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TESTING FOR CHANGE POINTS DUE TO A COVARIATE THRESHOLD IN QUANTILE REGRESSION

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Abstract: We develop a new procedure for testing change points due to a covariate threshold in regression quantiles. The proposed test is based on the CUSUM of the subgradient of the quantile objective function and requires fitting the model only under the null hypothesis. The critical values can be obtained by simulating the Gaussian process that characterizes the limiting distribution of the test statistic. The proposed method can be used to detect change points at a single quantile level or across multiple quantiles, and can accommodate both homoscedastic and heteroscedastic errors. Simulation study suggests that the proposed method has higher computational efficiency and comparable power with the existing likelihood-ratio-based method in the finite samples. The performance of the proposed method is further illustrated by the analysis of a blood pressure and body mass index data set.

Key words and phrases: Change point, covariate threshold, hypothesis testing, quantile regression, threshold regression model.

1. Introduction

In regression models, the regression functions are often assumed to take the same parametric form across the entire domain of interest. However, in some applications, regression functions may have different forms in different regions of the conditioning variables. For instance, blood pressure (BP) and body mass index (BMI) showed different relationships for those below and above a BMI threshold (Kaufman et al. (1997), Kerry et al. (2005), Tesfaye et al. (2007)); in acute HIV infection, CD4 counts increased rapidly in the first 2-4 weeks but gradually decreased afterwards (Ghosh and Vaida (2007)); the consumer price index showed a significantly positive correlation with the contribution of renewables to energy supply in the higher-economic growth regime, but no significant correlation in the lower-economic growth regime (Chang, Huang, and Lee (2009)). The questions in these applications can be addressed by threshold regression models that specify different regression functions in subsamples segmented by a continuous predictor, referred to as a *threshold variable* hereafter. In threshold regression models, the covariate effects are piecewise functions of the threshold variable with jumps occurring at the unknown change points. They can be viewed as special cases of varying coefficient models (Hastie and Tibshirani (1993)). Threshold regression models have wide applications in biostatistics, epidemiology, economics, and finance; see Huang (2009) and Pennell, Whitmore, and Lee (2010) for additional examples. In the threshold regression literature, most existing work focuses on mean regression models (Hansen (1996)). We focus on quantile regression.

Quantile regression, pioneered by Koenker and Bassett (1978), is a valuable alternative regression technique to mean regression for modeling the stochastic relationships between random variables. Using quantile regression, we can explore the stochastic relationships in a comprehensive way by studying different tails of the conditional distribution of the response variable. In addition, quantile regression offers an automatic approach for capturing the heteroscedasticity in the population. We refer to Koenker (2005) for a more comprehensive review of quantile regression.

A number of researchers have studied the detection and estimation of change points in quantile regression for time series data where the model structure changes after an unknown time point. Su and Xiao (2008) proposed a sup-Wald test for detecting the structural change of conditional distribution based on sequential quantile regression estimators. Qu (2008) developed two test procedures for detecting change points, one based on a subgradient statistic and the other based on a Wald-type statistic. Oka and Qu (2011) studied the estimation of multiple structural changes in conditional quantile functions. The situations discussed in these papers differ from threshold regression models, where the change point is due to an unknown threshold in the continuous threshold covariate.

There exists limited work for quantile regression with threshold effects. Caner (2002) studied least absolute deviation estimators in threshold linear regression models. Kato (2009) developed extended convexity arguments and applied the theory to establish the asymptotic properties of a proposed Wald-type test for median threshold regression model. The proposed test relies on an identically distributed error assumption and it requires estimating the unknown error density function even for homoscedastic errors. Lee, Seo, and Shin (2011) proposed a sup-likelihood-ratio-based method for testing the existence of threshold effects in regression models, which includes quantile regression as a special case. They also discussed estimation for a bent line quantile regression, where a threshold covariate has different effects in two segments separated at an unknown point and the other covariates have constant effects across the entire domain. Therefore, bent line regression can be viewed as a special case of this threshold regression model. Galvao, Montes-Rojas, and Olmo (2011) studied estimation for threshold quantile autoregressive models and established asymptotics for the threshold and regression parameter estimators. Galvao et al. (2013) developed a uniform test of linearity against threshold effects in quantile regression model for stationary time series processes. Cai and Stander (2008) and Cai (2010) discussed forecasting for quantile threshold autoregressive time series models with known threshold values.

In this article, we propose a new procedure for testing the presence of change points due to covariate thresholding in quantile regression. We focus on the general threshold regression models of Lee, Seo, and Shin (2011) that allow a set of covariates to have heterogeneous effects across the domain of the threshold covariate. Our proposed test statistic is based on a CUSUM process of subgradients obtained by fitting the quantile regression model under the null hypothesis of no threshold effects. We study the asymptotic properties of the proposed test statistics under both null and local alternative models, and develop a convenient procedure for calculating the critical values by simulating the Gaussian process. The proposed test can accommodate both homoscedastic and heteroscedastic errors, and it can be used to test for the presence of change points occurring at either a specified single quantile level or at multiple quantiles. Our numerical studies suggest that the proposed test has similar power as the likelihood-ratio test of Lee, Seo, and Shin (2011) in finite samples when sizes are controlled at the nominal levels, but that the former is less sensitive to the choice of bandwidth. In addition, since it requires only fitting the null model, the proposed method is computationally more efficient than the likelihood-ratio test that involves grid search over the domain of the threshold covariate for fitting the alternative model.

The rest of the paper is organized as follows. In Section 2, we present the proposed testing procedure at one quantile level, together with the asymptotic properties of the proposed test statistic and the calculation of critical values. In Section 3, we discuss the extension of testing change points at multiple quantiles. We assess the finite sample performance of the proposed method through a simulation study in Section 4 and the analysis of a blood pressure and body mass index data set in Section 5. Proofs are given in the Supplementary Material.

2. Testing for Threshold Effects at a Single Quantile Level

2.1. Model setup and testing procedure

At a given quantile level $\tau \in (0,1)$, we consider the threshold quantile regression model

$$Q_Y(\tau|\mathbf{X},\mathbf{Z},U) = \mathbf{X}^T\boldsymbol{\beta}_0(\tau) + \mathbf{Z}^T\boldsymbol{\alpha}_0(\tau)I(U > u_0), \tag{2.1}$$

where $Q_Y(\tau|\cdot)$ is the τth conditional quantile of the response variable Y, $\mathbf{X}$ is a p-dimensional design vector with the first element as 1 corresponding to the intercept, U is the univariate threshold variable that might be an element of $\mathbf{X}$, $\mathbf{Z}$ is a q-dimensional design vector, $\boldsymbol{\beta}_0(\tau)$, and $\boldsymbol{\alpha}_0(\tau)$ are the unknown regression coefficients with u_0 the unknown threshold parameter lying in the region (0,1). Without loss of generality, we assume that U and $\mathbf{Z}$ are subsets of $\mathbf{X}$, and $U \in [0,1]$. Suppose $\{(Y_i, \mathbf{X}_i); i = 1, \ldots, n\}$ is a random sample of $(Y, \mathbf{X})$.

