Statistica Sinica **13**(2003), 1269-1280

ON PARTIAL SUFFICIENCY AND INVARIANCE

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Abstract: This paper studies the relationship between partial sufficiency (in the sense of Fraser) and invariance, both in terms of σ-fields and of statistics, with application of the main result to the problem of testing statistical hypotheses. Some examples are given to illustrate these results.

Key words and phrases: Invariance, partial sufficiency, sufficiency.

1. Introduction and Mathematical Definitions

Two well-known methods of reduction of statistical experiments are sufficiency, where no information is lost, and invariance, where the loss of information is justified by an argument of symmetry. The relationship between the two is studied in Hall, Wijsman and Ghosh (1965), in the invariant case, and in Berk (1972) in the almost invariant case. A version of the main theorem of Hall, Wijsman and Ghosh (1965), they refer to it as a theorem of Stein, can be found in Nogales, Oyola and Pérez (2000). We also cite Berk, Nogales and Oyola (1996) and Nogales and Oyola (1996), where some remarks are made on the conditional independence of the sufficient and invariant or almost invariant σ-fields given their intersection.

In this paper, a similar study is realized for the Fraser approach to the concept of partial sufficiency. We distinguish the invariant and almost invariant cases, and the results are stated in terms of σ-fields as well as in terms of statistics. An application to testing hypotheses and three examples are given to illustrate our results on the relationship between partial sufficiency and invariance. The reader is referred to the classical paper of Basu (1978), reproduced in Ghosh (1988), where a detailed study of the evolution of the concept of partial sufficiency can be found.

Let us fix the notations to be used throughout the paper. $(\Omega, \mathcal{A}, \mathcal{P})$ is a statistical experiment, i.e., $(\Omega, \mathcal{A})$ is a measurable space and $\mathcal{P}$ a family of probability measures on $(\Omega, \mathcal{A})$. We suppose

$$\mathcal{P} = \{P_{\theta,\phi} \colon (\theta, \phi) \in \Theta \times \Phi\}, \tag{1}$$

where Θ and Φ are nonempty sets. θ is considered the parameter of interest, while ϕ is a nuisance parameter. The family $\mathcal{P}$ will be supposed identifiable, in the sense that $P_{\theta,\phi} \neq P_{\theta',\phi'}$ if $(\theta,\phi) \neq (\theta',\phi')$.

Given $\phi \in \Phi$, we write $\mathcal{P}_\phi = \{P_{\theta,\phi}\colon \theta \in \Theta\}$; $\mathcal{N}$ (resp., $\mathcal{N}_\phi$) will denote the family of the $\mathcal{P}$-null (resp., $\mathcal{P}_\phi$-null) events. For two statistics, f and g, we write $f \sim g$ (resp., $f \overset{\phi}{\sim} g$) if $\{f \neq g\}$ belongs to $\mathcal{N}$ ($\mathcal{N}_\phi$, resp.) In this case, f and g are said to be $\mathcal{P}$-equivalent (resp., $\mathcal{P}_\phi$-equivalent). For a sub-σ-field $\mathcal{B} \subset \mathcal{A}$, $[\mathcal{B}]^+$ denotes the class of the $\mathcal{B}$-measurable and non-negative functions, and we write $\overline{\mathcal{B}}$ (resp. $\overline{\mathcal{B}}^\phi$) for the completion of $\mathcal{B}$ with the $\mathcal{P}$-null (resp., $\mathcal{P}_\phi$-null) sets. Given two sub-σ-fields $\mathcal{C}$ and $\mathcal{D}$ of $\mathcal{A}$, we write $\mathcal{C} \sim \mathcal{D}$ when $\overline{\mathcal{C}} = \overline{\mathcal{D}}$.

