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CONSTRUCTION OF MIXED-LEVEL SUPERSATURATED DESIGNS BY THE SUBSTITUTION METHOD

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Abstract: Supersaturated design (SSD) has received much interest because of its potential in factor screening experiments. Most studies focus on the construction and analysis of symmetrical SSDs. This paper considers the construction of asymmetrical (or mixed-level) SSDs. A new construction method, called the substitution method, for $E(f_{NOD})$ optimal and nearly-optimal SSDs is proposed. The basic idea of this method is to divide the rows of one design into several blocks, and then substitute the levels of another design with these blocks. The two designs can be SSDs and saturated orthogonal arrays, and their selections are investigated in detail. The properties of the designs generated are also discussed, and many new designs are tabulated for practical use.

Key words and phrases: Coincidence number, equidistant design, mixed-level, orthogonal array, supersaturated design.

1. Introduction

The supersaturated design (SSD) is a kind of factorial design in which the number of runs is not enough to estimate all the main effects. Sometimes scientists and engineers meet in conducting experiments in which, from a large number of factors, they need to screen out a few significant ones in a relatively small number of experimental runs. In such situations SSDs may be useful. Booth and Cox (1962) examined these designs systematically and proposed the $E(s^2)$ criterion, but such designs were not studied further until the appearance of work by Lin (1993) and Wu (1993). SSDs have become increasingly popular in recent years because of their potential in factor screening experiments. Most studies have focused on the construction and analysis of symmetrical SSDs. However, much practical experience indicates that mixed-level SSDs also have wide use.

Researches on mixed-level SSDs include the early work by Fang, Lin, and Liu (2000, 2003) who proposed the $E(f_{NOD})$ criterion and the FSOA method for constructing mixed-level SSDs, and by Yamada and Matsui (2002) and Yamada and Lin (2002) who used χ^2 to evaluate mixed-level SSDs. Fang, Ge, Liu, and Qin (2004a) and Koukouvinos and Mantas (2005) constructed many $E(f_{NOD})$ optimal mixed-level SSDs. Li, Liu, and Zhang (2004) derived a lower bound of χ^2 along with the sufficient and necessary condition for achieving it. Recent work on mixed-level SSD includes Xu (2003), Xu and Wu (2005), Liu, Fang, and Hickernell (2006), Yamada, Matsui, Matsui, Lin, and Tahashi (2006), Ai, Fang, and He (2007), Zhang, Zhang, and Liu (2007), Tang, Ai, Ge, and Fang (2007), Chen and Liu (2008a,b), Liu and Lin (2009), and Liu and Zhang (2009).

The main purpose of this article is to provide a construction method, called the substitution method, for mixed-level SSDs. Section 2 proposes the general construction method for SSDs and gives an illustrative example. The method is carried out by substituting the levels of a support design by the row-blocks of a blocked design. In Section 3, the selections of the equidistant blocked and support designs are investigated and the properties of the generated designs are discussed. Section 4 extends the method to the case of non-equidistant designs, and provides general steps for selecting blocked and support designs. The last section contains some further discussions. Many new designs are tabulated in the Appendix for practical use.

The rest of this section is devoted to the $E(f_{NOD})$ criterion that will be used. A q_i-level design of n runs and m_i factors, denoted by $D(n; q_i^{m_i})$, is an $n \times m_i$ matrix $D_i = (d_{kj})$, where each column takes values from a set of q_i symbols, say $\{1, \ldots, q_i\}$; a mixed-level design with n runs and $m = \sum_{i=1}^t m_i$ factors, denoted by $D(n; {q_1}^{m_1} \cdots {q_t}^{m_t})$, is an $n \times \sum_{i=1}^t m_i$ matrix $D = [D_1, \ldots, D_t]$. D is called an orthogonal array of strength 2, denoted by $L_n({q_1}^{m_1} \cdots {q_t}^{m_t})$, if for any two columns all possible level-combinations appear equally often. When $\sum_{i=1}^t (q_i - 1)m_i = n-1$, D is called a saturated design, and when $\sum_{i=1}^t (q_i - 1)m_i > n-1$, the design is called a *supersaturated design* (SSD). Two columns are called fully aliased if one column can be obtained from the other by permuting levels. A design is called balanced if each column of the design has the equal occurrence property of the levels. Throughout the paper, we only consider balanced designs.

For a $D(n; {q_1}^{m_1} \cdots {q_t}^{m_t})$, we suppose the ith and jth columns $\boldsymbol{d}_i$ and $\boldsymbol{d}_j$ have q_i and q_j levels, respectively. The $E(f_{NOD})$ criterion is

$$E(f_{NOD}) = \frac{\sum_{1 \le i < j \le m} f_{NOD}(\boldsymbol{d}_i, \boldsymbol{d}_j)}{\binom{m}{2}}, \quad \text{where}$$

$$f_{NOD}(\boldsymbol{d}_i, \boldsymbol{d}_j) = \sum_{a=1}^{q_i} \sum_{b=1}^{q_j} \left(n_{ab}(\boldsymbol{d}_i, \boldsymbol{d}_j) - \frac{n}{q_i q_j} \right)^2,$$

and $n_{ab}(\boldsymbol{d}_i, \boldsymbol{d}_j)$ is the number of (a, b)-pairs in $(\boldsymbol{d}_i, \boldsymbol{d}_j)$. Let λ_{hl} be the coincidence number between the hth and lth rows. Fang, Lin, and Liu (2000, 2003) expressed

$E(f_{NOD})$ in terms of λ_{hl}'s as

$$E(f_{NOD}) = \frac{\sum_{h,l=1,h\neq l}^{n} \lambda_{hl}^2}{m(m-1)} + C_f, \text{ where} \tag{1.1}$$

$$C_f = \frac{nm}{m-1} - \frac{n^2}{m(m-1)}\left(\sum_{i=1}^{t}\frac{m_i}{q_i} + \sum_{i=1}^{t}\frac{m_i(m_i-1)}{q_i^2} + \sum_{i=1}^{t}\sum_{j=1,j\neq i}^{t}\frac{m_i m_j}{q_i q_j}\right),$$

and obtained a lower bound of it. Then the lower bound was improved by Fang, Ge, Liu, and Qin (2004a) as

$$E(f_{NOD}) \geq \frac{n(n-1)}{m(m-1)}\Big[(\lfloor\lambda\rfloor + 1 - \lambda)(\lambda - \lfloor\lambda\rfloor) + \lambda^2\Big] + C_f. \tag{1.2}$$

Equality is achieved in (1.2) if and only if all the values of λ_{hl} $(h \neq l)$ take at most two different values $\lfloor\lambda\rfloor$ and $\lfloor\lambda\rfloor + 1$, where $\lfloor\lambda\rfloor$ denotes the integer part of λ, and

$$\lambda = \frac{n\sum_{i=1}^{t} m_i/q_i - m}{n-1}. \tag{1.3}$$

Note that $m - \lambda_{hl}$ is the *Hamming distance* between the hth and lth rows, a design with equal Hamming distances, $m - \lambda$, is called an *equidistant design*, and a design with Hamming distances $m - \lfloor\lambda\rfloor$ and $m - \lfloor\lambda\rfloor - 1$ is called a *weak equidistant design* (Zhang, Fang, Li, and Sudjianto (2005)). Obviously both designs are $E(f_{NOD})$ optimal.