We are interested in testing the existence of a threshold effect at the quantile level τ. We consider the null and alternative hypotheses

$$H_0 : \boldsymbol{\alpha}_0(\tau) = \mathbf{0} \ \text{ for any } u_0 \in (0,1) \ \ vs. \ \ H_1 : \boldsymbol{\alpha}_0(\tau) \neq \mathbf{0} \ \text{ for some } u_0 \in (0,1).$$

Under the null hypothesis, the threshold parameter $\boldsymbol{\alpha}_0(\tau)$ is not identifiable, while $\boldsymbol{\beta}_0(\tau)$ can be estimated by

$$\hat{\boldsymbol{\beta}}(\tau) = \arg \min_{\boldsymbol{\beta}(\tau) \in R^p} \sum_{i=1}^{n} \rho_\tau\{Y_i - \mathbf{X}_i^T \boldsymbol{\beta}(\tau)\},$$

where $\rho_\tau(r) = r\{\tau - I(r < 0)\}$ is the quantile check function. The building block of our test statistic is

$$\mathbf{R}_n\{u, \tau, \hat{\boldsymbol{\beta}}(\tau)\} = n^{-1/2} \sum_{i=1}^{n} \psi_\tau\{Y_i - \mathbf{X}_i^T \hat{\boldsymbol{\beta}}(\tau)\} \mathbf{Z}_i I(U_i \leq u), \tag{2.2}$$

where $\psi_\tau(r) = \tau - I(r < 0)$. To test for the presence of an unknown change point, we search over all possible candidates $u \in (0,1)$ and define the test statistic as

$$T_n(\tau) = \sup_{u \in (0,1)} \|\mathbf{R}_n\{u, \tau, \hat{\boldsymbol{\beta}}(\tau)\}\|, \tag{2.3}$$

where $\|\cdot\|$ stands for the L_2 norm.

Here $\mathbf{R}_n\{u, \tau, \hat{\boldsymbol{\beta}}(\tau)\}$ is the negative subgradient of the quantile objective function corresponding to the subsample with U_i below the threshold u. Therefore, our proposed test statistic can be viewed as a CUSUM statistic based on signs of quantile residuals; see Bai (1996) for a discussion of CUSUM test statistic in least squares regression. Under the null hypothesis, $\hat{\boldsymbol{\beta}}(\tau)$ is a root-n consistent estimator of $\boldsymbol{\beta}_0(\tau)$ and $\mathbf{R}_n\{u, \tau, \hat{\boldsymbol{\beta}}(\tau)\}$ converges to a Gaussian process with mean zero. On the other hand, if there exists a threshold effect, the significant difference of $\hat{\boldsymbol{\beta}}(\tau)$ and $\boldsymbol{\beta}_0(\tau)$ makes the estimated residuals consistently fall below or above zero for a subsample and thus forces the statistic $T_n(\tau)$ to take a large value. Our proposed test statistic depends on a score-type statistic, and thus requires fitting the model only under H_0. This is in contrast with the likelihood ratio approach in Lee, Seo, and Shin (2011), which requires fitting alternative models under all possible partitions of the domain $(0,1)$ of the threshold variable.

2.2. Asymptotic properties

We study the asymptotic properties of the proposed test statistic under the null and local alternative models. Let $f(\cdot|\mathbf{X})$ and $F(\cdot|\mathbf{X})$ denote the conditional density and distribution functions of Y given $\mathbf{X}$, respectively; we write $f(\cdot|\mathbf{X}_i)$ and $F(\cdot|\mathbf{X}_i)$ as $f_i(\cdot)$ and $F_i(\cdot)$. Let $f_U(u)$ and $F_U(u)$ be the density and distribution functions of U, $\mathbf{Z}_u = \mathbf{Z}I(U \le u)$, $\mathbf{S}_{\mathbf{z}}(u) = E\left[\mathbf{Z}_u\mathbf{X}^T f\{F^{-1}(\tau|\mathbf{X})|\mathbf{X}\}\right]$, and $\mathbf{S} = \mathbf{S}_{\mathbf{x}}(1) = E[\mathbf{X}\mathbf{X}^T f\{F^{-1}(\tau|\mathbf{X})|\mathbf{X}\}]$. We make the following assumptions.

A1 The density $f_i(\cdot)$ is continuous, uniformly bounded away from zero and infinity, and has a bounded first derivative in the neighborhood of $F_i^{-1}(\tau)$ for $i = 1, \ldots, n$.

A2 The density $f_U(u)$ is continuous.

A3 (a) The design vector satisfies $\max_{1\le i\le n} \|\mathbf{X}_i\| = O_p(n^{1/10}/\log n)$; (b) The matrix $\mathbf{S}$ is positive definite; (c) $E\|\mathbf{X}\|^6$ is bounded.

Assumption A1 is standard in quantile regression literature. Assumption A3(a) imposes some conditions on the moments of the covariates that are needed to establish the asymptotic representation of $\hat{\boldsymbol{\beta}}(\tau)$ and the uniform convergence of the subgradient process under the null and alternative hypotheses. Assumption A3(b) is needed to obtain the asymptotic representation of $\hat{\boldsymbol{\beta}}(\tau)$. Assumption A3(c), due to Bai (1996), ensures the stochastic equicontinuity of the subgradient process $\mathbf{R_n}\{\mathbf{u}, \tau, \boldsymbol{\beta}_{\mathbf{0}}(\tau)\}$.

Theorem 1. *Suppose* A1–A3 *hold. Under the null hypothesis H_0,*

$$T_n(\tau) \Rightarrow \sup_{u\in(0,1)} \|\boldsymbol{R}(u)\|, \ \textit{as } n \to \infty, \tag{2.4}$$

where " $\Rightarrow$ " denotes the weak convergence, $\boldsymbol{R}(u)$ is a Gaussian process with mean 0 and covariance function $\mathbf{W}(u,u') = \tau(1-\tau)\{E(\mathbf{Z}_{u'}\mathbf{Z}_u^T) - E(\mathbf{Z}_u\mathbf{X}^T)\mathbf{S}^{-1}\mathbf{S}_z^T(u') -\mathbf{S}_{\mathbf{z}}(u)\mathbf{S}^{-1}E(\mathbf{X}\mathbf{Z}_{u'}^T) + \mathbf{S}_{\mathbf{z}}(u)\mathbf{S}^{-1}E(\mathbf{X}\mathbf{X}^T)\mathbf{S}^{-1}\mathbf{S}_z^T(u')\}$.

