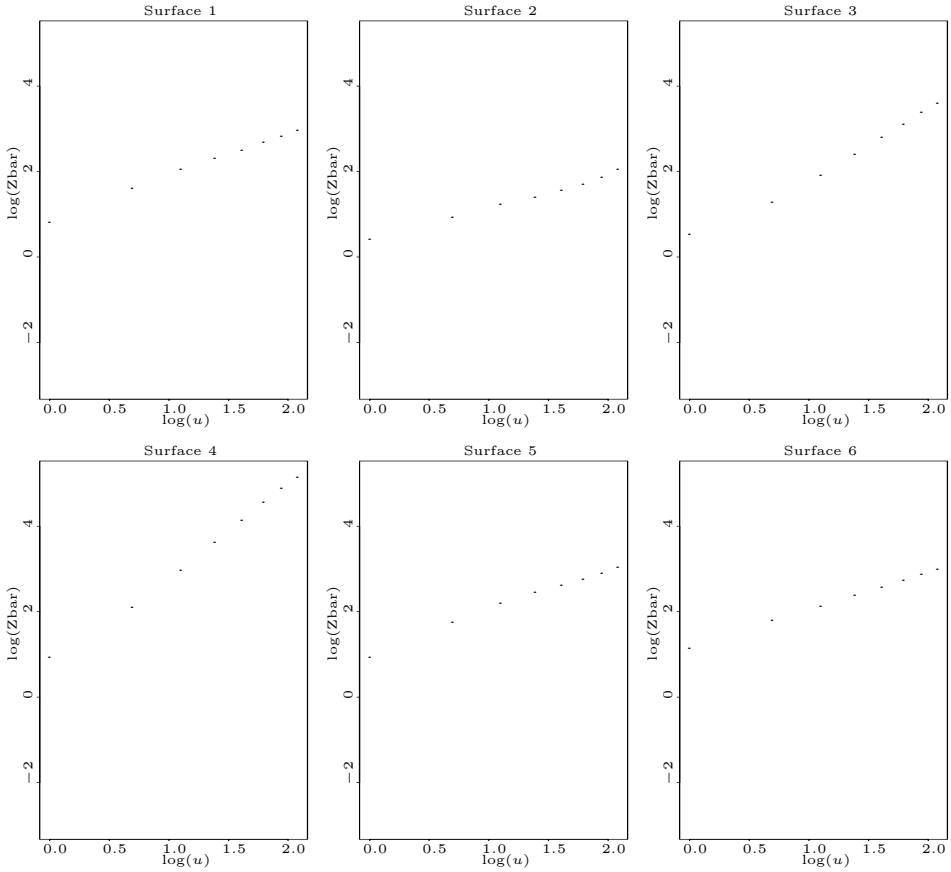


Figure 1. Perspective plots for food wraps 4 and 5.

(a) The six food wrap datasets.



(b) Three simulated surfaces: isotropic Gaussian; anisotropic Gaussian; isotropic χ^2_1.

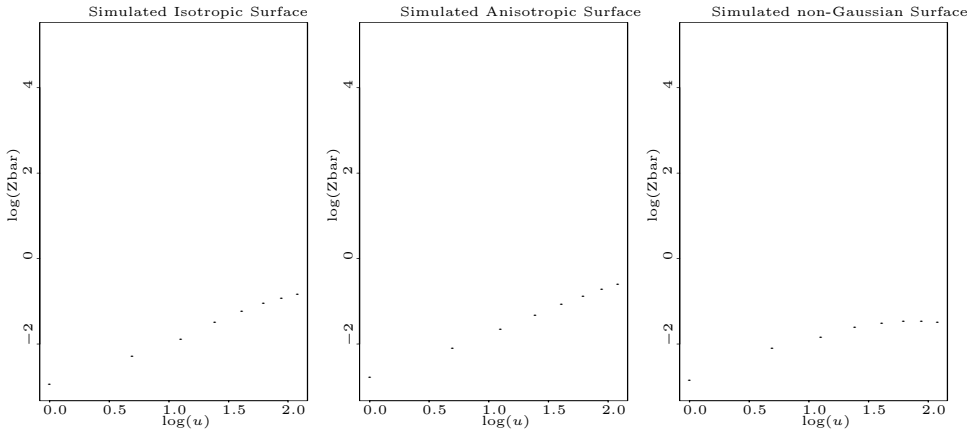


Figure 2. Log-log plots based on (4.1).

All unknown parameters in the covariance function are estimated for $m = 2, \ldots, 8$. The different choices of m give very similar results. Hence in Table 4 we have only presented results obtained with $m = 4$. We first note that $\hat{\alpha}_1$ and $\hat{\alpha}_2$ are very close. Computing the estimated fractal dimension, $\hat{D}$, based on the relationship (2.7), we can compare our estimates of fractal dimension with those given in Davies and Hall (1998). Overall, our estimates and standard errors are fairly similar to theirs, even though the two approaches are quite different.

Table 4. Numerical results for the six food wrap surfaces.

Surface	$\hat{\alpha}_1$ $\{s.e.\}$	$\hat{\alpha}_2$ $\{s.e.\}$	$\hat{D}$	$hats$	$\hat{\psi}$
1	1.09 {0.022}	1.11 {0.018}	2.44 (2.50)	0.46 (0.41)	0.26 (0.12)
2	0.73 {0.019}	0.73 {0.018}	2.63 (2.48)	0.57 (0.67)	0.14 (0.12)
3	1.35 {0.022}	1.21 {0.036}	2.39 (2.31)	0.77 (0.72)	−0.10 (−0.18)
4	1.95 {0.025}	1.76 {0.047}	2.12 (2.19)	0.97 (0.69)	−0.04 (−0.04)
5	1.11 {0.022}	1.12 {0.019}	2.44 (2.46)	0.60 (0.60)	0.23 (0.17)
6	0.89 {0.021}	0.91 {0.017}	2.55 (2.58)	0.72 (0.59)	0.09 (0.09)

Notes: for each surface, $\hat{\alpha}_1$ and $\hat{\alpha}_2$ are the OLS and GLS estimators of α; the estimated standard errors were obtained using Corollary 3.3; and in all cases, $m = 4$. The quantity $\hat{D}$ is given by $3 - \hat{\alpha}_2/2$, and the estimates $\hat{s}$ and $\hat{\psi}$ are calculated using (5.1)–(5.3). The numbers in curved brackets $(\cdot)$ in the $\hat{D}$, $\hat{s}$ and $\hat{\psi}$ columns are the corresponding Davies and Hall (1999) estimates.