Another class of design criteria relies on maximum $f_{NOD}(\boldsymbol{d}_i, \boldsymbol{d}_j)$ values (Koukouvinos and Mantas (2005) and Chen and Liu (2008a)):

$$\max f_{NOD}^{q_u,q_u} = \max\Big\{f_{NOD}(\boldsymbol{d}_i, \boldsymbol{d}_j) | 1 \leq i < j \leq m, \text{ both } \boldsymbol{d}_i \text{ and } \boldsymbol{d}_j \text{ have } q_u \text{ levels}\Big\},$$

$$\max f_{NOD}^{q_u,q_v} = \max\Big\{f_{NOD}(\boldsymbol{d}_i, \boldsymbol{d}_j) | 1 \leq i, j \leq m, \boldsymbol{d}_i \text{ has } q_u \text{ levels}, \boldsymbol{d}_j \text{ has } q_v \text{ levels}, q_u \neq q_v\Big\}.$$

The $\chi^2(D)$ criterion defined by Yamada and Lin (1999) and Yamada and Matsui (2002) is in fact to minimize $\chi^2(D) = \sum_{1\leq i<j\leq m} q_i q_j f_{NOD}(\boldsymbol{d}_i, \boldsymbol{d}_j)/n$. There are also several other criteria for evaluating mixed-level SSDs, such as the (i) discrete discrepancy (Fang, Lin, and Liu (2000, 2003)), (ii) minimum moment aberration (Xu (2003)). (iii) generalized minimum aberration (Xu and Wu (2001, 2005)), and (iv) minimum projection uniformity (Hickernell and Liu (2002)) criteria. It is obvious that $E(f_{NOD})$ and $\chi^2(D)$ are equivalent in the symmetric case, and it has been shown that they are extensions of existing criteria

for symmetric SSDs; they are closely related to the other four criteria for mixed-level SSDs. In this paper, we mainly adopt $E(f_{NOD})$ to evaluate the newly constructed SSDs.

2. The General Construction Method

Since Fang, Lin, and Liu (2000, 2003) proposed the $E(f_{NOD})$ criterion, there have been several papers concerning the construction of $E(f_{NOD})$-optimal SSDs, such as Fang, Ge, and Liu (2002, 2004) Fang, Ge, Liu, and Qin (2004b), Aggarwal and Gupta (2004), Koukouvinos and Stylianou (2004), Georgiou and Koukouvinos (2006) and Georgiou, Koukouvinos, and Mantas (2006) on multi-level SSDs, Fang, Ge, Liu, and Qin (2004a), Koukouvinos and Mantas (2005), and Chen and Liu (2008a,b) on mixed-level SSDs. As we can see from the previous section and Fang, Ge, Liu, and Qin (2004a), there are strict restrictions on the run size n, level sizes $q_1, \ldots, q_t$, and the respective numbers of factors $m_1, \ldots, m_t$, for a design to attain the lower bound of $E(f_{NOD})$. For many cases, even though the λ in (1.3) is an integer, this lower bound may not be achieved and there is still a need for further research on the construction of SSDs with small values of $E(f_{NOD})$, in particular for the mixed-level case. We now introduce a general construction method for SSDs. The main steps of the construction are as follows.

Step 1. Select a design $D(n; q_1^{m_1} \cdots q_t^{m_t})$ with $m = \sum_{i=1}^t m_i$ and another design $D(kp; p^r)$. (Note that p can be different from $q_1, \ldots, q_t$, but should be a positive divisor of n.)

Step 2. Divide the n rows of $D(n; q_1^{m_1} \cdots q_t^{m_t})$ into p blocks with n/p rows and m columns each; denote these p blocks by $b_1, \ldots, b_p$, respectively.

Step 3. Substitute the p levels, say $1, \ldots, p$, in $D(kp; p^r)$ by $b_1, \ldots, b_p$, respectively, then obtain a design $D(kn; q_1^{m_1 r} \cdots q_t^{m_t r})$.

We call this method the *substitution method*, and call the two designs $D(n; q_1^{m_1} \cdots q_t^{m_t})$ and $D(kp; p^r)$ the *blocked design* and the *support design*, respectively.

We know that equidistant designs are a special kind of $E(f_{NOD})$ optimal designs, and they include saturated $L_n(q^m)$'s with $m = (n-1)/(q-1)$. Such designs can be selected as blocked and support designs. In this paper we construct mixed-level SSDs from the known equidistant designs (most of which are SSDs), and investigate their properties.

Example 1. Table 1 shows an equidistant design $D(6; 2^1 3^3)$, constructed from an $L_9(3^4)$ via the FSOA method, proposed by Fang, Lin, and Liu (2003), and Table 2 tabulates an equidistant design $D(12; 6^{11})$ due to Lu, Hu, and Zheng (2003). Take the $D(6; 2^1 3^3)$ as a blocked design and divide its six rows into six blocks as indicated in Table 1, i.e., the ith row forms b_i for $i = 1, \ldots, 6$. Take the

Table 1. A blocked design $D(6; 2^1 3^3)$

Block	1	2	3	4
b_1	1	1	1	1
b_2	1	2	2	2
b_3	1	3	3	3
b_4	2	1	2	3
b_5	2	2	3	1
b_6	2	3	1	2

Table 2. An SSD $D(12; 6^{11})$

1	2	3	4	5	6	7	8	9	10	11
1	1	1	1	1	1	1	1	1	1	1
1	2	2	2	2	2	2	2	2	2	2
2	1	3	3	3	3	3	3	3	3	2
2	2	1	4	4	4	4	4	4	4	3
3	3	2	1	5	5	5	5	4	3	4
4	4	3	2	1	6	6	5	5	4	5
3	5	4	3	2	1	6	6	6	5	3
5	3	5	4	3	2	1	6	5	6	6
6	4	6	5	4	3	2	1	6	6	4
4	6	6	6	5	4	3	2	1	5	6
5	6	4	5	6	5	4	3	2	1	5
6	5	5	6	6	6	5	4	3	2	1