We next establish the asymptotic properties of the proposed test statistic under the local alternative model,

$$Q_{Y_i}(\tau|\mathbf{X}_i) = \mathbf{X}_i^T\boldsymbol{\beta}_0(\tau) + n^{-1/2}\mathbf{Z}_i^T\boldsymbol{\alpha}_0(\tau)I(U_i > u_0), i = 1, \ldots, n, \tag{2.5}$$

where $u_0 \in (0,1)$ is the change point and $\boldsymbol{\alpha}_0(\tau) \ne 0$ is a q-dimensional vector.

Theorem 2. *Suppose* A1–A3 *hold. Under the local alternative model* (2.5)

$$T_n(\tau) \Rightarrow \sup_{u\in(0,1)} \|\boldsymbol{R}(u) + \boldsymbol{q}\{u, \boldsymbol{\alpha}_0(\tau)\}\|, \ \textit{as } n \to \infty, \tag{2.6}$$

where $\boldsymbol{R}(u)$ is the Gaussian process in Theorem 1, *and* $\boldsymbol{q}\{u,\boldsymbol{\alpha}_0(\tau)\} = -\mathbf{S}_z(u)\mathbf{S}^{-1}\boldsymbol{Q}\{\boldsymbol{\alpha}_0(\tau)\} + \boldsymbol{P}\{u,\boldsymbol{\alpha}_0(\tau)\}$, *where* $\boldsymbol{Q}\{\boldsymbol{\alpha}_0(\tau)\} = E\big[\mathbf{X}\mathbf{Z}^T\boldsymbol{\alpha}_0(\tau)I(U > u_0)f\{F^{-1}(\tau|\mathbf{X})|\mathbf{X}\}\big]$ *and* $\boldsymbol{P}\{u,\boldsymbol{\alpha}_0(\tau)\} = E\big[\mathbf{Z}\mathbf{Z}^T\boldsymbol{\alpha}_0(\tau)I(u_0 < U \le u)f\{F^{-1}(\tau|\mathbf{X})|\mathbf{X}\}\big]$.

Here $\mathbf{q}\{u, \boldsymbol{\alpha}_0(\tau)\} = 0$ for all u under H_0, and it is a nonzero function of u under the local alternative, so the proposed test statistic can be used to distinguish the alternative with a change point due to a threshold effect from the null hypothesis of no change point. Moreover the power of the test $T_n(\tau)$ approaches one when the order of the threshold effect under the alternative model is arbitrarily close to $n^{-1/2}$.

Corollary 1. *Suppose* A1−A3 *hold. Under the local alternative model* $Q_Y(\tau|\mathbf{X}_i) = \mathbf{X}_i^T\boldsymbol{\beta}_0(\tau) + n^{-1/2}a_n\mathbf{Z}_i^T\boldsymbol{\alpha}_0(\tau)I(U_i > u_0)$ *with* $u_0 \in (0,1)$ *and* $a_n \to \infty$, *we have* $\lim_{n\to\infty} P\{\|T_n(\tau)\| \geq t\} = 1$ *for all* $t > 0$.

2.3. Calculation of critical values

As the asymptotic null distribution of $T_n(\tau)$ is nonstandard, and depends on the unknown density function for heteroscedastic errors, critical values for $T_n(\tau)$ cannot be tabulated for general cases. We overcome this difficulty by simulating the asymptotic representation of $\mathbf{R}_n\{u, \tau, \hat{\boldsymbol{\beta}}(\tau)\}$.

Let $\{e_i; i = 1, \ldots, n\}$ be a random sample with τth quantile zero, and let $\{\omega_i; i = 1, \ldots, n\}$ be a random sample independent of e_i with zero mean, unit variance, and a finite third moment. Define

$$\mathbf{R}_n^*(u) = n^{-1/2}\sum_{i=1}^{n}\omega_i\psi_\tau(e_i)\{I(U_i \leq u)\mathbf{Z}_i - \mathbf{S}_{\mathbf{z},n}(u)\mathbf{S}_n^{-1}\mathbf{X}_i\}, \tag{2.7}$$

where $\mathbf{S}_{z,n}(u) = n^{-1}\sum_{i=1}^n \mathbf{Z}_i\mathbf{X}_i^T I(U_i \leq u)K_{h_n}\{Y_i - \mathbf{X}_i^T\hat{\boldsymbol{\beta}}(\tau)\}$, $\mathbf{S}_n = n^{-1}\sum_{i=1}^n \mathbf{X}_i\mathbf{X}_i^T K_{h_n}\{Y_i - \mathbf{X}_i^T\hat{\boldsymbol{\beta}}(\tau)\}$, $K_{h_n}(\cdot) = h_n^{-1}K(\cdot/h_n)$, $K(\cdot)$ is a kernel function, and h_n is a positive bandwidth. In our numerical studies, we follow the suggestion of Lee, Seo, and Shin (2011) and use Silverman's rule of thumb (Silverman (1986)), $h_n = 1.06\hat{\sigma}n^{-1/5}$, where $\hat{\sigma}$ is the sample standard deviation of $\{Y_i - \mathbf{X}_i^T\hat{\boldsymbol{\beta}}(\tau); i = 1, \ldots, n\}$. The sensitivity analysis in Section 3 shows that the performance of the proposed test procedure is stable with $h_n = c\hat{\sigma}n^{-1/5}$ for $c \in [0.1,\ 3]$.

Here are the asymptotic properties of $\mathbf{R}_n^*(u)$. We make the following assumptions on the kernel function and the bandwidth.

A4 The function $K(\cdot)$ is a symmetric kernel function with compact support; it satisfies $\int K(u)du = 1$ and has a bounded first derivative.

A5 The positive bandwidth h_n satisfies $h_n \to 0$ and $h_n n^{1/5}\sqrt{\log n} \to \infty$ as $n \to \infty$.

Theorem 3. *Suppose* A1−A5 *hold. Under* H_0, $\boldsymbol{R}_n^*(u)$ *converges to the Gaussian process* $\boldsymbol{R}(u)$ *defined in Theorem* 1 *as* $n \to \infty$.

Theorem 3 suggests that we can calculate the critical values by the quantiles of $\max_u \|\mathbf{R}_n^*(u)\|$. The detailed computing procedure is as follows:

Step 1. Generate $\{e_i; i=1,\ldots,n\}$ as a random sample with the τth quantile zero and $\{\omega_i, i=1,\ldots,n\}$ as a random sample (independent of e_i) with mean 0, variance 1, and a finite third moment. We generate e_i independently from $N(0,1)-\Phi^{-1}(\tau)$ with Φ being the cumulative distribution function of $N(0,1)$, and ω_i from the two-point mass distribution with equal probability at 1 and -1.