Moreover, we can also compare parameters in the covariance function that characterise other features of food wraps. Let us first recall their choice of M and show how our parameters, e_1, e_2 and e_{12} relate to their parametrisation, with the latter given by $M(\theta) = (c_0/2)[1 + s\cos\{2(\theta - \psi)\}]^{\alpha/2}$, where $s \in [0, 1]$ is a measure of isotropy (isotropic surfaces will have $s = 0$). The quantities ψ and c_0 are the "maximal roughness orientation" and "average topothesy" respectively.

With elementary manipulations, we find that $c_0 = 2^{1-(\alpha/2)}(e_1^2 + e_2^2)^{\alpha/2}$,

$$\psi = \frac{1}{2}\arctan\Big(\frac{2e_{12}}{e_1^2 - e_2^2}\Big) \quad \text{and} \quad s = \Big(1 - \frac{4(e_1^2 e_2^2 - e_{12}^2)}{(e_1^2 + e_2^2)^2}\Big)^{1/2}.$$

To obtain estimates for ψ and s, we simply substitute the estimates of e_1, e_2 and e_{12} described in Appendix A. The quantity $\hat{c}_0$ is calculated based on information obtained in both the ordinary and generalised least squares. Apart from the estimate $\hat{c}_0$, which depends on the somewhat arbitrary choice of scale, our estimates for all other parameters are very close to the Davies and Hall (1999) estimates for all surfaces; see Table 4. The estimated standard errors in Table 4 for $\hat{\alpha}_1$ and $\hat{\alpha}_2$ were obtained using Corollary 3.3. One could also obtain estimates of the standard errors of the other parameter estimates in Table 4 by applying Taylor expansion arguments to the formulae in Appendix A. We have not done so,

principally because we view the other parameter estimates as "quick–and–dirty", whose main purpose is for substitution in the GLS weight matrix.

The results in Table 4 suggest that all six food wrap surfaces are anisotropic.

7. Proofs

We begin by stating a key elementary result. Let n be a multi-index with positive components which converge to infinity according to (2.3), $\mathcal{I}_n$ the set (of cardinality N) defined by $\mathcal{I}_n = \{i : 0 \le i < n\}$, and L the generic symbol for a function which is slowly varying at infinity. Then

$$\sum_{-n<h<n} \{1 + ||h||\}^{-\rho} = \begin{cases} O(1) & \text{if} \quad \rho > d, \\ O\{L(N)\} & \text{if} \quad \rho = d, \\ O(N^{(d-\rho)/d}) & \text{if} \quad \rho < d. \end{cases} \tag{7.1}$$

An equivalent statement of (7.1) will sometimes be more convenient for our purposes:

$$N^{-2} \sum_{i\in\mathcal{I}_n} \sum_{j\in\mathcal{I}_n} (1 + ||i-j||)^{-\rho} = \begin{cases} O(N^{-1}) & \text{if} \quad \rho > d, \\ O\{N^{-1}L(N)\} & \text{if} \quad \rho = d, \\ O(N^{-\rho/d}) & \text{if} \quad \rho < d. \end{cases} \tag{7.2}$$

7.1. Proof of results in Section 3

Proof of Lemma 3.1. For $0 \le r \in \mathbf{Z}^d$, define $r! = \prod_{\ell=1}^{d}(r[\ell]!)$. It follows directly from the defining property of increments that if increments $\mathbf{a}$ and $\mathbf{b}$ have orders $p' \ge p$ and $q' \ge q$, respectively, and the multi-index r satisfies $|r| \le p+q+1$, then

$$\begin{aligned} \sum_{j,k} a_j^u b_k^v (k-j)^r &= \sum_{j,k} a_j^u b_k^v \sum_{0\le h\le r} \frac{r!}{h!(r-h)!}(-1)^h j^h k^{r-h} \\ &= \sum_{0\le h\le r} (-1)^h \frac{r!}{h!(r-h)!}\Big(\sum_j j^h a_j^u\Big)\Big(\sum_k k^{r-h} b_k^v\Big) \\ &= 0, \end{aligned} \tag{7.3}$$

because for all multi-indices $0 \le h \le r$ such that $|r| \le p+q+1$, either $|h| \le p$ in which case $\sum_j j^h a_j^u = 0$, or $|r-h| \le q$ in which case $\sum_k k^{r-h} b_k^v = 0$. (Note that when $d \ge 2$ and $\mathbf{a} \ne \mathbf{b}$, it is possible that (7.3) also holds true for larger values of $|r|$, depending on the particular choices of $\mathbf{a}$ and $\mathbf{b}$.) Now

$$\begin{aligned} \sigma^{uv}_{N,\mathbf{a},\mathbf{b}}(h) &= \operatorname{cov}\{Y^u_{N,\mathbf{a}}(i), Y^v_{N,\mathbf{b}}(i+h)\} \\ &= N^{\alpha/d} \sum_{j,k} a_j^u b_k^v \gamma\Big(\frac{h+k-j}{n}\Big). \end{aligned} \tag{7.4}$$