Table 3. Generated SSD $D(12; 2^{11} 3^{33})$

1	2	3	4	5	6	7	8	9	10	11
1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1
1 1 1 1	1 2 2 2	1 2 2 2	1 2 2 2	1 2 2 2	1 2 2 2	1 2 2 2	1 2 2 2	1 2 2 2	1 2 2 2	1 2 2 2
1 2 2 2	1 1 1 1	1 3 3 3	1 3 3 3	1 3 3 3	1 3 3 3	1 3 3 3	1 3 3 3	1 3 3 3	1 3 3 3	1 2 2 2
1 2 2 2	1 2 2 2	1 1 1 1	2 1 2 3	2 1 2 3	2 1 2 3	2 1 2 3	2 1 2 3	2 1 2 3	2 1 2 3	1 3 3 3
1 3 3 3	1 3 3 3	1 2 2 2	1 1 1 1	2 2 3 1	2 2 3 1	2 2 3 1	2 2 3 1	2 1 2 3	1 3 3 3	2 1 2 3
2 1 2 3	2 1 2 3	1 3 3 3	1 2 2 2	1 1 1 1	2 3 1 2	2 3 1 2	2 2 3 1	2 2 3 1	2 1 2 3	2 2 3 1
1 3 3 3	2 2 3 1	2 1 2 3	1 3 3 3	1 2 2 2	1 1 1 1	2 3 1 2	2 3 1 2	2 3 1 2	2 2 3 1	1 3 3 3
2 2 3 1	1 3 3 3	2 2 3 1	2 1 2 3	1 3 3 3	1 2 2 2	1 1 1 1	2 3 1 2	2 2 3 1	2 3 1 2	2 3 1 2
2 3 1 2	2 1 2 3	2 3 1 2	2 2 3 1	2 1 2 3	1 3 3 3	1 2 2 2	1 1 1 1	2 3 1 2	2 3 1 2	2 1 2 3
2 1 2 3	2 3 1 2	2 3 1 2	2 3 1 2	2 2 3 1	2 1 2 3	1 3 3 3	1 2 2 2	1 1 1 1	2 2 3 1	2 3 1 2
2 2 3 1	2 3 1 2	2 1 2 3	2 2 3 1	2 3 1 2	2 2 3 1	2 1 2 3	1 3 3 3	1 2 2 2	1 1 1 1	2 2 3 1
2 3 1 2	2 2 3 1	2 2 3 1	2 3 1 2	2 3 1 2	2 3 1 2	2 2 3 1	2 1 2 3	1 3 3 3	1 2 2 2	1 1 1 1

$D(12; 6^{11})$ as a support design and substitute its six levels $1, \ldots, 6$ by $b_1, \ldots, b_6$, respectively, to obtain an SSD $D(12; 2^{11} 3^{33})$ as shown in Table 3. It can be easily checked that the coincidence number between any two distinct rows of this generated design is 14, and thus the design is $E(f_{NOD})$ optimal, the $E(f_{NOD})$ value is 4.4651, the lower bound given in (1.2). It is known that in the blocked design $D(6; 2^1 3^3)$, the 2-level factor is orthogonal to any of the 3-level factors. It can be verified that if we divide the 44 columns of this generated design into 11 sub-designs of 4 sequential columns each, then within any sub-design the 2-level factor is still orthogonal to any of the 3-level factors.

Remark 1. Note that our method can always produce $E(f_{NOD})$ optimal designs by selecting appropriate blocked and support designs. In the following section, we set guidelines for producing them.

3. Selection of The Blocked and Support Designs and Properties of The Generated Design

The previous section presented the construction procedure for mixed-level SSDs. Throughout this section, both the blocked and support designs are *equidistant designs.* As for how to divide the rows of $D(n; q_1^{m_1} \cdots q_t^{m_t})$ into p blocks, we have the following.

Theorem 1. *Let the blocked and support designs be equidistant designs.*

(1). *If $n > p$ and $k > 1$, no matter how the rows of the blocked design $D(n; q_1^{m_1} \cdots q_t^{m_t})$ are divided, the generated design has only two different values of coincidence numbers between its rows, and its $E(f_{NOD})$ is a constant.*

(2). *If $n = p$ or $k = 1$, no matter how the rows of the design $D(n; q_1^{m_1} \cdots q_t^{m_t})$ are divided, the generated design has constant coincidence numbers between its rows, and thus is $E(f_{NOD})$ optimal.*

Proof. (1). From the definition of equidistant design in Section 1, we know that the coincidence numbers of the blocked design $D(n; q_1^{m_1} \cdots q_t^{m_t})$ take the constant λ given in (1.3) and those of the support design $D(kp; p^r)$ are

$$\lambda^* = \frac{r(k-1)}{(kp-1)}. \tag{3.1}$$

The mr columns of the generated design $D(kn; q_1^{m_1 r} \cdots q_t^{m_t r})$ can be divided into r sub-designs $C_1, \ldots, C_r$ of m sequential columns each, and the kn rows can be divided into kp row-groups where, in each sub-design, there are k replicates of the blocks $b_1, \ldots, b_p$. For the hth row of b_i and the lth row of b_j, the coincidence number takes the value m if $i = j$ and $h = l$, and the value λ otherwise. From the construction method, we know that the number of coincident blocks between any two distinct row-groups of the generated design is λ^*.

Now consider the coincidence number between the ith and jth rows ($i \neq j$) of the generated design. Based on the above discussion, it can be easily shown that

$$\lambda_{ij} = \begin{cases} \lambda^* m + (r - \lambda^*)\lambda, & \text{if } i = j \mod \frac{n}{p}; \\ r\lambda, & \text{otherwise} , \end{cases} \tag{3.2}$$

and the frequencies of these two values are $N_1 = kn(kp-1)$ and $N_2 = kn(kn-kp)$, respectively. Thus $\sum_{i,j=1}^{kn} \lambda_{ij}^2$ is a constant no matter how the design $D(n; q_1^{m_1} \cdots q_t^{m_t})$ is divided, so $E(f_{NOD})$ is constant.

(2). If $n = p$, the λ_{ij}'s for $i \neq j$ can only take the first value given in (3.2), and if $k = 1$, we have $\lambda^* = 0$ from (3.1) and thus $\lambda_{ij} = r\lambda$. In either of these cases, $E(f_{NOD})$ achieves its lower bound and the generated design is $E(f_{NOD})$ optimal.