Step 2. Simulate the process

$$\mathbf{R}_n^*(u) = n^{-1/2}\sum_{i=1}^{n}\omega_i\psi_\tau(e_i)\left\{I(U_i\le u)\mathbf{Z}_i - \mathbf{S}_{z,n}(u)\mathbf{S}_n^{-1}\mathbf{X}_i\right\},$$

and obtain the test statistic $T_n^*(\tau)=\sup_{u\in(0,1)}\|\mathbf{R}_n^*(u)\|$.

Step 3. Repeat Steps 1−2 J times to get $T_{n1}^*(\tau),\ldots,T_{nJ}^*(\tau)$. Calculate the critical value for a level α test by the $(1-\alpha)$th sample quantile of $\{T_{nj}^*(\tau); j=1,\ldots,J\}$.

Remark 1. In (2.5), let $\epsilon = Y - \mathbf{X}^T\boldsymbol{\beta}_0(\tau) - n^{-1/2}\mathbf{Z}^T\boldsymbol{\alpha}_0(\tau)\ I(U>u_0)$ denote the error term whose τth quantile is zero conditional on $\mathbf{X}$. If $f_\epsilon(\cdot|\mathbf{X})$ is the conditional density function of ϵ given $\mathbf{X}$, then we have $f\{F^{-1}(\tau|\mathbf{X})|\mathbf{X}\} = f_\epsilon(0|\mathbf{X})$. Hence for (2.5) with *i.i.d.* (independent and identically distributed) errors, the density functions in $\mathbf{S}(u)$ and $\mathbf{S}^{-1}$ cancel out and do not need to be estimated. The process $\mathbf{R}_n^*(u)$ can then be simplified as

$$\mathbf{R}_n^*(u) = n^{-1/2}\sum_{i=1}^{n}\omega_i\psi_\tau(e_i)\{I(U_i\le u)\mathbf{Z}_i - \mathbf{S}_{\mathbf{z},1n}(u)\mathbf{S}_{1n}^{-1}\mathbf{X}_i\}, \tag{2.8}$$

where $\mathbf{S}_{\mathbf{z},1n}(u) = n^{-1}\sum_{i=1}^n \mathbf{Z}_i\mathbf{X}_i^T I(U_i\le u)$ and $\mathbf{S}_{1n} = n^{-1}\sum_{i=1}^n \mathbf{X}_i\mathbf{X}_i^T$.

2.4. Comparison to the sup-likelihood-ratio-type test of Lee, Seo, and Shin (2011)

Lee, Seo, and Shin (2011) developed a general sup-likelihood-ratio-type (LRT) test for testing threshold effects in regression models including quantile regression. We carry out a systematic comparison of our proposed sup-score-type (SS) test with the LRT test.

The LRT test statistic in Lee, Seo, and Shin (2011) is

$$LRT_n = \sup_{u\in(0,1)} n\{Q_n(u) - \tilde{Q}_n\}, \tag{2.9}$$

where $\tilde{Q}_n = \min\limits_{\boldsymbol{\beta}(\tau)} n^{-1}\sum_{i=1}^n \rho_\tau\{Y_i - \mathbf{X}_i^T\boldsymbol{\beta}(\tau)\}$ and $Q_n(u) = \min\limits_{\boldsymbol{\beta}(\tau),\boldsymbol{\alpha}(\tau)} n^{-1}\sum_{i=1}^n$ $\rho_\tau\{Y_i - \mathbf{X}_i^T\boldsymbol{\beta}(\tau) - \mathbf{Z}_i^T\boldsymbol{\alpha}(\tau)I(U_i>u)\}$ are the minimum values of the quantile objective functions under the null and the alternative hypothesis with the change point $u_0 = u$, respectively.

The proposed SS method is based on the score-type test statistic obtained by only fitting the null model, while the LRT method is based on the pseduo-likelihood-ratio-type test statistic and requires fitting both the null and alternative models.

The asymptotic distributions of LRT_n under the null and local alternative models are given next, where (i) comes from (3.6) of Lee, Seo, and Shin (2011), and the proof of (ii) is provided in the Supplementary Material.

Proposition 1. (i) *Under* A1−A3 *and* H_0, $LRT_n \Rightarrow (1/2)\{\sup_{u\in(0,1)} \mathcal{G}(u)^T \boldsymbol{V}(u)^{-1}\mathcal{G}(u) - \mathcal{G}_1^T \boldsymbol{V}_1^{-1}\mathcal{G}_1\}$ *as* $n \to \infty$, *where* $\mathcal{G}(u)$ *is a mean-zero Gaussian process with covariance kernel* $\widetilde{\mathbf{W}}(u,u') = \tau(1-\tau)E(\widetilde{\mathbf{X}}_u\widetilde{\mathbf{X}}_{u'}^T)$, $\widetilde{\mathbf{X}}_u = (\mathbf{X}^T, \mathbf{Z}^T I(U > u))^T$ *being the vector of covariates under the alternative hypothesis,* $\boldsymbol{V}(u) = E[\widetilde{\mathbf{X}}_u\widetilde{\mathbf{X}}_u^T\ f\{\mathbf{X}^T\boldsymbol{\beta}_0(\tau)|\mathbf{X}\}]$, $\mathcal{G}_1$ *and* $\boldsymbol{V}_1$ *denote the first* p *elements of* $\mathcal{G}$ *and the first* $p\times p$ *block of* $\boldsymbol{V}(u)$. (ii) *Under* A1−A5 *and the local alternative model* (2.5), $LRT_n \Rightarrow (1/2)(\sup_{u\in(0,1)}[\mathcal{G}(u) + \widetilde{\boldsymbol{q}}_L\{u, \boldsymbol{\alpha}_0(\tau)\}]^T \boldsymbol{V}(u)^{-1}\ [\mathcal{G}(u) + \widetilde{\boldsymbol{q}}_L\{u, \boldsymbol{\alpha}_0(\tau)\}] - (\mathcal{G}_1 + \widetilde{\boldsymbol{q}}_1)^T \mathbf{V}_1^{-1}\ (\mathcal{G}_1 + \widetilde{\boldsymbol{q}}_1))$ *as* $n \to \infty$, *where* $\widetilde{\boldsymbol{q}}_L\{u, \boldsymbol{\alpha}_0(\tau)\} = (E[\mathbf{X}\mathbf{Z}^T\boldsymbol{\alpha}_0(\tau)I(U > u_0)f\{\mathbf{X}^T\boldsymbol{\beta}_0(\tau)\ |\mathbf{X}\}]^T,\ E[\mathbf{Z}\mathbf{Z}^T\boldsymbol{\alpha}_0(\tau)I\{U > \max(u, u_0)\}f\{\mathbf{X}^T\boldsymbol{\beta}_0(\tau)|\mathbf{X}\}]^T)^T$, *and* $\tilde{\boldsymbol{q}}_1 = E[\mathbf{X}\mathbf{Z}^T\ \boldsymbol{\alpha}_0(\tau)I(U > u_0)f\{\mathbf{X}^T\boldsymbol{\beta}_0(\tau)|\mathbf{X}\}]$.