Using $\mathcal{A}^{(d)}_{p+q+2}$ in (2.5), we may Taylor expand γ about h as follows:

$$\gamma\Big(\frac{h+k-j}{n}\Big) = \sum_{s=0}^{p+q+1} \sum_{|r|=s} \frac{1}{r!}\gamma^{(r)}\Big(\frac{h}{n}\Big)\Big(\frac{k-j}{n}\Big)^r + \sum_{|r|=p+q+2} \frac{1}{r!}\gamma^{(r)}\Big(\frac{h^*}{n}\Big)\Big(\frac{k-j}{n}\Big)^r, \tag{7.5}$$

where $h^* = h + \eta(k-j)$ for some $\eta \in [0,1]$. Now (7.3) implies that when we substitute (7.5) into (7.4), the first sum on the right hand side of (7.5) is annihilated and we obtain

$$\sigma^{uv}_{N,\mathbf{a},\mathbf{b}} = N^{\alpha/d} \sum_{j,k} a^u_j b^v_k \sum_{|r|=p+q+2} \gamma^{(r)}\Big(\frac{h^*}{n}\Big)\frac{1}{r!}\Big(\frac{k-j}{n}\Big)^r. \tag{7.6}$$

Finally, we note that condition $\mathcal{A}^{(d)}_{p+q+2}$ in (2.5) and condition (2.3) imply that, for each non-negative multi-index r satisfying $|r| = p+q+2$,

$$N^{\alpha/d}\gamma^{(r)}\Big(\frac{h^*}{n}\Big)\Big(\frac{k-j}{n}\Big)^r = O\{(1+||h||)^{\alpha-p-q-2}\}, \tag{7.7}$$

and the conclusion of the lemma follows from (7.6) and (7.7).

Proof of Theorem 3.2. To establish the first part of the theorem, we apply (7.1) and Lemma 3.1, with $\mathbf{a} = \mathbf{b}$ and $p \geq 1$, to the right hand side of (3.16). See the proof of part (ii) of Theorem 3.1 in Kent and Wood (1997) for further details.

We now establish asymptotic normality of $\bar{\mathbf{Z}}_N$. Fix an m-vector $\mathbf{f}$ and define the $mN \times mN$ diagonal matrix Γ_N by $\Gamma_N = \mathrm{diag}\{\mathbf{f}^T, \ldots, \mathbf{f}^T\}$ for $N \geq 1$. Define the mN-vector $\mathbf{W}_N = (\mathbf{Y}^T_N(j), j \in \mathcal{I}_n)$, where $\mathbf{Y}_N(j) = (Y^u_N(j), u = 1, \ldots, m)$; it does not matter how we arrange the components in $\mathbf{W}_N$.

By construction, we have $N^{-1}\mathbf{W}^T_N\Gamma_N\mathbf{W}_N = \mathbf{f}^T\bar{\mathbf{Z}}_N$. If we can show that

$$S_N \equiv N^{1/2}\mathbf{f}^T(\bar{\mathbf{Z}}_N - E\bar{\mathbf{Z}}_N) = N^{-1/2}\{\mathbf{W}^T_N\Gamma_N\mathbf{W}_N - E(\mathbf{W}_N\Gamma_N\mathbf{W}_N)\} \tag{7.8}$$

is asymptotically normal for each fixed $\mathbf{f}$, then the conclusion will follow immediately from an application of the Cramér-Wold device (see Billingsley (1968)).

Let V_N denote the covariance matrix of $\mathbf{W}_N$. Note that each entry of V_N is of the form $\sigma^{uv}_N(h)$ with suitable choices of u, v and h. Standard manipulations involving quadratic forms in normal variables show that the cumulant generating function of S_N in (7.8) is given by

$$\kappa_N(\theta) \equiv \log\left(E\, e^{\theta S_N}\right) = -\frac{1}{2}\sum_{q=1}^{mN}\{\log\left(1-\theta\lambda_{q,N}\right) + \theta\lambda_{q,N}\}, \tag{7.9}$$

where $\lambda_{q,N}$ is the qth eigenvalue of $\Lambda_N = 2N^{-1/2}\, V_N^{1/2}\, \Gamma_N\, V_N^{1/2}$, and $V_N^{1/2}$ is the symmetric positive definite square root of V_N.

The key step in our proof is to show that

$$\operatorname{tr}(\Lambda_N^4) = \sum_{j=1}^{mN} \lambda_{q,N}^4 \to 0 \quad \text{as} \quad N \to \infty. \tag{7.10}$$

Before proving (7.10), we show that (7.10) is sufficient to ensure asymptotic normality of S_N.

First, note that (7.10) implies that $\max_{1\le q\le mN} \lambda_{q,N} \to 0$ as $N \to \infty$, and that $\delta = \sup_{N\ge 1} \max_{1\le q\le mN} |\lambda_{q,N}|$ is finite. Expanding (7.9) about $\theta = 0$ using Taylor's theorem, we obtain

$$\kappa_N(\theta) = \frac{1}{2} \sum_{q=1}^{mN} \Big\{ \frac{1}{2}\,\theta^2\, \lambda_{q,N}^2 + \frac{1}{3}\,\theta^3\, \lambda_{q,N}^3 + \frac{1}{4}\theta^4\, \lambda_{q,N}^4 (1 - \theta^* \lambda_{q,N})^{-4} \Big\} \tag{7.11}$$

for some θ^* which satisfies $0 \le |\theta^*| \le |\theta|$. If we restrict attention to $|\theta| \le \frac{1}{2}\delta^{-1}$, then $(1 - \theta^*\lambda_{q,N}) > 1/2$ for all q and N. For such θ,

$$\kappa_N(\theta) = \frac{1}{2}\theta^2 \Big(\frac{1}{2} \sum \lambda_{q,N}^2 \Big) + O\Big(\Big| \sum\ \lambda_{q,N}^3 \Big| \Big) + O\Big(\sum \lambda_{q,N}^4 \Big)\,.$$

Now

$$\frac{1}{2} \sum \lambda_{q,N}^2 = \frac{1}{2} \operatorname{tr}(\Lambda_N^2) = 2\, N^{-1} \operatorname{tr}\{(V_N\, \Gamma_N)^2\} = N\, \mathbf{f}^T \Phi_N \mathbf{f}, \tag{7.12}$$

where Φ_N is defined above (3.14). Also, by the first part of the theorem, $\zeta = N \lim_{N\to\infty} \mathbf{f}^T \Phi_N \mathbf{f} = \mathbf{f}^T \Phi_0 \mathbf{f}$ exists and is finite. Moreover, (7.10) implies that

$$\Big| \sum \lambda_{q,N}^r \Big| \le \Big(\max_{1\le q\le mN} |\lambda_{q,N}| \Big)^{r-2} \sum \lambda_{q,N}^2 \to 0 \quad \text{as} \quad N \to \infty \tag{7.13}$$

for $r \ge 3$. Putting (7.11)-(7.13) together, we see that for each $\theta \in (-(2\delta)^{-1}, (2\delta)^{-1})$, $\kappa_N(\theta) \to \theta^2\zeta/2$ as $N \to \infty$. Thus, if (7.10) holds, $\kappa_N(\theta)$ converges pointwise to the cumulant generating function of an $N(0,\zeta)$ variable in a neighbourhood of the origin, which is sufficient to ensure that $S_N \to N(0,\zeta)$ in distribution.