Let $\mathcal{B}$ be a sub-σ-field of $\mathcal{A}$ (resp., $S : (\Omega,\mathcal{A}) \to (\Omega_S,\mathcal{A}_S)$ be a statistic). Given a non-negative real statistic f on $(\Omega,\mathcal{A})$, we consider the conditional expectation $E_{\theta,\phi}(f|\mathcal{B})$ (resp., $E_{\theta,\phi}(f|S)$) as the equivalence class of all measurable functions $g : (\Omega,\mathcal{B}) \to \bar{\mathbb{R}}$ (resp., $g : (\Omega_S,\mathcal{A}_S) \to \bar{\mathbb{R}}$) such that $\int_B f dP_{\theta,\phi} = \int_B g dP_{\theta,\phi}$, for all $B \in \mathcal{B}$ (resp., $\int_{S^{-1}A_S} f dP_{\theta,\phi} = \int_{A_S} g dP^S_{\theta,\phi}$, for all $A_S \in \mathcal{A}_S$, where $P^S_{\theta,\phi}$ denotes the probability distribution of S with respect to $P_{\theta,\phi}$, defined on $\mathcal{A}_S$ by $P^S_{\theta,\phi}(A_S) = P_{\theta,\phi}(S^{-1}A_S)$). For an event $A \in \mathcal{A}$, the conditional probabilities $P_{\theta,\phi}(A|\mathcal{B})$ and $P_{\theta,\phi}(A|S)$ are defined as the conditional expectations $E_{\theta,\phi}(I_A|\mathcal{B})$ and $E_{\theta,\phi}(I_A|S)$, resp., where I_A denotes the indicator function of A.

The sub-σ-field $\mathcal{B}$ of $\mathcal{A}$ is said to be θ-oriented when the restriction $P^{\mathcal{B}}_{\theta,\phi}$ of the probability $P_{\theta,\phi}$ to the σ-field $\mathcal{B}$ does not depend on ϕ. $\mathcal{B}$ is said to be specific θ-sufficient if it is sufficient for the statistical experiment $(\Omega,\mathcal{A},\mathcal{P}_\phi)$ for every $\phi \in \Phi$. $\mathcal{B}$ is said to be partially θ-sufficient (in the sense of Fraser (1956)) if it is θ-oriented and specific θ-sufficient. A statistic $S : (\Omega,\mathcal{A},\mathcal{P}) \to (\Omega_S,\mathcal{A}_S)$ is said to be θ-oriented, specific θ-sufficient or partially θ-sufficient if its induced σ-field $S^{-1}(A_S)$ is.