The critical value can be obtained by simulating the asymptotic distribution of LRT_n under the null hypothesis; see Section 3.2 of Lee, Seo, and Shin (2011) for details. The limiting distribution of the LRT test statistic has a nonstandard form different from that of the SS test statistic, and this makes it difficult to compare two statistics explicitly. However, we can compare the asymptotic local powers of two methods analytically by simulating the limiting distributions of $T_n(\tau)$ and LRT_n. Our analysis in Section 4.2 shows that two methods have similar asymptotic local power for the various designs considered.

3. Testing for Threshold Effects at Multiple Quantiles

In many cases, different magnitudes of changes may occur at different quantiles. It is then more informative to incorporate multiple quantiles instead of only one quantile level to identify the threshold effects. We use $\mathcal{T} = [\omega_1, \omega_2]$ to represent a closed set of quantiles of interest. We focus on the null and alternative hypotheses:

$$H_0^*:\ \boldsymbol{\alpha}_0(\tau) = 0\ \text{ for any } u_0 \in (0,1) \text{ and for all } \tau \in \mathcal{T},$$
$$H_1^*:\ \boldsymbol{\alpha}_0(\tau) \neq 0\ \text{ for some } u_0 \in (0,1)\ \text{ and some } \tau \in \mathcal{T},$$

which imply that there is no threshold effect for any quantile in the set $\mathcal{T}$ under the null while there exist threshold effects at some quantiles under the alternative.

We consider the test statistic

$$T_n = \sup_{\tau\in\mathcal{T}} \sup_{u\in(0,1)} \|R_n\{u,\tau,\hat{\boldsymbol{\beta}}(\tau)\}\|,$$

where $\hat{\boldsymbol{\beta}}(\tau)$ is the coefficient estimate at the τth quantile obtained under the null hypothesis. A large T_n indicates that there likely exist threshold effects for some $\tau \in \mathcal{T}$.

To establish the limiting distribution of T_n, we strength the condition A1 as follows.

A6 Assumption A1 holds uniformly in $\tau \in \mathcal{T}$.

Theorem 4. *Suppose* A2−A6 *hold. Under* H_0^*, *we have*

$$T_n \Rightarrow \sup_{\tau\in\mathcal{T}} \sup_{u\in(0,1)} \|\boldsymbol{R}(u,\tau)\|, \ \text{as } n\to\infty,$$

where $\boldsymbol{R}(u,\tau)$ *is a* q*-vector of independent Gaussian processes with zero mean and covariance function*

$$E\{\boldsymbol{R}(u,\tau)\boldsymbol{R}(u',\tau')\} = \{\min(\tau,\tau')-\tau\tau'\}\{E(\mathbf{Z}_u\mathbf{Z}_{u'})-E(\mathbf{Z}_u\mathbf{X}^T)\mathbf{S}^{-1}\mathbf{S}_z^T(u') \\ -\mathbf{S}_{\mathbf{z}}(u)\mathbf{S}^{-1}E(\mathbf{X}\mathbf{Z}_{u'}^T)+\mathbf{S}_{\mathbf{z}}(u)\mathbf{S}^{-1}E(\mathbf{X}\mathbf{X}^T)\mathbf{S}^{-1}\mathbf{S}_z^T(u')\big\}. \quad (3.1)$$

If the local alternative model (2.5) *holds for all quantiles* $\tau\in\mathcal{T}$, *then*

$$T_n \Rightarrow \sup_{\tau\in\mathcal{T}} \sup_{u\in(0,1)} \|\boldsymbol{R}(u,\tau)+\boldsymbol{q}\{u,\boldsymbol{\alpha}_0(\tau)\}\|, \ \text{as } n\to\infty,$$

where $\boldsymbol{q}\{u,\boldsymbol{\alpha}_0(\tau)\}$ *is defined in Theorem* 2.

To obtain critical values of the test statistic across quantiles, the simulation method described in Section 2.3 can be slightly modified, as follows. Let $\tau_1,\ldots,\tau_m$ be a grid of quantile levels from $\mathcal{T}$. We can generate the subgradient process as

$$\mathbf{R}_{nk}^*\{u,\tau_k,\boldsymbol{\beta}(\tau_k)\} = n^{-1/2}\sum_{i=1}^n \omega_i\psi_{\tau_k}\{e_i-Q_e(\tau_k)\}\left\{I(U_i\le u)\mathbf{Z}_i-\mathbf{S}_{\mathbf{z},n}(u)\mathbf{S}_n^{-1}\mathbf{X}_i\right\},$$

for $k=1,\ldots,m$ and calculate $T_n^* = \max_{k=1,\ldots,m}\sup_{u\in(0,1)}\|\mathbf{R}_{nk}^*\{u,\tau_k,\boldsymbol{\beta}(\tau_k)\}\|$, where e_i are *i.i.d.* random variables with the τth quantile $Q_e(\tau)$, and ω_i are *i.i.d.* random variables independent of e_i with zero mean, unit variance, and finite third moment.

4. Simulation Study

4.1. Setup

The simulated data sets were generated as

$$Y_i = 1 + X_i + U_i - \alpha_0 X_i I(U_i > 0.5) + \sigma(X_i, U_i)\epsilon_i, \tag{4.1}$$

where $X_i \sim Uniform(-2, 2)$, $U_i \sim Uniform(0, 1)$, ϵ_i were independent and identically distributed random errors, $\sigma(X_i, U_i)$ measures the heteroscedasticity, and α_0 controls the degree of departure from the null hypothesis with $\alpha_0 = 0$ representing no threshold effects in the model. We considered four cases. In Case 1, $\sigma(X_i, U_i) = 1$ corresponds to a homoscedastic model. Cases 2-4 have heteroscedastic errors with $\sigma(X_i, U_i) = 1 + 0.3X_i$ in Cases 2 and 4, and $\sigma(X_i, U_i) = 1 + 0.3X_i + 0.3U_i$ in Case 3. The random errors ϵ_i were generated from $N(0, 1)$ in Cases 1−3 and from t_4 distribution in Case 4. The sample size was $n = 200$ and 500. For all scenarios, the simulation was repeated 500 times.