We now establish (7.10). Direct calculation shows that

$$\begin{aligned} \operatorname{tr}(\Lambda_N^4) &= 16N^{-2} \operatorname{tr}\{(V_N^{1/2}\, \Gamma_N\, V_N^{1/2})^4\} \\ &= 16N^{-2} \operatorname{tr}\{(V_N\, \Gamma_N)^4\} \\ &= 16N^{-2} \sum_{u_1=1}^{m} \cdots \sum_{u_4=1}^{m} f_{u_1} \ldots f_{u_4}\, \Delta_N(u_1, \ldots, u_4), \end{aligned} \tag{7.14}$$

where

$$\begin{aligned}&\Delta_N(u_1,\dots,u_4)\\ &=\sum_{i_1\in\mathcal{I}_n}\dots\sum_{i_4\in\mathcal{I}_n}\sigma_N^{u_1u_2}(i_1-i_2)\sigma_N^{u_2u_3}(i_2-i_3)\sigma_N^{u_3u_4}(i_3-i_4)\sigma_N^{u_4u_1}(i_4-i_1);\quad(7.15)\end{aligned}$$

see e.g. Taqqu (1975) for related calculations (though the result proved there is different than Theorem 3.2). For each triple (h_1,h_2,h_3) which satisfies $-n<h_a<n$, $1\le a\le 3$, the cardinality of the set

$$\#\{(i_1,i_2,i_3,i_4):i_a\in\mathcal{I}_n\,,a=1,\dots,4\,;h_a=i_a-i_{a+1}\,,1\le a\le 3\}$$

is bounded above by N. It follows that

$$\begin{aligned}&|\Delta_N(u_1,\dots,u_4)|\\ &\le N\sum_{-n<h_1<n}\dots\sum_{-n<h_3<n}|\sigma_N^{u_1u_2}(h_1)\,\sigma_N^{u_2u_3}(h_2)\,\sigma_N^{u_3u_4}(h_3)\,\sigma_N^{u_1u_4}(h_1+h_2+h_3)|\,.\end{aligned}$$

When the increment concerned has order $p\ge 1$ and condition $\mathcal{A}_4^{(d)}$ in (2.5) holds, then it follows from Lemma 3.1 that for some constants C_1 and C_2,

$$\begin{aligned}|\Delta_N(u_1,\dots,u_4)|&\le C_1\,N\prod_{a=1}^{3}\Big(\sum_{-n<h<n}|\sigma_N^{u_au_{a+1}}(h)|\Big)\\ &\le C_1\,N\prod_{a=1}^{3}\Big(\sum_{h\in\mathbf{Z}^d}|\sigma_N^{u_au_{a+1}}(h)|\Big)\\ &\le C_2\,N\Big(\sum_{h\in\mathbf{Z}^d}(1+||h||)^{\alpha-2p-2}\Big)^3\\ &=O(N)\,,\end{aligned}$$

since $\alpha-2p-2<-d$ when $p\ge 1$, $0<\alpha<2$ and $d=1,2$. Thus, $\mathrm{tr}\,(\Lambda_N^4)=O(N^{-1})$, and the theorem is proved.

7.2. Proof of Theorem 5.1

Before giving a proof, we state the "diagram formula". This formula, which plays a very useful role in our proof, gives an explicit expression in terms of variances and covariances for the expectation of an arbitrary product of Hermite polynomials in correlated Gaussian variables. We now recall the definition of the Hermite polynomials: for integer $a\ge 0$, the Hermite polynomial of degree a is given by $H_a(x)=(-1)^a\exp(x^2/2)\mathcal{D}^a\exp(-x^2/2)$ where $\mathcal{D}\equiv d/dx$. So, for example, $H_0(x)=1$, $H_1(x)=x$ and $H_2(x)=x^2-1$.

Lemma 7.1. ("Diagram Formula"). *Let $U_1, \dots, U_K$ ($K \geq 2$) be Gaussian variables with $E(U_k) = 0$, $E(U_k^2) = 1$ and $\operatorname{cov}(U_k, U_\ell) = \sigma_{k,\ell}$ for $1 \leq k, \ell \leq K$. Let $a_1, \dots, a_K$ be positive integers. If $a_1 + a_2 + \cdots + a_K$ is even and equals $2q$, say, then*

$$E\{H_{a_1}(U_1) \dots H_{a_K}(U_K)\} = \frac{a_1! \dots a_K!}{2^q q!} \sum \sigma_{k_1,\ell_1} \dots \sigma_{k_q,\ell_q}$$

where summation is over all indices $k_1, \ell_1, \dots k_q, \ell_q \in \{1, \dots, K\}$ such that (i) *$k_1 \neq \ell_1, \dots k_q \neq \ell_q$, and* (ii) *there are precisely a_1 indices 1, a_2 indices $2, \dots, a_K$ indices K. When $a_1 + \cdots + a_K$ is odd, the expectation of the above product is zero.*

See Taqqu (1977), Major (1981) and Breuer and Major (1983). In the proof of Theorem 5.1, we need the following easy consequence of Lemma 7.1.