Let us briefly recall invariance. A transformation on the set Ω is a bijective map from Ω onto itself. We say a group G of bimeasurable transformations on $(\Omega,\mathcal{A})$ leaves $(\Omega,\mathcal{A},\mathcal{P})$ invariant when, for all $g \in G$ and $P \in \mathcal{P}$, the probability distribution P^g of g with respect to P remains in $\mathcal{P}$; we also say that G leaves the family $\mathcal{P}$ invariant. The G-orbit of a point $\omega \in \Omega$ is the set $\{g(\omega)\colon g \in G\}$. A statistic $f : (\Omega,\mathcal{A},\mathcal{P}) \to (\Omega',\mathcal{A}')$ is said to be G-invariant if $f \circ g = f$, for all $g \in G$ (i.e., if it is constant on every orbit); f is said to be almost G-invariant (resp., ϕ-almost G-invariant for a given $\phi \in \Phi$) if $\{f \neq f \circ g\} \in \mathcal{N}$ (resp., $\{f \neq f \circ g\} \in \mathcal{N}_\phi$), for all $g \in G$. f is said to be maximal G-invariant if it is G-invariant and takes different values on different orbits. An event $A \in \mathcal{A}$ is said to be G-invariant, almost G-invariant or ϕ-almost G-invariant when its indicator function I_A is. In the next, $\mathcal{A}_I$, $\mathcal{A}_A$ and $\mathcal{A}^\phi_A$ will denote, respectively, the σ-fields of the G-invariant, almost G-invariant and ϕ-almost G-invariant events. Obviously, every G-invariant or almost G-invariant statistic is $\mathcal{A}_I$-measurable or $\mathcal{A}_A$-measurable, resp. We can find, in Florens, Mouchart and Rolin (1990), regular conditions to guarantee the converse implications and to establish clearly the relationship between the maximal G-invariance of a statistic $U : (\Omega, \mathcal{A}) \to (\Omega_U, \mathcal{A}_U)$ and its induced σ-field $U^{-1}(\mathcal{A}_U)$. In particular, if $(\Omega, \mathcal{A})$ and $(\Omega_U, \mathcal{A}_U)$ are standard Borel spaces (recall that a measurable space is said to be a standard Borel space if there exists a bimeasurable map from it onto a Borel set of $\mathbb{R}$) and the orbits are measurable, then U is a maximal G-invariant function if, and only if, $U^{-1}(\mathcal{A}_U) = \mathcal{A}_I$. Obviously, the completion of $\mathcal{A}_I$ is always included in $\mathcal{A}_A$. In Lehmann (1986) we can find a theorem of Stein stating sufficient conditions to guarantee the converse contention, i.e., that every almost G-invariant real statistic is $\mathcal{P}$-equivalent to some G-invariant one. Namely, this is the case when the family $\mathcal{P}$ is dominated by a σ-finite measure and remains invariant under the action of a locally compact topological group $(G, \mathcal{G})$, which acts measurably on $(\Omega, \mathcal{A})$. These conditions are easily checked in the examples of the last section. A sub-σ-field $\mathcal{B}$ is said to be stable (resp., essentially stable) when $g\mathcal{B} = \mathcal{B}$ (resp., $g\mathcal{B} \sim \mathcal{B}$) for all $g \in G$. A statistic $S : (\Omega, \mathcal{A}, \mathcal{P}) \to (\Omega_S, \mathcal{A}_S)$ is said to be stable when $S^{-1}(\mathcal{A}_S)$ is; S is said to be equivariant when all the points in Ω with the same image by S have also the same image by $S \circ g$, for all $g \in G$. For an equivariant, stable and surjective statistic S, a group $G^S = \{g^S : g \in G\}$ of bimeasurable transformations on $(\Omega_S, \mathcal{A}_S)$ is induced as follows: $g^S \circ S = S \circ g$, for all $g \in G$; it is readily shown that G^S leaves invariant the family $\mathcal{P}^S := \{P^S : P \in \mathcal{P}\}$ of the probability distributions of S with respect to the probability measures of the family $\mathcal{P}$.

To finish this section, recall the main results on the relationship between sufficiency and invariance related to those we present below.

The first one was published in Hall, Wijsman and Ghosh (1965) and attributed to Stein. It is the version for σ-fields of this theorem: Suppose the statistical experiment $(\Omega, \mathcal{A}, \mathcal{P})$ remains invariant under the action of the group G, and let $\mathcal{B}$ be a sufficient sub-σ-field of $\mathcal{A}$ satisfying the conditions A(i) $\mathcal{B}$ is stable, A(ii) $\mathcal{B} \cap \mathcal{A}_I \sim \mathcal{B} \cap \mathcal{A}_A$. Then, $\mathcal{B} \cap \mathcal{A}_I$ is sufficient for $\mathcal{A}_I$.

The statistics version of the theorem of Stein can be found in the first part of the paper of Hall, Wijsman and Ghosh (1965) and, in the present form, in Nogales, Oyola and Pérez (2000). The following diagram illustrates the statement of the result:

$$
\begin{tikzcd}
 & (\Omega_S, \mathcal{A}_S) \arrow[dr, "U_S"] & \\
(\Omega, \mathcal{A}) \arrow[ur, "S"] \arrow[dr, "U"'] & & (\Omega', \mathcal{A}') \\
 & (\Omega_U, \mathcal{A}_U) \arrow[ur, "S_U"'] &
\end{tikzcd}
$$