The $E(f_{NOD})$ optimality of the generated design in Example 1 can be directly verified since $n = p = 6$. The next theorem considers the selection of the support design in order to get a generated design with a smaller $E(f_{NOD})$.

Theorem 2. *Given a blocked design $D(n; q_1^{m_1} \cdots q_t^{m_t})$, suppose there exist support designs $D(kp_i; {p_i}^r)$, $i = 1, 2$ with $k > 1$ and $p_1 < p_2$. Then the $D(kn; q_1^{m_1 r} \cdots q_t^{m_t r})$ generated from $D(kp_2; {p_2}^r)$ has a smaller $E(f_{NOD})$ than the one generated from $D(kp_1; {p_1}^r)$.*

Proof. For the blocked design, the coincidence numbers between its rows take the constant λ given in (1.3). For any $D(kp_i; {p_i}^r)$, the coincidence numbers take the value $\lambda_i^* = r(k-1)/(kp_i - 1)$, $i = 1, 2$. Then, similar to the proof of Theorem 1, the corresponding $D(kn; q_1^{m_1 r} \cdots {q_t}^{m_t r})$ takes the coincidence number values $\lambda_{i1}^* = \lambda_i^* m + (r - \lambda_i^*)\lambda$ and $\lambda_{i_2}^* = r\lambda$, with frequencies $N_{i1} = kn(kp_i - 1)$ and $N_{i2} = kn(kn - kp_i)$, respectively, $i = 1, 2$. Since $p_1 < p_2$, we can easily have

$$\begin{aligned} \lambda_1^* > \lambda_2^*, \quad \lambda_{11}^* > \lambda_{21}^* > \lambda_{22}^* = \lambda_{12}^*, \\ N_{11} < N_{21}, \quad N_{12} > N_{22}, \quad N_{11}\lambda_{11}^* + N_{12}\lambda_{12}^* = N_{21}\lambda_{21}^* + N_{22}\lambda_{22}^*. \end{aligned}$$

Then, based on the expression of $E(f_{NOD})$ in (1.1) and the majorization theory recently used in Zhang, Fang, Li, and Sudjianto (2005) and Liu, Fang, and Hickernell (2006), we get the assertion of the theorem.

From the construction method, we know that the largest possible value of p for the support design is n; then, based on Theorem 2 and conclusion (2) of Theorem 1, we have the following.

Corollary 1. *Given a blocked design $D(n; q_1^{m_1} \cdots q_t^{m_t})$, if there exists a support design $D(kn; n^r)$ with $k > 1$, then among all the possible $D(kn; q_1^{m_1 r} \cdots q_t^{m_t r})$ generated from $D(kp_i; {p_i}^r)$, the one corresponding to $D(kn; n^r)$ has the smallest $E(f_{NOD})$, and only this generated design achieves the lower bound of $E(f_{NOD})$.*

The above results tell us how to divide a blocked design and how to select a good support design in terms of the $E(f_{NOD})$ criterion. As for the near-orthogonality of columns of the generated designs, one has the following.

Theorem 3. *Let $\boldsymbol{b}_i$, $\boldsymbol{s}_i$, and $\boldsymbol{g}_i$ be the ith columns of the blocked design $D(n; q_1^{m_1} \cdots q_t^{m_t})$, support design $D(kp; p^r)$, and generated design $D(kn; q_1^{m_1 r} \cdots q_t^{m_t r})$, respectively, and $b_l = (b_{l1}, \ldots, b_{lm})$ be the lth block of the blocked design with $n = p$.*

(1). $f_{NOD}(\boldsymbol{g}_{i'}, \boldsymbol{g}_{j'}) = k^2 f_{NOD}(\boldsymbol{b}_i, \boldsymbol{b}_j)$. *In particular if the blocked design is an $L_n(q_1^{m_1} \cdots q_t^{m_t})$, $f_{NOD}(\boldsymbol{g}_{i'}, \boldsymbol{g}_{j'}) = 0$, where $i' = (u-1)m+i$, $j' = (u-1)m+j$, $u = 1, \ldots, r$, $i, j = 1, \ldots, m$, $i \neq j$.*

(2). *When* $n = p$, $n_{ab}(\boldsymbol{g}_{i'}, \boldsymbol{g}_{j'}) = \sum_S n_{hl}(\boldsymbol{s}_u, \boldsymbol{s}_v)$. *In particular if the support design is an* $L_{kn}(n^r)$, $f_{NOD}(\boldsymbol{g}_{i'}, \boldsymbol{g}_{j'}) = 0$, *where* $S = \{(h, l) : b_{hi} = a, b_{lj} = b\}$, $i' = (u-1)m + i$, $j' = (v-1)m + j$, $u, v = 1, \ldots, r$, $u \neq v$, $i, j = 1, \ldots, m$.

(3). *If the blocked and support designs are an* $L_n(q_1^{m_1} \cdots q_t^{m_t})$ *and an* $L_{kn}(n^r)$, *respectively, then the generated design is an* $L_{kn}(q_1^{m_1 r} \cdots q_t^{m_t r})$. *Furthermore, if the blocked design is a saturated* $L_n(q^m)$ *and the* $L_{kn}(n^r)$ *is saturated, then the generated design is a saturated* $L_{kn}(q^{mr})$.

Remark 2. The properties given in this theorem are valid for both equidistant and non-equidistant blocked and support designs. Note that the $E(f_{NOD})$ criterion is not enough to prevent the existence of fully aliased columns in the design. The mr columns of the generated design can be divided into r sub-designs of m sequential columns each. Conclusion (1) of this theorem guarantees that the near-orthogonality measured by f_{NOD} of the blocked design is well kept within each sub-design of the generated design. Since it can be easily shown that $f_{NOD}(\boldsymbol{d}_i, \boldsymbol{d}_j) = \sum_{a=1}^{q_i} \sum_{b=1}^{q_j} (n_{ab}(\boldsymbol{d}_i, \boldsymbol{d}_j))^2 - n^2/(q_i q_j)$, conclusion (2) ensures that the near-orthogonality between columns of the support design is well kept between sub-designs of the generated design. Conclusion (3) is a special case which ensures the production of orthogonal generated designs.

These properties are very helpful when we have some prior information about the activeness of the factors; one can allocate the possibly active factors, or the ones of interest, into orthogonal columns in order to avoid aliasing among those factors when screening them.

4. Extensions to Non-equidistant Designs

In the above discussion, the blocked and support designs are equidistant designs, but such designs may not exit for some given parameters. In such cases, we can replace the blocked and/or support designs with non-equidistant designs, such as *weak equidistant designs*.