We considered two variations of the proposed testing procedure, *SIID* and *SNID*, for which the critical values were obtained through simulation by assuming *i.i.d.* and non-*i.i.d.* errors, respectively. For comparison, we included two variations of the sup-likelihood-ratio-type method, *LIID* and *LNID* for *i.i.d.* and non-*i.i.d.* errors, respectively. Different from *SIID*, the density function in *LIID* not can be cancelled out and thus the bandwidth parameter is still involved in the kernel estimation; see (3.8) and (3.9) in Lee, Seo, and Shin (2011) for more details. For all the methods considered, the lower and upper 10% quantiles of U were trimmed when maximizing over U, and we used the Gaussian kernel function for the *SNID*, *LIID* and *LNID* methods.

4.2. Type I error and sensitivity analysis

Table 1 summarizes the Type I errors of the four testing procedures in Cases 1-4 with $n = 200$ and 500 at quantile levels $\tau = 0.1$, 0.5, and 0.9. The nominal significance level is set as 5%. For *SIID* and *SNID*, we also report the Type I errors for detecting the threshold effects at three quantiles jointly (referred to as "Multiple" in the table). The score-based methods *SIID* and *SNID* give slightly inflated Type I errors for the smaller sample size $n = 200$. However, for $n = 500$, the *SNID* method maintains the Type I errors close to the nominal level for all scenarios considered. Even though the *SIID* method assumes *i.i.d.* errors, the method is quite robust against the violation of this assumption. For $n = 500$, the Type I errors of *SIID* are close to the nominal level in Case 1 with homoscedastic errors, and they are also reasonable (slightly inflated for a few scenarios) in Cases 2−4 with heterocedastic errors. Across all scenarios considered, the sup-likelihood-ratio-based methods *LIID* and *LNID* give deflated Type I errors for both $n = 200$ and $n = 500$, and this over-conservativeness was also observed in the Supplementary Material of Lee, Seo, and Shin (2011, p.19).

To assess the sensitivity of *SNID*, *LIID*, and *LNID* to the bandwidth, we let $h_n = cn^{-1/5}\hat{\sigma}$, and plot in Figure 1 the Type I errors of three methods against

Table 1. Type I errors of different testing procedures in Cases 1−4.

Sample size	τ	Methods	Case 1	Case 2	Case 3	Case 4
n=200	$\tau = 0.1$	*SIID*	0.052	0.058	0.054	0.062
		SNID	0.040	0.038	0.020	0.046
		LIID	0.030	0.022	0.024	0.024
		LNID	0.014	0.016	0.012	0.010
	$\tau = 0.5$	*SIID*	0.082	0.086	0.082	0.074
		SNID	0.068	0.072	0.070	0.058
		LIID	0.016	0.022	0.018	0.022
		LNID	0.008	0.014	0.006	0.006
	$\tau = 0.9$	*SIID*	0.058	0.078	0.070	0.098
		SNID	0.048	0.054	0.052	0.056
		LIID	0.018	0.024	0.030	0.038
		LNID	0.004	0.014	0.012	0.016
	Multiple	*SIID*	0.084	0.084	0.084	0.076
		SNID	0.068	0.072	0.080	0.056
n=500	$\tau = 0.1$	*SIID*	0.038	0.060	0.060	0.070
		SNID	0.036	0.046	0.038	0.038
		LIID	0.018	0.022	0.020	0.028
		LNID	0.012	0.012	0.008	0.022
	$\tau = 0.5$	*SIID*	0.042	0.064	0.056	0.062
		SNID	0.048	0.056	0.046	0.056
		LIID	0.030	0.026	0.030	0.034
		LNID	0.028	0.014	0.016	0.012
	$\tau = 0.9$	*SIID*	0.052	0.050	0.064	0.074
		SNID	0.046	0.038	0.054	0.044
		LIID	0.032	0.022	0.032	0.032
		LNID	0.018	0.016	0.012	0.024
	Multiple	*SIID*	0.048	0.060	0.054	0.074
		SNID	0.042	0.038	0.036	0.052

$c \in [0.1,\ 3]$ in Cases 1-4 at the single quantile level $\tau = 0.5$. The LRT methods *LIID* and *LNID* are sensitive to the choice of the bandwidth: for all scenarios considered, the Type I errors of *LIID* fall below the nominal level across the entire region of c considered, and the Type I errors of *LNID* are generally close to the nominal level for smaller c but not for large c. The acceptable region of c for the LRT methods appears to depend on n, the distribution, and the quantile level of study, and it would be difficult to find a constant c that works universally well for the LRT methods. The score-based test *SNID* is more robust against the choice of h_n, and it gives reasonable Type I errors across a wide range of c, especially for the sample size $n = 500$.

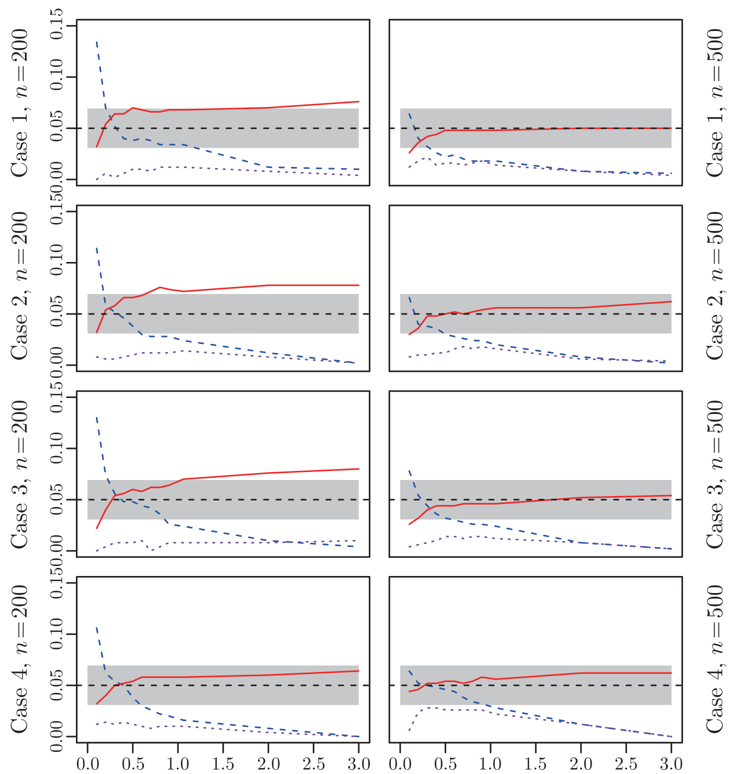

Figure 1. Type I errors of *SNID*, *LIID* and *LNID* at $\tau = 0.5$ against c, the constant involved in the bandwidth $h_n = cn^{-1/5}\hat{\sigma}$. The dashed horizontal line corresponds to the nominal level of 5%, and the shade area corresponds to the 95% confidence interval of Type I error of valid tests of size 5%. The solid, dashed and dotted lines represent Type I errors from the *SNID*, *LNID* and *LIID* methods, respectively.