Lemma 7.2. *Choose $L \in \{1, \dots, K-1\}$. In the notation of Lemma* 7.1*, if $a_1 + \cdots + a_k$ is even,*

$$\begin{aligned} &\operatorname{cov}\{H_{a_1}(U_1) \cdots H_{a_L}(U_L), H_{a_{L+1}}(U_{L+1}) \cdots H_{a_K}(U_K)\} \\ &= \frac{a_1! \cdots a_K!}{2^q q!} \sum \sigma_{k_1,\ell_1} \cdots \sigma_{k_q,\ell_q}, \end{aligned}$$

where the summation is over all $k_1, \ell_1, \dots, k_q, \ell_q \in \{1, \dots, K\}$ such that (i) *and* (ii) *of Lemma* 7.1 *are satisfied and, in addition, the following holds: for some $r \in \{1, \dots, q\}$,*

$$\min\{k_r, \ell_r\} \leq L \quad \text{and} \quad \max\{k_r, \ell_r\} > L.$$

If $a_1 + \cdots + a_K$ is odd, the above covariance is zero.

Proof of Theorem 5.1. Using $g(\cdot) = (\cdot)^2$ in the definition of $\bar{Z}_u$ in (5.1), putting $W_{i,j} = N^{\alpha/(2d)}(X_{i+j} - X_i)$, and using the definition of $Y_{u,i}$ given at the beginning of section 5, we obtain

$$\begin{aligned} \bar{Z}_u &= N^{-1} \sum_{i \in \mathcal{I}_n} \Big\{ 2X_i \Big(\sum_j a_j^u W_{i,j} \Big) + N^{-\alpha/(2d)} \sum_j a_j^u W_{i,j}^2 \Big\}^2 \\ &= N^{-1} \sum_{i \in \mathcal{I}_n} \Big\{ 4X_i^2 Y_{u,i}^2 + 4N^{-\alpha/(2d)} X_i Y_{u,i} \sum_j a_j^u W_{i,j}^2 + N^{-\alpha/d} \Big(\sum_j a_j^u W_{i,j}^2 \Big)^2 \Big\} \\ &= N^{-1} \sum_{i \in \mathcal{I}_n} \Big\{ 4X_i^2 \mu_u + R_{0,u,i} + N^{-\alpha/(2d)} R_{1,u,i} + N^{-\alpha/d} R_{2,u,i} \Big\} \\ &= \bar{G}\mu_u + \bar{R}_{0,u} + N^{-\alpha/(2d)} \bar{R}_{1,u} + N^{-\alpha/d} \bar{R}_{2,u}, \end{aligned} \tag{7.16}$$

where $R_{0,u,i} = 4X_i^2(Y_{u,i}^2 - \mu_u)$, $R_{1,u,i} = 4X_i Y_{u,i} \sum_j a_j^u W_{i,j}^2$,

$$R_{2,u,i} = \Big\{ \sum_j a_j^u W_{i,j}^2 \Big\}^2 \text{ and } \bar{R}_{q,u} = N^{-1} \sum_{i \in \mathcal{I}_n} R_{q,u,i} \quad (q = 0, 1, 2;\ u = 1, \dots, m).$$

Noting that $\bar{G}$ is finite and strictly positive with probability one, and taking logarithms, we obtain

$$\log(\bar{Z}_u) = \log(\bar{G}) + \log(\mu_u) + \log\Big\{1 + \frac{\bar{R}_{0,u} + N^{-\alpha/(2d)}\bar{R}_{1,u} + N^{-\alpha/d}\bar{R}_{2,u}}{\bar{G}\mu_u}\Big\}.$$

Now if (U_N) is any sequence of random variables converging to 0 in probability, then $\log(1 + U_N) = U_N\{1 + o_p(1)\}$. Consequently, if it can be shown that

$$\bar{R}_{0,u} = o_p(1) \quad \text{and} \quad \bar{R}_{q,u} = O_p(1), \quad q = 1, 2, \quad u = 1, \ldots, m. \tag{7.17}$$

Then it will follow from (3.18), (3.19) and (7.17) that

$$\begin{aligned}
\bar{G}(\hat{\alpha} - \alpha_N) &= \bar{G}\Big[\Big\{\sum_{u=1}^{m} L_u \log(\bar{Z}_u)\Big\} - \alpha_N\Big] \\
&= \bar{G}\Big\{\sum_u L_u \log(\bar{G}) + \sum_u L_u \log(\mu_u) - \alpha_N\Big\} \\
&\quad + \sum_u L_u \mu_u^{-1}\{\bar{R}_{0,u} + N^{-\alpha/(2d)}\bar{R}_{1,u} + N^{-\alpha/d}\bar{R}_{2,u}\}\{1 + o_p(1)\} \\
&= 0 + \alpha_N - \alpha_N \\
&\quad + \sum_u L_u \mu_u^{-1}\{\bar{R}_{0,u} + N^{-\alpha/(2d)}\bar{R}_{1,u} + N^{-\alpha/d}\bar{R}_{2,u}\}\{1 + o_p(1)\} \\
&= T_N\{1 + o_p(1)\},
\end{aligned} \tag{7.18}$$

where

$$\begin{aligned}
T_N &= \sum_u L_u \mu_u^{-1}\{\bar{R}_{0,u} + N^{-\alpha/(2d)}\bar{R}_{1,u} + N^{-\alpha/d}\bar{R}_{2,u}\} \\
&= \bar{R}_0 + N^{-\alpha/(2d)}\bar{R}_1 + N^{-\alpha/d}\bar{R}_2,
\end{aligned} \tag{7.19}$$

with the obvious definitions for $\bar{R}_0$, $\bar{R}_1$ and $\bar{R}_2$. The truth of statement (7.17) is a direct consequence of steps (a), (b) and (c) given below. Using (5.2), (7.18) and the positivity of $\bar{G}$, it is seen that $\hat{\alpha} - \alpha = \bar{G}^{-1}T_N + O(N^{-\beta/d}) + o_p(T_N)$ as required.