Theorem 4. *Given an equidistant blocked design* $D(n; {q_1}^{m_1} \cdots {q_t}^{m_t})$ *and a support design* $D(kp; p^r)$ *with different values of coincidence numbers between its rows, then no matter how the blocked design is divided, the generated design* $D(kn; {q_1}^{m_1 r} \cdots {q_t}^{m_t r})$ *has a constant* $E(f_{NOD})$.

Proof. For the blocked design, the coincidence numbers between its rows take the constant λ given in (1.3). Suppose, among the $kp(kp-1)$ coincidence numbers of $D(kp; p^r)$, there are N_i with the value λ_i^* for $i = 1, \ldots, l$, hence $\sum_{i=1}^{l} N_i = kp(kp-1)$. Then the coincidence numbers of the generated design take $l+1$ values: $\beta_i = \lambda_i^* m + (r - \lambda_i^*)\lambda$, for $i = 1, \ldots, l$, and $\beta_{l+1} = r\lambda$, with frequencies

$M_i = N_i n/p$, for $i = 1, \ldots, l$, and $M_{l+1} = kn(kn-kp)$, respectively. The assertion follows.

Note that, when $n = p$, $M_{l+1} = 0$ and there are only l values of coincidence numbers in the generated design.

Example 2. Suppose we want to construct an SSD $D(12; 2^{10}3^{30})$ with the $D(6; 2^1 3^3)$ shown in Table 1 as the blocked design. For the support design, we take the $D(12; 6^{10})$ formed by the first ten columns of the design in Table 2. It can be seen that this $D(12; 6^{10})$ is a weak equidistant design, and thus an $E(f_{NOD})$ optimal one. The generated design $D(12; 2^{10}3^{30})$ consists of the first ten sub-designs of the design shown in Table 3. It has two values of coincidence numbers, 10 and 13, and its $E(f_{NOD})$ is 4.46 with the lower bound being 4.42.

Theorem 2 and Corollary 1 suggest selecting the equidistant support designs with large level sizes (run sizes), in particular the n-level ones if they exist. However, for some cases, such equidistant designs may not exist though there may exist equidistant designs with smaller level sizes.

Theorem 5. *Given an equidistant blocked design $D(n; q_1^{m_1} \cdots q_t^{m_t})$, suppose there exist a support design $D(kp; p^r)$ with λ^* coincidences between its rows, where $k > 1$ and $p < n$, and another support design $D(kn; n^r)$ with $\lambda_1^*, \ldots, \lambda_l^*$ ($l \geq 2$) coincidences satisfying*

$$0 \leq \lambda_1^* < \cdots < \lambda_l^* \leq \lambda^*. \tag{4.1}$$

Then if there exist at least two strict inequalities in (4.1), *the $D(kn; q_1^{m_1 r} \cdots q_t^{m_t r})$ generated from $D(kn; n^r)$ has a smaller $E(f_{NOD})$ than the one generated from $D(kp; p^r)$ ($p < n$); otherwise, the two generated designs have the same $E(f_{NOD})$.*

Proof. For the blocked design, the coincidence numbers between its rows take the constant λ given in (1.3). It can be easily shown that the design generated from the $D(kp; p^r)$ takes two values of coincidence numbers: $\lambda_{11}^* = r\lambda$ and $\lambda_{12}^* = \lambda^* m + (r - \lambda^*)\lambda$, and the other generated design takes l different values: $\lambda_{2i}^* = \lambda_i^* m + (r - \lambda_i^*)\lambda$, for $i = 1, \ldots, l$. Since (4.1) holds,

$$\lambda_{11}^* \leq \lambda_{21}^* < \cdots < \lambda_{2l}^* \leq \lambda_{12}^*. \tag{4.2}$$

The condition that there exist at least two strict inequalities in (4.1) implies that there exist at least two strict inequalities in (4.2). Thus, based on the expression of $E(f_{NOD})$ in (1.1) and majorization theory, the assertions follow.

Remark 3. When $l > 2$, there are at least two strict inequalities in (4.1) as well as in (4.2); when $l = 2$, if $0 < \lambda_1^*$ and/or $\lambda_l^* < \lambda^*$, there exist at least two strict inequalities in (4.1); if $\lambda_1^* = 0$ and $\lambda_l^* = \lambda^*$, the two generated designs perform the same in terms of $E(f_{NOD})$. Note that when $l = 1$, it can be shown that

$0 < \lambda_1^* < \lambda^*$, i.e., the condition that there exist at least two strict inequalities in (4.1) holds; this is the case of equidistant support designs and the design generated from $D(kn; n^r)$ is $E(f_{NOD})$ optimal (cf. Corollary 1).

Remark 4. When there exist no equidistant blocked designs, we can choose weak equidistant designs or other $E(f_{NOD})$ optimal or nearly-optimal ones as blocked designs. As we can see from the proof of Theorem 1, for example for the case of $k = 1$ (i.e., $\lambda^* = 0$), if we have a non-equidistant blocked design with l different coincidence numbers $\lambda_1, \ldots, \lambda_l$, then the generated design will have coincidence numbers in the form of $\sum_{i=1}^{l} r_i\lambda_i$ with $\sum_{i=1}^{l} r_i = r$. From (1.1), these values should spread as equally as possible, as should $\lambda_1, \ldots, \lambda_l$, hence we prefer to select the best blocked design in terms of $E(f_{NOD})$.

Based on the above discussion, we have the following steps for the selections of the blocked design $D(n; q_1^{m_1} \cdots q_t^{m_t})$ and support design $D(kp; p^r)$ for constructing a $D(kn; q_1^{m_1 r} \cdots q_t^{m_t r})$.

Step 1. Select two equidistant designs $D(n; q_1^{m_1} \cdots q_t^{m_t})$ and $D(kp; p^r)$ satisfying $n = p$ or $k = 1$ if they exist (cf. Theorem 1 and Corollary 1).

Step 2. Select an equidistant blocked design $D(n; q_1^{m_1} \cdots q_t^{m_t})$ and a weak equidistant support design $D(kn; n^r)$ (i.e., $n = p$) according to Theorem 5 and Remark 3.

Step 3. Select a weak equidistant blocked design $D(n; q_1^{m_1} \cdots q_t^{m_t})$ and an equidistant support design $D(p; p^r)$ (i.e., $k = 1$) according to Remark 4.

Step 4. If (weak) equidistant designs are difficult to obtain, select other good designs in terms of $E(f_{NOD})$.