The result itself goes as follows: Let $(\Omega, \mathcal{A})$, $(\Omega_S, \mathcal{A}_S)$, $(\Omega_U, \mathcal{A}_U)$ and $(\Omega', \mathcal{A}')$ be standard Borel spaces, and G be a group of bimeasurable transformations on $(\Omega, \mathcal{A})$ leaving invariant the statistical experiment $(\Omega, \mathcal{A}, \mathcal{P})$. Let $S : (\Omega, \mathcal{A}) \to (\Omega_S, \mathcal{A}_S)$ be a surjective, equivariant, stable and sufficient statistic. Let $U : (\Omega, \mathcal{A}) \to (\Omega_U, \mathcal{A}_U)$ and $U_S : (\Omega_S, \mathcal{A}_S) \to (\Omega', \mathcal{A}')$ be maximal invariant statistics for the groups G and G^S, resp. Suppose that, for every almost G^S-invariant real statistics, there exists a G^S-invariant real statistic which is $\mathcal{P}^S$-equivalent to it. Then, there exists a sufficient statistic $S_U : (\Omega_U, \mathcal{A}_U) \to (\Omega', \mathcal{A}')$ such that $S_U \circ U = U_S \circ S$.

If the principle of invariance is understood as a reduction to the σ-field $\mathcal{A}_A$ of the almost invariant events, the following theorem of Berk (1972) solves the problem considered: If $\mathcal{B}$ is sufficient and essentially stable, $\mathcal{B} \cap \mathcal{A}_A$ is sufficient for $\mathcal{A}_A$.

Finally, we consider the special case in which $P^g = P$, for all $P \in \mathcal{P}$ and all $g \in G$, i.e., when every $g \in G$ is a model preserving transformation. We say the statistical experiment $(\Omega, \mathcal{A}, \mathcal{P})$ is strongly invariant. We have the following result of Farrell (see, for example Ghosh (1988)): If $\mathcal{P}$ is dominated by a σ-finite measure and the transformations in G are model preserving, $\mathcal{A}_A$ is sufficient. Moreover, if $\mathcal{B}$ is sufficient and essentially stable, $\mathcal{B} \cap \mathcal{A}_A$ is sufficient.

2. Partial Sufficiency and Almost Invariance

In order to obtain a similar result to the theorem of Berk for partial sufficiency, we make use of the following lemmas.

Lemma 1. *If $\mathcal{B}$ is a σ-field θ-oriented and essentially stable, $\mathcal{B} \cap \mathcal{A}_A^{\phi} = \mathcal{B} \cap \mathcal{A}_A$ for every $\phi \in \Phi$.*

Proof. Fix $\phi \in \Phi$. Let $\mathcal{F}$ be the class of the events $A \in \mathcal{A}$ such that $P_{\theta,\phi'}(A)$ does not depend on $\phi' \in \Phi$. By hypothesis $\mathcal{B} \subset \mathcal{F}$. First, we prove that $\overline{\mathcal{B}} \subset \mathcal{F}$. Indeed, given $B \in \mathcal{B}$, $N \in \mathcal{N}$, $\phi' \in \Phi$ and $\theta \in \Theta$, we have that

$$\begin{aligned} P_{\theta,\phi'}(B \triangle N) &= P_{\theta,\phi'}(B) + P_{\theta,\phi'}(N) - 2P_{\theta,\phi'}(B \cap N) \\ &= P_{\theta,\phi'}(B) = P_{\theta,\phi}(B) = P_{\theta,\phi}(B \triangle N), \end{aligned}$$

which proves that $B \triangle N \in \mathcal{F}$.

Next, we note that $\mathcal{F} \cap \mathcal{N}_\phi = \mathcal{N}$, as the $\mathcal{P}_\phi$-null events of $\mathcal{F}$ are $\mathcal{P}$-null and $\mathcal{N} \subset \mathcal{F}$.

To finish the proof of the lemma, we take $B \in \mathcal{B} \cap \mathcal{A}_A^{\phi}$ and $g \in G$. Then, $I_B \circ g \in [g^{-1}\mathcal{B}]^+ \subset [\overline{\mathcal{B}}]^+$, by the essential stability of $\mathcal{B}$. Hence $N_g := \{I_B \neq I_B \circ g\} \in \overline{\mathcal{B}} \cap \mathcal{N}_\phi \subset \mathcal{F} \cap \mathcal{N}_\phi = \mathcal{N}$. This shows that $B \in \mathcal{A}_A$ and gives the proof.