The remainder of the proof is concerned with the calculation of $E(T_N)$ and $\mathrm{Var}\,(T_N)$. In Appendix B, it is proved that

$$E(T_N) = 2DN^{-\alpha/d}\sum_{u=1}^{m} u^{\alpha} L_u + O(N^{-(\alpha+\beta)/d}), \tag{7.20}$$

where $\beta > 0$ is the quantity given in (2.5) and D, which depends on N only through $nN^{-1/d}$, is given by

$$D = -\frac{\sum_{j,k} a_j a_k f(j-k)^2}{\sum_{j,k} a_j a_k f(j-k)}, \quad \text{where} \quad f(t) = \Big|\Big|\frac{t}{\nu}\Big|\Big|^{\alpha} M\Big(\frac{t}{\nu}\Big/\Big|\Big|\frac{t}{\nu}\Big|\Big|\Big), \tag{7.21}$$

$t = j, k, j-k$, and $\nu = nN^{-1/d}$. Inspection of D shows that it will be non-zero except (possibly) in exceptional circumstances. Therefore, since $\sum_u u^\alpha L_u$ is non-zero in general, $E(T_N)$ is typically asymptotic to $const.N^{-\alpha/d}$.

In the final part of the proof we determine the order of $\mathrm{Var}\,(T_N)$. From (7.19) we have

$$\mathrm{Var}\,(T_N) = \sum_{u,v=1}^{m} L_u \mu_u^{-1} L_v \mu_v^{-1} \sum_{q,r=0}^{2} \mathrm{cov}(N^{-\alpha q/(2d)} \bar{R}_{q,u}, N^{-\alpha r/(2d)} \bar{R}_{r,v}). \quad (7.22)$$

Since the (u, v)-sum is finite, we only need to examine the order of

$$\mathrm{cov}(N^{-\alpha q/(2d)} \bar{R}_{q,u}, N^{-\alpha r/(2d)} \bar{R}_{r,v}), \qquad q, r = 0, 1, 2.$$

For convenience, let us write $W_i = W_{i,j}$ and $Y_i = Y_{u,i}$ (since the dependence of the W's and Y's on the omitted subscript does not affect the validity of any of the order statements below). Consider the joint covariance structure of the X, Y and W fields. We write $\sigma_{XX}(i-j) = \mathrm{cov}(X_i, X_j)$, $\sigma_{XY}(i-j) = \mathrm{cov}(X_i, Y_j)$ and so on. Using Lemma 3.1 combined with elementary calculations we obtain the following:

$$\sigma_{WW}(i-j) = O\{(1 + ||i-j||)^{\alpha-2}\}; \qquad \sigma_{XX}(i-j) = O(1); \quad (7.23)$$

$$\sigma_{YY}(i-j) = O\{(1 + ||i-j||)^{\alpha-2p-2}\}, \quad (7.24)$$

where p is the order of the increment used;

$$\sigma_{WX}(i-j) = O\{N^{-\alpha/(2d)}(1 + ||i-j||)^{\alpha-1}\}, \quad (7.25)$$

$$\sigma_{XY}(i-j) = O\{N^{-\alpha/(2d)}(1 + ||i-j||)^{\alpha-p-1}\}, \quad (7.26)$$

and

$$\sigma_{WY}(i-j) = O\{(1 + ||i-j||)^{\alpha-p-2}\}. \quad (7.27)$$

All the above statements hold uniformly for fixed $i, j \in \mathcal{I}_n$ as N increases.

We complete the proof by evaluating the order of the terms in (7.22) one by one; only selected details are given. The results below are obtained by repeated use of Lemma 7.2 combined with (7.2) and (7.23)-(7.27).

(a) $\mathrm{Var}\,(\bar{R}_{0,u})$. From the definition of $\bar{R}_{0,u}$ we have

$$\mathrm{Var}\,(\bar{R}_{0,u}) = 16\gamma_0 \mu_u^2 N^{-2} \sum_{i,j \in \mathcal{I}_n} \mathrm{cov}[\{H_2(X_i/\sqrt{\gamma_0}) + 1\} H_2(Y_i/\sqrt{\mu_u}), \{H_2(X_j/\sqrt{\gamma_0}) + 1\} H_2(Y_j/\sqrt{\mu_u})],$$

where $\gamma_0 = \gamma(0) = \mathrm{Var}\,(X_i)$. Now

$$\mathrm{cov}[\{H_2(X_i/\sqrt{\gamma_0}) + 1\} H_2(Y_i/\sqrt{\mu_u}), \{H_2(X_j/\sqrt{\gamma_0}) + 1\} H_2(Y_j/\sqrt{\mu_u})]$$

$$
\begin{aligned}
&= \operatorname{cov}\{H_2(X_i/\sqrt{\gamma_0})H_2(Y_i/\sqrt{\mu_u}), H_2(X_j/\sqrt{\gamma_0})H_2(Y_j/\sqrt{\mu_u})\} \\
&\quad +2\operatorname{cov}\{H_2(X_i/\sqrt{\gamma_0})H_2(Y_i/\sqrt{\mu_u}), H_2(Y_j/\sqrt{\mu_u})\} \\
&\quad +\operatorname{cov}\{H_2(Y_i/\sqrt{\mu_u}), H_2(Y_j/\sqrt{\mu_u})\}.
\end{aligned}
$$

Several applications of Lemma 7.2 show that

$$
\begin{aligned}
&\operatorname{cov}\{H_2(X_i/\sqrt{\gamma_0})H_2(X_j/\sqrt{\gamma_0})H_2(Y_j/\sqrt{\mu_u})\} \\
&= O\{\sigma_{YY}(i-j)^2\} + O\{N^{-\alpha/d}\sigma_{YY}(i-j)\} + O\{N^{-\alpha/d}\sigma_{XY}(i-j)^2\} \\
&\quad +O\{\sigma_{XY}(i-j)^2\sigma_{YY}(i-j)\},
\end{aligned}
$$

$$
\operatorname{cov}\{H_2(X_i/\sqrt{\gamma_0})H_2(Y_i/\sqrt{\mu_u}), H_2(Y_j/\sqrt{\mu_u})\} = O\{N^{-\alpha/(2d)}\sigma_{YY}(i-j)\sigma_{XY}(i-j)\},
$$

and

$$
\operatorname{cov}\{H_2(Y_i/\sqrt{\mu_u}), H_2(Y_j/\sqrt{\mu_u})\} = O\{\sigma_{YY}(i-j)^2\}.
$$