Remark 5. Note that, if there exist more than one candidate for any set of given parameters, one looks to select a design with smaller maximum f_{NOD} values in order to keep the $\max f_{NOD}^{q_u,q_u}$ and $\max f_{NOD}^{q_u,q_v}$ values of the generated design as small as possible (cf. Theorem 3).

Appendices A and B tabulate some $E(f_{NOD})$ optimal designs which are constructed from two equidistant designs $D(n; q_1^{m_1} \cdots q_t^{m_t})$ and $D(kn; n^r)$. Note that the orthogonal arrays used in the tables can be found from the sites maintained by Dr. N. J. A. Sloane (`http://www.research.att.com/~njas/oadir/`) and Dr. W. F. Kuhfeld (`http://support.sas.com/techsup/technote/ts723.html`). Each equidistant blocked design in Appendix B is obtained by adding an n-level column to the corresponding blocked design in Appendix A, this column is a permutation of the n levels. All the designs in these two tables are new except for those designs marked with * that can also be constructed by other methods, e.g., those proposed by Georgiou and Koukouvinos (2006) and Georgiou, Koukouvinos, and Mantas (2006). In particular, the designs $D(81; 3^{40})$ and

$D(256; 4^{85})$ (corresponding to the two 1's marked with $^\circ$) are saturated orthogonal arrays (cf. (3) of Theorem 3). Since all the selected source designs perform well in terms of their maximum f_{NOD} values, there are no fully aliased columns in these generated designs, and they also have the properties mentioned in Theorem 3.

Many other $E(f_{NOD})$ optimal or nearly-optimal designs can be constructed using this new method, especially for the case of $k = 1$. For this case, the support designs $D(p; p^r)$ can be uniform designs and orthogonal Latin hypercube designs (Fang, Li, and Sudjianto (2006), Steinberg and Lin (2006) and Pang, Liu, and Lin (2009)).

5. Further Discussion

The substitution method is easy to perform, and the generated designs have low values of $E(f_{NOD})$. The near-orthogonality between columns in terms of $\max f_{NOD}^{q_u,q_u}$ and $\max f_{NOD}^{q_u,q_v}$ of the blocked design is well kept within each sub-design of the generated design, and that of the support design is well kept between sub-designs of the generated design. In some cases, the coincidence numbers of the generated design may take more that two different values; from (1.1), the coincidence numbers should spread as equally as possible. In order to improve these designs with respect to $E(f_{NOD})$, we can use them as starting designs for the Robin Hood Swap algorithm of Zhang, Fang, Li, and Sudjianto (2005), or the NOA algorithm of Nguyen (1996).

The method proposed in this paper includes several existing methods as special cases. When the blocked design is an $L_n(q^m)$ and the support design is a $D(n; n^r)$, our method reduces to the collapsing method proposed by Fang, Lin, and Ma (2000) and the row permutation method of Lu and Sun (2001), Aggarwal and Gupta (2004), Koukouvinos and Stylianou (2004), and Georgiou, Koukouvinos, and Mantas (2006); when the blocked and support designs are both q-level equidistant designs, our method reduces to a method proposed by Lu and Wan (2007).

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Appendix A: Some $E(f_{NOD})$ optimal designs.

Blocked design	[Source]	Support design	[Source]	Generated design
$D(6;3^5)$	[FGL04†]	$D(12;6^{11})$	[LHZ03]	$D(12;3^{55})^*$
$D(6;3^5)$	[FGL04]	$D(12;6^{22})$	[GK06]	$D(12;3^{110})$
$D(6;3^5)$	[FGL04]	$D(12;6^{33})$	[GK06]	$D(12;3^{165})$
$D(6;3^5)$	[FGL04]	$D(18;6^{17})$	[LHZ03]	$D(18;3^{85})$
$D(6;3^5)$	[FGL04]	$D(18;6^{34})$	[GK06]	$D(18;3^{170})$
$D(6;3^5)$	[FGL04]	$D(24;6^{23})$	[LHZ03]	$D(24;3^{115})$
$D(6;3^5)$	[FGL04]	$D(30;6^{29})$	[LFXY02]	$D(30;3^{145})$
$D(6;3^5)$	[FGL04]	$D(36;6^{35})$	[LFXY02]	$D(36;3^{175})$
$D(6;3^5)$	[FGL04]	$D(42;6^{41})$	[LFXY02]	$D(42;3^{205})$
$D(9;3^{4k})$	[FGL04]	$L_{81}(9^{10})$	[OA‡]	$D(81;3^{40k})$, $k=1^\circ,\ldots,7$
$D(9;3^{8k})$	[GKM06]	$L_{81}(9^{10})$	[OA]	$D(81;3^{80k})$, $k=4,5,6$
$D(8;4^7)$	[FGL02]	$D(32;8^{31})$	[FGLQ04b]	$D(32;4^{217})$
$D(8;4^{7k})$	[GK06]	$D(32;8^{31})$	[FGLQ04b]	$D(32;4^{217k})$, $k=2,\ldots,6$
$D(8;4^7)$	[FGL02]	$L_{64}(8^9)$	[OA]	$D(64;4^{63})^*$
$D(8;4^{7k})$	[GK06]	$L_{64}(8^9)$	[OA]	$D(64;4^{63k})^*$, $k=2,\ldots,6$
$D(16;4^{5k})$	[FGLQ04b]	$L_{256}(16^{17})$	[OA]	$D(256;4^{85k})$, $k=1^\circ,\ldots,7$
$D(6;2^1 3^3)$	[FLL03]	$D(12;6^{11})$	[LHZ03]	$D(12;2^{11}3^{33})$
$D(6;2^1 3^3)$	[FLL03]	$D(12;6^{22})$	[GK06]	$D(12;2^{22}3^{66})$
$D(6;2^1 3^3)$	[FLL03]	$D(12;6^{33})$	[GK06]	$D(12;2^{33}3^{99})$
$D(6;2^1 3^3)$	[FLL03]	$D(18;6^{17})$	[LHZ03]	$D(18;2^{17}3^{51})$
$D(6;2^1 3^3)$	[FLL03]	$D(18;6^{34})$	[GK06]	$D(18;2^{34}3^{102})$
$D(6;2^1 3^3)$	[FLL03]	$D(24;6^{23})$	[LHZ03]	$D(24;2^{23}3^{69})$
$D(6;2^1 3^3)$	[FLL03]	$D(30;6^{29})$	[LFXY02]	$D(30;2^{29}3^{87})$
$D(6;2^1 3^3)$	[FLL03]	$D(36;6^{35})$	[LFXY02]	$D(36;2^{35}3^{105})$
$D(6;2^1 3^3)$	[FLL03]	$D(42;6^{41})$	[LFXY02]	$D(42;2^{41}3^{123})$
$D(8;2^1 4^4)$	[FLL03]	$D(32;8^{31})$	[FGLQ04b]	$D(32;2^{31}4^{124})$
$D(8;2^8 4^4)$	[KM05]	$D(32;8^{31})$	[FGLQ04b]	$D(32;2^{248}4^{124})$
$D(8;2^1 4^4)$	[FLL03]	$L_{64}(8^9)$	[OA]	$D(64;2^9 4^{36})$
$D(8;2^8 4^4)$	[KM05]	$L_{64}(8^9)$	[OA]	$D(64;2^{72}4^{36})$
$D(16;2^1 8^8)$	[FLL03]	$L_{256}(16^{17})$	[OA]	$D(256;2^{17}8^{136})$

† FGL04: Fang, Ge, and Liu (2004); etc.