4.3. Power analysis

For power analysis, we consider the local alternative model (4.1) with $\alpha_0 = n^{-1/2}\delta$, where $\delta = 1, \ldots, 9$. We showed in Section 4.2 that the LRT methods tend to have deflated Type I errors. For fairness of the power comparison, for both SS and LRT methods, we calculated the critical values as the 95th percentiles of the corresponding test statistics across 500 simulated data generated from the null hypothesis. This ensures that both SS and LRT tests have exactly 5% Type I error. For comparison, we also include the asymptotic local powers of the SS and LRT methods, obtained by simulating the limiting distributions of $T_n(\tau)$ and LRT_n using the results in Theorem 2 and Proposition 1, respectively.

Figure 2 plots the asymptotic and finite-sample Monte Carlo local powers of SS and LRT tests across δ at $\tau = 0.5$, referred to as ASLP and MCLP,

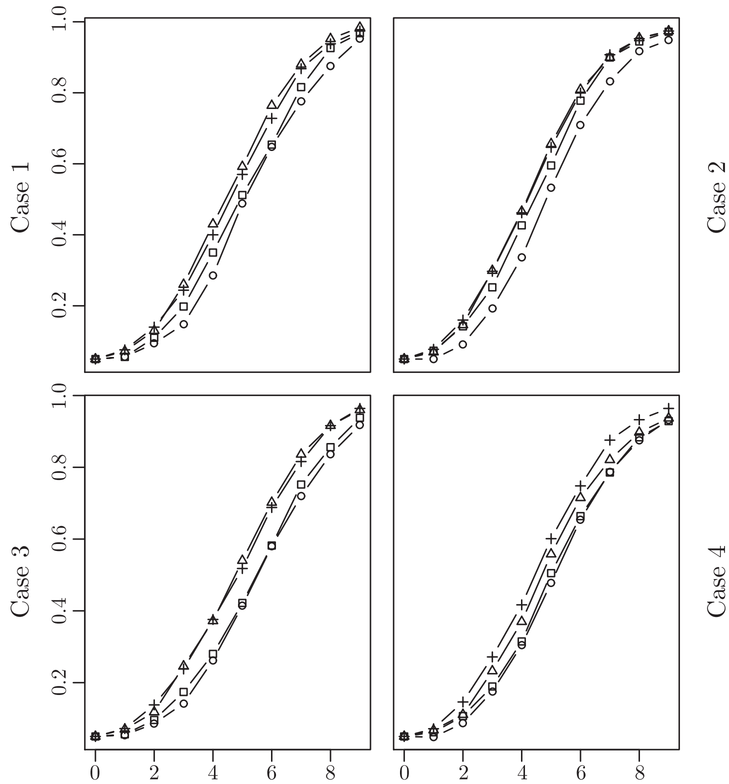

Figure 2. The asymptotic local power (ASLP) and Monte Carlo local power (MCLP) obtained by simulation (500 repetitions) with $n = 2{,}000$ from the SS and LRT methods against δ for the local alternative model with $\alpha_0 = n^{-1/2}\delta$ at $\tau = 0.5$. The lines with circles, squares, crosses and triangles represent $ASLP(LRT)$, $MCLP(LRT)$, $ASLP(SS)$, and $MCLP(SS)$ methods, respectively.

respectively. The finite-sample local powers were obtained through Monte Carlo simulation by running the test procedures for data generated from model (4.1) with $\alpha_0 = n^{-1/2}\delta$ and $n = 2{,}000$. The results show that as δ increases, the local powers of both SS and LRT methods increase gradually to one. The MCLP and ASLP agree generally well with each other, and this validates the theoretical results. In addition, the asymptotic local powers of the SS and LRT methods appear close to each other for all four cases considered.

4.4. Comparison of computational efficiency

To compare the computational efficiency of the SS and LRT methods, we report in Table 2 the average computing time (in seconds) of each test procedure for analyzing one simulated data set at median. The *SIID* tends to be faster than

Table 2. The average computing time (in seconds) of different testing procedures for analyzing one simulated data set at median.

	SIID	*SNID*	*LIID*	*LNID*
	$n = 200$			
Case 1	0.60	16.45	33.28	36.38
Case 2	0.58	17.13	30.26	34.58
Case 3	0.54	17.45	30.21	34.16
Case 4	0.56	17.54	40.64	31.60
	$n = 500$			
Case 1	1.51	21.65	115.31	159.57
Case 2	1.47	20.75	97.18	105.36
Case 3	1.34	21.27	94.66	110.42
Case 4	1.40	21.83	122.95	112.83

SNID as it does not require estimating the unknown density function. Since the SS test requires fitting the model only under null hypothesis, both *SIID* and *SNID* take less time than the LRT methods, and this computational efficiency is more obvious for larger samples.

5. Blood Pressure and Body Mass Index Study

The relationship between blood pressure and body mass index has long been an important topic in public health studies. Many studies have reported positive associations between BP and BMI in different populations (He et al. (1994); Droyvold et al. (2005); Tesfaye et al. (2007)). However, some researchers have shown that a linear relationship may not hold across the entire BMI range (Bunker et al. (1995); Kaufman et al. (1997); Kerry et al. (2005)).

We use the data from the National Health and Nutrition Examination Survey (NHANES) to illustrate the performance of our proposed method. The NHANES program started in the early 1960s and was designed to assess the health and nutritional status of adults and children in the United States. We study the relationship between systolic blood pressure (SBP) and BMI at different quantiles after accounting for gender and age effects, and examine whether the relationship is stable across the range of BMI. In this study, the analysis at high quantiles is of great interest since high blood pressure is known to be an important potential cause of heart and vascular diseases. We consider the survey data of non-hispanic black people collected in year 2009-2010, including 683 males and 695 females. For easier demonstration, we analyze data for females and males separately.

First we carried out hypothesis tests to assess if BMI has any threshold effects on the quantiles of BP. In the presence of threshold effects, we have the

Table 3. P-values from testing the existence of threshold effects based on different test procedures in the blood pressure and body mass index study. The last row provides the p-values from the joint testing at three quantile levels.