Consequently

$$
\begin{aligned}
\operatorname{Var}(\bar{R}_{0,u}) &= O\Big\{N^{-2}\sum_{i,j}\sigma_{YY}(i-j)^2\Big\} + O\Big\{N^{-2}\sum_{i,j}\sigma_{YY}(i-j)\sigma_{XY}(i-j)\Big\} \\
&= O\Big\{N^{-2}\sum_{i,j}\sigma_{YY}(i-j)^2\Big\} + O(N^{-1}), \\
&= O\Big\{N^{-2}\sum_{i,j}\sigma_{YY}(i-j)^2\Big\},
\end{aligned}
$$

which is compatible with the stated order of $\operatorname{Var}(T_N)$ in the theorem.
(b) $\operatorname{Var}(\bar{R}_{1,u})$. Arguing along the lines indicated in case (a), and using (7.2) and Lemma 7.2, we find that

$$
\begin{aligned}
&\operatorname{Var}(N^{-\alpha/(2d)}\bar{R}_{1,u}) \\
&= O\Big\{N^{-\alpha/d}N^{-2}\sum_{i,j\in\mathcal{I}_n}\sigma_{WW}(i-j)\Big\} + O\Big\{N^{-\alpha/d}N^{-2}\sum_{i,j\in\mathcal{I}_n}\sigma_{WX}(i-j)^2\Big\} \\
&= O(N^{-1}).
\end{aligned}
$$

Consequently $\operatorname{Var}(N^{-\alpha/d}\bar{R}_{1,u})$ does not violate any of the order statements in the theorem. In fact, using the elementary properties of increments and without further assumptions, we can show that $\operatorname{Var}(N^{-\alpha/(2d)}\bar{R}_{1,u}) = o(N^{-1})$.
(c) $\operatorname{Var}(N^{-\alpha/d}\bar{R}_{2,u})$. In this case, we have

$$
\operatorname{Var}(N^{-\alpha/d}\bar{R}_{2,u}) = O\Big\{N^{-2\alpha/d}N^{-2}\sum_{i,j\in\mathcal{I}_n}\sigma_{WW}(i-j)^2\Big\} = o(N^{-1}),
$$

and once again this is compatible with the requirements of the theorem.

(d) Covariance terms. Steps (a), (b) and (c) above take care of the variance terms in (7.22). To account for the covariance terms in (7.22), we use the Cauchy-Schwartz inequality as follows:

$$\begin{aligned}
&\operatorname{cov}(N^{-\alpha q/(2d)}\bar{R}_{q,u}, N^{-\alpha r/(2d)}\bar{R}_{r,v}) \\
&= O[\{\operatorname{Var}(N^{-\alpha q/(2d)}\bar{R}_{q,u})\operatorname{Var}(N^{-\alpha r/(2d)}\bar{R}_{r,v})\}^{1/2}] \\
&= O\Big\{N^{-2}\sum_{i,j\in\mathcal{I}_n}\sigma_{YY}(i-j)^2\Big\},
\end{aligned}$$

as required. The proof is now complete.

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Appendix

A. Calculation of the Weight Matrix

We now give explicit definitions of the estimators of e_1,e_2 and e_{12} in (4.5) which were used in our numerical work. Consider the mean sums of squares corresponding to the increments (3.4)-(3.8):

$$\bar{A}^u_H = N^{-1}\sum_{i\in\mathcal{I}_n}\Big\{X\Big(\frac{i_{u,0}}{n}\Big)+X\Big(\frac{i_{-u,0}}{n}\Big)-2X\Big(\frac{i_{0,0}}{n}\Big)\Big\}^2,$$

$$\bar{A}^u_V = N^{-1}\sum_{i\in\mathcal{I}_n}\Big\{X\Big(\frac{i_{0,u}}{n}\Big)+X\Big(\frac{i_{0,-u}}{n}\Big)-2X\Big(\frac{i_{0,0}}{n}\Big)\Big\}^2,$$

$$\bar{A}^u_{D_+} = N^{-1}\sum_{i\in\mathcal{I}_n}\Big\{X\Big(\frac{i_{u,u}}{n}\Big)+X\Big(\frac{i_{-u,-u}}{n}\Big)-2X\Big(\frac{i_{0,0}}{n}\Big)\Big\}^2,$$

$$\bar{A}^u_{D_-} = N^{-1}\sum_{i\in\mathcal{I}_n}\Big\{X\Big(\frac{i_{-u,u}}{n}\Big)+X\Big(\frac{i_{u,-u}}{n}\Big)-2X\Big(\frac{i_{0,0}}{n}\Big)\Big\}^2,$$

with $\bar{A}^u_\square$ defined as in (4.2). From (3.13) we have

$$E(\bar{A}^u_*) = C_* u^\alpha\{1+o(1)\}, \tag{A1}$$

where $* = H, V, D_+, D_-, \square$. In the above, u ranges from 1 to m and $i_{q,r} = (i[1]+q, i[2]+r)$, where $i \in \mathbf{Z}^2$ and $q, r \in \mathbf{Z}$. Direct calculation shows that

$$C_H = (8 - 2^{1+\alpha})(e_1/n[1])^\alpha, \quad C_V = (8 - 2^{1+\alpha})(e_2/n[2])^\alpha \tag{A2}$$

and

$$C_{D_\pm} = (8 - 2^{1+\alpha})\Big\{\frac{e_1^2}{n[1]^2} + \frac{e_2^2}{n[2]^2} \pm \frac{2e_{12}}{n[1]n[2]}\Big\}^{\alpha/2}. \tag{A3}$$

From (A2),

$$e_1 = n[1]\Big(\frac{C_H}{8 - 2^{1+\alpha}}\Big)^{1/\alpha} \quad \text{and} \quad e_2 = n[2]\Big(\frac{C_V}{8 - 2^{1+\alpha}}\Big)^{1/\alpha}. \tag{A4}$$

In the case of e_{12}, we note from (A1) that $E(\bar{A}^u_{D_+})/E(\bar{A}^u_{D_-}) = C_{D_+}/C_{D_-}$. Using (A3), making e_{12} the subject for each u, and then averaging over u, we obtain

$$e_{12} = \frac{n[1]n[2]}{2}\Big(\frac{e_1^2}{n[1]^2} + \frac{e_2^2}{n[2]^2}\Big)m^{-1}\sum_{u=1}^{m}\Big(\frac{\rho_u - 1}{\rho_u + 1}\Big), \tag{A5}$$

where $\rho_u = \{E(\bar{A}^u_{D_+})/E(\bar{A}^u_{D_-})\}^{2/\alpha}$.