‡ OA: orthogonal array.

* Designs that can also be constructed via other existing methods.

° Saturated orthogonal arrays.

Appendix B: More $E(f_{NOD})$ optimal designs.

Blocked design	[Source§]	Support design	[Source]	Generated design
$D(6;3^5 6^1)$	[FGL04†]	$D(12;6^{11})$	[LHZ03]	$D(12;3^{55}6^{11})$
$D(6;3^5 6^1)$	[FGL04]	$D(12;6^{22})$	[GK06]	$D(12;3^{110}6^{22})$
$D(6;3^5 6^1)$	[FGL04]	$D(12;6^{33})$	[GK06]	$D(12;3^{165}6^{33})$
$D(6;3^5 6^1)$	[FGL04]	$D(18;6^{17})$	[LHZ03]	$D(18;3^{85}6^{17})$
$D(6;3^5 6^1)$	[FGL04]	$D(18;6^{34})$	[GK06]	$D(18;3^{170}6^{34})$
$D(6;3^5 6^1)$	[FGL04]	$D(24;6^{23})$	[LHZ03]	$D(24;3^{115}6^{23})$
$D(6;3^5 6^1)$	[FGL04]	$D(30;6^{29})$	[LFXY02]	$D(30;3^{145}6^{29})$
$D(6;3^5 6^1)$	[FGL04]	$D(36;6^{35})$	[LFXY02]	$D(36;3^{175}6^{35})$
$D(6;3^5 6^1)$	[FGL04]	$D(42;6^{41})$	[LFXY02]	$D(42;3^{205}6^{41})$
$D(9;3^{4k}9^1)$	[FGL04]	$L_{81}(9^{10})$	[OA‡]	$D(81;3^{40k}9^{10})$, $k=1,\ldots,7$
$D(9;3^{8k}9^1)$	[GKM06]	$L_{81}(9^{10})$	[OA]	$D(81;3^{80k}9^{10})$, $k=4,5,6$
$D(8;4^7 8^1)$	[FGL02]	$D(32;8^{31})$	[FGLQ04b]	$D(32;4^{217}8^{31})$
$D(8;4^{7k}8^1)$	[GK06]	$D(32;8^{31})$	[FGLQ04b]	$D(32;4^{217k}8^{31})$, $k=2,\ldots,6$
$D(8;4^7 8^1)$	[FGL02]	$L_{64}(8^9)$	[OA]	$D(64;4^{63}8^9)$
$D(8;4^{7k}8^1)$	[GK06]	$L_{64}(8^9)$	[OA]	$D(64;4^{63k}8^9)$, $k=2,\ldots,6$
$D(16;4^{5k}16^1)$	[FGLQ04b]	$L_{256}(16^{17})$	[OA]	$D(256;4^{85k}16^{17})$, $k=1,\ldots,7$
$D(6;2^1 3^3 6^1)$	[FLL03]	$D(12;6^{11})$	[LHZ03]	$D(12;2^{11}3^{33}6^{11})$
$D(6;2^1 3^3 6^1)$	[FLL03]	$D(12;6^{22})$	[GK06]	$D(12;2^{22}3^{66}6^{22})$
$D(6;2^1 3^3 6^1)$	[FLL03]	$D(12;6^{33})$	[GK06]	$D(12;2^{33}3^{99}6^{33})$
$D(6;2^1 3^3 6^1)$	[FLL03]	$D(18;6^{17})$	[LHZ03]	$D(18;2^{17}3^{51}6^{17})$
$D(6;2^1 3^3 6^1)$	[FLL03]	$D(18;6^{34})$	[GK06]	$D(18;2^{34}3^{102}6^{34})$
$D(6;2^1 3^3 6^1)$	[FLL03]	$D(24;6^{23})$	[LHZ03]	$D(24;2^{23}3^{69}6^{23})$
$D(6;2^1 3^3 6^1)$	[FLL03]	$D(30;6^{29})$	[LFXY02]	$D(30;2^{29}3^{87}6^{29})$
$D(6;2^1 3^3 6^1)$	[FLL03]	$D(36;6^{35})$	[LFXY02]	$D(36;2^{35}3^{105}6^{35})$
$D(6;2^1 3^3 6^1)$	[FLL03]	$D(42;6^{41})$	[LFXY02]	$D(42;2^{41}3^{123}6^{41})$
$D(8;2^1 4^4 8^1)$	[FLL03]	$D(32;8^{31})$	[FGLQ04b]	$D(32;2^{31}4^{124}8^{31})$
$D(8;2^8 4^4 8^1)$	[KM05]	$D(32;8^{31})$	[FGLQ04b]	$D(32;2^{248}4^{124}8^{31})$
$D(8;2^1 4^4 8^1)$	[FLL03]	$L_{64}(8^9)$	[OA]	$D(64;2^9 4^{36}8^9)$
$D(8;2^8 4^4 8^1)$	[KM05]	$L_{64}(8^9)$	[OA]	$D(64;2^{72}4^{36}8^9)$
$D(16;2^1 8^8 16^1)$	[FLL03]	$L_{256}(16^{17})$	[OA]	$D(256;2^{17}8^{136}16^{17})$

§ The blocked design is obtained by adding an n-level column to the source.

† FGL04: Fang, Ge, and Liu (2004); etc.

‡ OA: orthogonal array.

References

Aggarwal, M. L. and Gupta, S. (2004). A new method of construction of multi-level supersaturated designs. *J. Statist. Plann. Inference* **121**, 127-134.

Ai, M. Y., Fang, K. T. and He, S. Y. (2007). $E(\chi^2)$-optimal mixed-level supersaturated designs. *J. Statist. Plann. Inference* **137**, 306-316.