The next lemma is a consequence of Berk's Theorem.

Lemma 2. *Let $\mathcal{B}$ be a specific θ-sufficient and essentially stable σ-field. Suppose that G leaves invariant every family $\mathcal{P}_\phi$, $\phi \in \Phi$. Then $\bigcap_{\theta\in\Theta} P_{\theta,\phi}(A|\mathcal{B} \cap \mathcal{A}_A^\phi) \neq \emptyset$ for all $\phi \in \Phi$ and all $A \in \mathcal{A}_A^\phi$.*

Proof. Given $\phi \in \Phi$ and $A \in \mathcal{A}_A^\phi$, let

$$q_A^\phi \in \bigcap_{\theta\in\Theta} P_{\theta,\phi}(A|\mathcal{B}).$$

Then, for every $B \in \mathcal{B}$, $N \in \mathcal{N}_\phi$, and $\theta \in \Theta$,

$$P_{\theta,\phi}(A \cap B) = \int_{B\triangle N} q_A^\phi \, dP_{\theta,\phi}. \tag{2}$$

Being $\mathcal{P}_\phi$ invariant under the action of G, for every $g \in G$ there exists a bijection $\overline{g}_1$ from Θ onto itself, such that $P_{\theta,\phi}^g = P_{\overline{g}_1(\theta,\phi)}$, for all $\theta \in \Theta$. Then, for all $\phi \in \Phi$, $N \in \mathcal{N}_\phi$ is stable, because if $N \in \mathcal{N}_\phi$ and $g \in G$,

$$P_{\theta,\phi}(gN) = P_{\overline{g}^{-1}(\theta,\phi)}(N) = P_{\overline{g}_1^{-1}(\theta),\phi}(N) = 0.$$

Thus, for all $B \in \mathcal{B}$, $N \in \mathcal{N}_\phi$, $g \in G$ and $\theta \in \Theta$, we have

$$\begin{aligned}\int_{B\triangle N} q_A^\phi \circ g \, dP_{\theta,\phi} &= \int_{gB\triangle gN} q_A^\phi \, dP_{\theta,\phi}^g = \int_{gB\triangle gN} q_A^\phi \, dP_{\overline{g}_1(\theta),\phi} \\ &\overset{(2)}{=} P_{\overline{g}_1(\theta),\phi}(A \cap gB) = P_{\theta,\phi}^g(A \cap gB) \\ &= P_{\theta,\phi}(g^{-1}(A) \cap B) = P_{\theta,\phi}(A \cap B) \\ &= P_{\theta,\phi}(A \cap (B\triangle N)).\end{aligned}$$

We have then proved that

$$\int_D q_A^\phi \circ g \, dP_{\theta,\phi} = P_{\theta,\phi}(A \cap D), \quad \int_D q_A^\phi \, dP_{\theta,\phi} = P_{\theta,\phi}(A \cap D), \quad \forall D \in \overline{\mathcal{B}}^\phi, \ \forall \theta \in \Theta.$$

Being $q_A^\phi \circ g$ $g^{-1}(\mathcal{B})$-measurable and $g^{-1}(\mathcal{B}) \sim \mathcal{B}$, we get that

$$q_A^\phi, \ q_A^\phi \circ g \in \bigcap_{\theta\in\Theta} P_{\theta,\phi}(A|\overline{\mathcal{B}}^\phi)$$

and, so, $q_A^\phi \overset{\phi}{\sim} q_A^\phi \circ g$, for all $g \in G$. Thus, q_A^ϕ is ϕ-almost G-invariant.

Now, we are ready to prove the desired result.