Quantile	*SIID*	*SNID*	*LIID*	*LNID*
		Male		
$\tau = 0.1$	0.002	0.032	0.000	0.010
$\tau = 0.5$	0.008	0.044	0.020	0.064
$\tau = 0.9$	0.016	0.016	0.012	0.208
Multiple	0.002	0.044		
		Female		
$\tau = 0.1$	0.434	0.206	0.546	0.686
$\tau = 0.5$	0.430	0.166	0.028	0.230
$\tau = 0.9$	0.186	0.098	0.094	0.912
Multiple	0.402	0.140		

quantile regression model

$$Q_{Y_i}(\tau|X_i, Z_i) = \begin{cases} a_1(\tau) + b_1(\tau)X_i + c_1(\tau)Z_i, & X_i \leq u_0(\tau), \\ a_2(\tau) + b_2(\tau)X_i + c_2(\tau)Z_i, & X_i > u_0(\tau), \end{cases} \tag{5.1}$$

where Y_i, X_i, and Z_i represent the systolic blood pressure, BMI, and age of the ith subject, respectively, and $u_0(\tau)$ is the unknown change point associated with the τth quantile. We considered three quantile levels $\tau = 0.1$, 0.5 and 0.9.

Table 3 summarizes the testing results from the proposed methods *SIID* and *SNID*, and the sup-likelihood-ratio-based methods *LIID* and *LNID* for males and females separately. For females, except the *LIID* method at $\tau = 0.5$, all four methods suggest that BMI has no significant threshold effect at three quantiles. For males, *SIID*, *SNID*, and *LIID* suggest that BMI has significant threshold effects at three quantiles, but the *LNID* method only detects the threshold effect at $\tau = 0.1$ at the significance level of 0.05. This agrees with the observation in the simulation study that the LRT methods tend to be more conservative than the proposed sup-score-based methods. In contrast, the likelihood-ratio test in Hansen (1996) for mean regression gives p-values 0.001 and 0.014 for the male and female groups, respectively. In the following, we focus our analysis on the male group.

For the quantile regression model (5.1) with a threshold effect, let $\boldsymbol{\theta}(\tau)$ be the collection of regression coefficients $\{a_k(\tau), b_k(\tau), c_k(\tau); k = 1, 2\}$ and the change point $u_0(\tau)$. Following Li et al. (2011), we can estimate $\boldsymbol{\theta}(\tau)$ by

$$\hat{\boldsymbol{\theta}}(\tau) = \arg\min \sum_{i=1}^{n} \rho_\tau\{Y_i - Q_{Y_i}(\tau|X_i, Z_i)\}. \tag{5.2}$$

Table 4. Estimation results in the study of BP and BMI association for the male group. The values in parentheses correspond to the bootstrap standard errors.

		First Segment			Second Segment			Change Point
		a_1	b_1	c_1	a_2	b_2	c_2	
$\tau = 0.1$	Est	64.291	1.310	0.341	113.105	-0.346	0.122	24.6
	SE	(8.570)	(0.474)	(0.063)	(4.935)	(0.164)	(0.056)	
$\tau = 0.5$	Est	62.433	1.855	0.590	102.842	0.157	0.393	21.7
	SE	(8.150)	(0.485)	(0.138)	(4.687)	(0.158)	(0.044)	
$\tau = 0.9$	Est	104.977	-0.086	1.129	115.623	0.168	0.589	22.4
	SE	(19.710)	(0.966)	(0.167)	(8.210)	(0.237)	(0.060)	
Mean	Est	74.958	1.126	0.667	107.728	0.060	0.377	22.4
	SE	(11.774)	(0.622)	(0.062)	(3.499)	(0.105)	(0.036)	

Note: Est and SE stand for Parameter estimate and Standard error, respectively.

We summarize the estimated parameters and corresponding standard errors (obtained by the paired bootstrap method with 200 bootstrap repetitions). in Table 4, and plot the estimated quantiles and mean functions of BP against BMI conditional on the average age in Figure 3. The estimated mean and median regression functions appear consistent with each other. The BMI cut point is around 22 kg/m^2 for both median and the upper quantile $\tau = 0.9$, while it is larger (24.6 kg/m^2) at the lower quantile $\tau = 0.1$, that is, for those with low blood pressures. Not surprisingly, age exhibits significantly positive effects on BP at all three quantiles. However, the effects of age become smaller for those males with BMI above the cut points. In addition, age tends to have larger effects on the upper quantile of the BP distribution. At lower and central quantiles, the effects of BMI are significantly positive before the change point, then become insignificant after the change point. At the upper quantile $\tau = 0.9$, the effects of BMI are not significant throughout the BMI range.

The consistency of BMI effects on BP has been examined in many medical studies for different populations. Past work suggested that the relationship between BMI and BP is likely nonlinear but this may vary between subgroups. By studying urban Nigerian civil servants in 1992, Bunker et al. (1995) suggested that BMI and BP were not correlated below the BMI threshold of 21.5 kg/m^2 but correlated above the threshold. In a study of low-BMI populations in Africa and the Caribbean, Kaufman et al. (1997) observed a threshold at $21 kg/m^2$ in the relationship between BMI and BP for women, but not for men. Kerry et al. (2005) found that for lean older and semi-urban women in West Africa, the effect of BMI on BP below the change point was greater than the effect above it.

Our analysis for US non-hispanic black subjects also suggests different BMI and BP relationships for different subgroups. The previous works all focus on

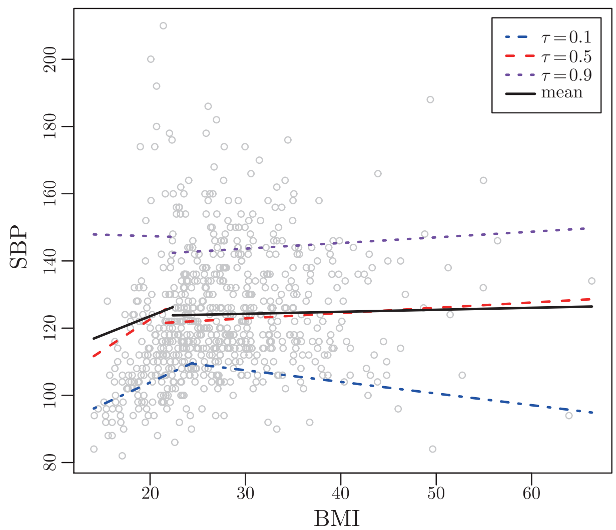

Figure 3. Estimated conditional quantile functions of blood pressure against body mass index for the male group at the mean age.

mean regression analysis. Our proposed methods offer an alternative way to test for the existence of the BMI threshold effect at different tails of the BP distribution, and thus can help assess the BP and BMI association from more angles. For instance, our analysis suggests that at the upper tails of the BP distribution, BP has weaker associations with BMI but stronger associations with age. Further investigation is needed to understand the reasons for this phenomenon and its health implications in obesity epidemic.

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Supplementary Material

Remarks and technical proofs are provided in the Supplementary Material.

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