Booth, K. H. V. and Cox, D. R. (1962). Some systematic supersaturated designs. *Technometrics* **4**, 489-495.

Chen, J. and Liu, M. Q. (2008a). Optimal mixed-level k-circulant supersaturated designs. *J. Statist. Plann. Inference* **138**, 4151-4157.

Chen, J. and Liu, M. Q. (2008b). Optimal mixed-level supersaturated design with general number of runs. *Statist. Probab. Lett.* **78**, 2496-2502.

Fang, K. T., Ge, G. N. and Liu, M. Q. (2002). Construction of $E(f_{NOD})$-optimal supersaturated designs via Room squares. *Calcutta Statist. Assoc. Bull.* **52**, 71-84.

Fang, K. T., Ge, G. N. and Liu, M. Q. (2004). Construction of optimal supersaturated designs by the packing method. *Sci. China Ser. A* **47**, 128-143.

Fang, K. T., Ge, G. N., Liu, M. Q. and Qin, H. (2004a). Combinatorial constructions for optimal supersaturated designs. *Discrete Math.* **279**, 191-202.

Fang, K. T., Ge, G. N., Liu, M. Q. and Qin, H. (2004b). Construction of uniform designs via super-simple resolvable t-designs. *Util. Math.* **66**, 15-32.

Fang, K. T., Li, R. and Sudjianto, A. (2006). *Design and Modeling for Computer Experiments.* Chapman and Hall/CRC, New York.

Fang, K. T., Lin, D. K. J. and Liu, M. Q. (2000). Optimal mixed-level supersaturated design and computer experiment. Technical Report MATH-286, Hong Kong Baptist University.

Fang, K. T., Lin, D. K. J. and Liu, M. Q. (2003). Optimal mixed-level supersaturated design. *Metrika* **58**, 279-291.

Fang, K. T., Lin, D. K. J. and Ma, C. X. (2000). On the construction of multi-level supersaturated designs. *J. Statist. Plann. Inference* **86**, 239-252.

Georgiou, S. and Koukouvinos, C. (2006). Multi-level k-circulant supersaturated designs. *Metrika* **64**, 209-220.

Georgiou, S., Koukouvinos, C. and Mantas, P. (2006). On multi-level supersaturated designs. *J. Statist. Plann. Inference* **136**, 2805-2819.

Hickernell, F. J. and Liu, M. Q. (2002). Uniform designs limit aliasing. *Biometrika* **89**, 893-904.

Koukouvinos, C. and Mantas, P. (2005). Construction of some $E(f_{NOD})$ optimal mixed-level supersaturated designs. *Statist. Probab. Lett.* **74**, 312-321.

Koukouvinos, C. and Stylianou, S. (2004). Optimal multi-level supersaturated designs constructed from linear and quadratic functions. *Statist. Probab. Lett.* **69**, 199-211.

Li, P. F., Liu, M. Q. and Zhang, R. C. (2004). Some theory and the construction of mixed-level supersaturated designs. *Statist. Probab. Lett.* **69**, 105-116.

Lin, D. K. J. (1993). A new class of supersaturated designs. *Technometrics* **35**, 28-31.

Liu, M. Q., Fang, K. T. and Hickernell, F. J. (2006). Connections among different criteria for asymmetrical fractional factorial designs. *Statist. Sinica* **16**, 1285-1297.

Liu, M. Q. and Lin, D. K. J. (2009). Construction of optimal mixed-level supersaturated designs. *Statist. Sinica* **19**, 197-211.

Liu, M. Q. and Zhang, L. (2009). An algorithm for constructing mixed-level k-circulant supersaturated designs. *Comput. Statist. Data Anal.*, **53**, 2465-2470.

Lu, X., Fang, K. T., Xu, Q. F. and Yin, J. X. (2002). Balance pattern and BP-optimal factorial designs. *Technical Report* **MATH-324**, Hong Kong Baptist University.

Lu, X., Hu, W. and Zheng, Y. (2003). A systematical procedure in the construction of multi-level supersaturated designs. *J. Statist. Plann. Inference* **115**, 287-310.

Lu, X. and Sun, Y. (2001). Supersaturated designs with more than two levels. *Chinese Ann. Math. B* **22**, 183-194.

Lu, X. and Wan, Z. C. (2007). A new method in the construction of multi-level supersaturated designs (in Chinese). Manuscript.

Nguyen, N. K. (1996). An algorithmic approach to constructing supersaturated designs. *Technometrics* **38**, 69-73.

Pang, F., Liu, M. Q. and Lin, D. K. J. (2009). A construction method for orthogonal Latin hypercube designs with prime power levels. *Statist. Sinica* **19**, 1721-1728.

Steinberg, D. M. and Lin, D. K. J. (2006). A construction method for orthogonal Latin hypercube designs. *Biometrika* **93**, 279-288.

Tang, Y., Ai, M. Y., Ge, G. N. and Fang, K. T. (2007). Optimal mixed-level supersaturated designs and a new class of combinatorial designs. *J. Statist. Plann. Inference* **137**, 2294-2301.

Wu, C. F. J. (1993). Construction of supersaturated designs through partially aliased interactions. *Biometrika* **80**, 661-669.

Xu, H. (2003). Minimum moment aberration for nonregular designs and supersaturated designs. *Statist. Sinica* **13**, 691-708.

Xu, H. and Wu, C. F. J. (2001). Generalized minimum aberration for asymmetrical fractional factorial designs. *Ann. Statist.* **29**, 1066-1077.

Xu, H. and Wu, C. F. J. (2005). Construction of optimal multi-level supersaturated designs. *Ann. Statist.* **33**, 2811-2836.

Yamada, S. and Lin, D. K. J. (1999). Three-level supersaturated designs. *Statist. Probab. Lett.* **45**, 31-39.

Yamada, S. and Lin, D. K. J. (2002). Construction of mixed-level supersaturated design. *Metrika* **56**, 205-214.

Yamada, S., Matsui, M., Matsui, T., Lin, D. K. J., and Tahashi, T. (2006). A general construction method for mixed-level supersaturated design. *Comput. Statist. Data Anal.* **50**, 254-265.

Yamada, S. and Matsui, T. (2002). Optimality of mixed-level supersaturated designs. *J. Statist. Plann. Inference* **104**, 459-468.

Zhang, A., Fang, K. T., Li, R. and Sudjianto, A. (2005). Majorization framework for balanced lattice designs. *Ann. Statist.* **33**, 2837-2853.

Zhang, Q. Z., Zhang, R. C. and Liu, M. Q. (2007). A method for screening active effects in supersaturated designs. *J. Statist. Plann. Inference* **137**, 2068-2079.

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