Our estimates of e_1,e_2 and e_{12} are based on estimates $\hat{\alpha}_*$ and $\hat{c}_*$ of α and c_* obtained using the approximate linear relationship $\log \bar{A}^u_* \simeq c_* + \alpha \log u$, where $u = 1, \ldots, m$,$c_* = \log C_*$ and $* = H, V, D_+, D_-, \square$. More specifically, we have

$$\hat{\alpha}_* = \frac{\sum_{u=1}^m \Big(\log u - m^{-1}\sum_{v=1}^m \log v\Big)\log \bar{A}^u_*}{\sum_{u=1}^m \Big(\log u - m^{-1}\sum_{v=1}^m \log v\Big)^2},$$

$$\hat{c}_* = m^{-1}\sum_{u=1}^{m}(\log \bar{A}^u_* - \hat{\alpha}_* \log u),$$

and $\hat{C}_* = \exp(\hat{c}_*)$. To estimate e_1 in (A4), we put $C_H = \hat{C}_H$ and $\alpha = \hat{\alpha}_H$; to estimate e_2 in (A4), we put $C_V = \hat{C}_V$ and $\alpha = \hat{\alpha}_V$; and to estimate e_{12} in (A5), we use the estimates of e_1 and e_2 already obtained, and estimate ρ_u and α by $\hat{\rho}_u = \bar{A}^u_{D_+}/\bar{A}^u_{D_-}$ and $(\hat{\alpha}_{D_+} + \hat{\alpha}_{D_-})/2$, respectively.

B. Calculation of (7.20)

Since $\sum_j a_j = 0$, it follows that $Y_{u,i} = \sum_j a^u_j W_{i,j}$. By direct calculation, we find that

$$E(T_N) = \sum_{u=1}^{m} L_u \mu_u^{-1} \sum_{j,k} a^u_j a^u_k E\Big[4X_i^2\{W_{i,j}W_{i,k} - E(W_{i,j}W_{i,k})\} + 4N^{-\alpha/(2d)}X_i W_{i,j}W^2_{i,k} + N^{-\alpha/d}W^2_{i,j}W^2_{i,k}\Big]. \tag{B1}$$

Using the diagram formula with correlated, zero-mean Gaussian variables U_1, U_2, U_3, we have

$$E\{H_2(U_1)H_1(U_2)H_1(U_3)\} = 2\mathrm{cov}(U_1, U_2)\mathrm{cov}(U_1, U_3)$$

and

$$E\{H_2(U_1)H_2(U_2)\} = 2\{\mathrm{cov}(U_1, U_2)\}^2.$$

Consequently, starting from (B1) and performing some elementary calculations, we obtain

$$\begin{aligned} E(T_N) = \sum_u L_u \mu_u^{-1} \sum_{j,k} a_j^u a_k^u \Big[& 8\rho_{XW}(j)\rho_{XW}(k) \\ & + N^{-\alpha/(2d)}\{8\rho_{XW}(k)\rho_{WW}(j,k) + 4\rho_{XW}(j)\rho_{WW}(k,k)\} \\ & + N^{-\alpha/d}\{2\rho_{WW}(j,k)^2 + \rho_{WW}(j,j)\rho_{WW}(k,k)\}\Big], \end{aligned} \tag{B2}$$

where $\rho_{XW}(j) = \mathrm{cov}(X_i, W_{i,j})$ and $\rho_{WW}(j,k) = \mathrm{cov}(W_{i,j}, W_{i,k})$ do not depend on i because of stationarity of the underlying process.

From the definition of $W_{i,j}$ and using (2.4), we have

$$\begin{aligned} \rho_{XW}(j) &= \mathrm{cov}\{X_i, N^{\alpha/(2d)}(X_{i+j} - X_i)\} \\ &= N^{\alpha/(2d)}\{\gamma(j/n) - \gamma(0)\} \\ &= -N^{-\alpha/(2d)} f(j) + O(N^{-(\alpha+2\beta)/(2d)}), \end{aligned} \tag{B3}$$

where $f(j)$ is defined in (7.21). By a similar argument we obtain

$$\rho_{WW}(j,k) = f(j) + f(k) - f(j-k) + O(N^{-\beta/d}), \tag{B4}$$

and from (3.13) we have

$$\mu_u = -u^\alpha \sum_{j,k} a_j a_k f(j-k) + O(N^{-\beta/d}) = Cu^\alpha + O(N^{-\beta/d}). \tag{B5}$$

Substituting (B3)–(B5) into (B2), using the fact that $f(j-k) = f(k-j)$, and suppressing the $O(N^{-(\alpha+\beta)/d})$ term until the final line, we obtain

$$\begin{aligned} E(T_N) &= 2C^{-1}N^{-\alpha/d} \sum_u u^{-\alpha} L_u \sum_{j,k} a_j^u a_k^u \Big[4f(j)f(k) \\ &\quad - 4f(k)\{f(j) + f(k) - f(j-k)\} - 4f(j)f(k) \\ &\quad + \{f(j) + f(k) - f(j-k)\}^2 + 2f(j)f(k)\Big] \\ &= 2C^{-1}N^{-\alpha/d} \sum_u u^\alpha L_u \sum_{j,k} a_j a_k f(j-k)\{f(j-k) + 2f(k) - 2f(j)\} \\ &= 2DN^{-\alpha/d} \sum_u u^\alpha L_u + O(N^{-(\alpha+\beta)/d}), \end{aligned}$$

as required.

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