Theorem 3. *If G leaves invariant every family $\mathcal{P}_\phi$ and $\mathcal{B}$ is a partially θ-sufficient and essentially stable σ-field, then $\mathcal{B} \cap \mathcal{A}_A$ is partially θ-sufficient for $\mathcal{A}_A$.*

Proof. From the previous lemma we get that, for every $A \in \mathcal{A}_A$ and $\phi \in \Phi$, $\cap_{\theta\in\Theta} P_{\theta,\phi}(A|\mathcal{B} \cap \mathcal{A}_A^{\phi}) \neq \emptyset$. By Lemma 1, we have that $\mathcal{B} \cap \mathcal{A}_A^{\phi} = \mathcal{B} \cap \mathcal{A}_A$, for all $\phi \in \Phi$. Hence $\bigcap_{\theta\in\Theta} P_{\theta,\phi}(A|\mathcal{B} \cap \mathcal{A}_A) \neq \emptyset$. So, $\mathcal{B} \cap \mathcal{A}_A$ is specific θ-sufficient for $\mathcal{A}_A$. This finishes the proof, as $\mathcal{B} \cap \mathcal{A}_A$ also is θ-oriented.

Under the condition A(ii), we can obtain the analogous result for invariance as an easy consequence of this theorem.

Corollary 4. *Let $\mathcal{B}$ be a partially θ-sufficient and essentially stable σ-field. Suppose that G leaves invariant every family $\mathcal{P}_\phi$, $\phi \in \Phi$, and that* A(ii) *is satisfied.*

(i) *For all $\phi \in \Phi$ and $A \in \mathcal{A}_I$, there exists a G-invariant version in* $\bigcap_{\theta\in\Theta} P_{\theta,\phi}(A|\mathcal{B})$.

(ii) $\mathcal{B} \cap \mathcal{A}_I$ *is partially θ-sufficient for $\mathcal{A}_I$.*

Proof. We prove part (i): given $\phi \in \Phi$ and $A \in \mathcal{A}_I$, Lemmas 1 and 2 show that there exists an $\mathcal{A}_A$-measurable common version q_A^{ϕ} of $P_{\theta,\phi}(A|\mathcal{B})$, where $\theta \in \Theta$. Then, by A(ii), there exists a $\mathcal{B} \cap \mathcal{A}_I$-measurable function p_A^{ϕ} $\mathcal{P}$-equivalent to q_A^{ϕ}. Thus, this is a G-invariant common version of $P_{\theta,\phi}(A|\mathcal{B})$, where $\theta \in \Theta$. Part (ii) is an obvious corollary of previous theorem.

The proposition (i) can be easily extended to positive $\mathcal{A}_I$-measurable functions, obtaining the next corollary.

Corollary 5. *Under the same conditions, for all $\phi \in \Phi$ and $\psi \in [\mathcal{A}_I]^+$, there exists an invariant version $p_\psi^{\phi} \in \cap_{\theta\in\Theta} E_{\theta,\phi}(\psi|\mathcal{B})$.*

Now we obtain a version of Stein's Theorem for partial sufficiency.

Theorem 6. *Let $(\Omega, \mathcal{A})$, $(\Omega_S, \mathcal{S})$, $(\Omega_U, \mathcal{A}_U)$ and $(\Omega', \mathcal{A}')$ be standard Borel spaces, G a group of bimeasurable transformations leaving invariant the statistical experiment $(\Omega, \mathcal{A}, \mathcal{P})$, $S : (\Omega, \mathcal{A}) \to (\Omega_S, \mathcal{A}_S)$ a surjective, equivariant, stable and partially θ-sufficient statistic, and $U : (\Omega, \mathcal{A}) \to (\Omega_U, \mathcal{A}_U)$ and $U_S : (\Omega_S, \mathcal{A}_S) \to (\Omega', \mathcal{A}')$ maximal invariant statistics for the groups G and G^S, resp. Suppose that for every almost G^S-invariant real statistics there exists a G^S-invariant equivalent real statistic, and that every family $\mathcal{P}_\phi$ remains invariant under the action of G. Then, there exists a partially θ-sufficient statistic $S_U : (\Omega_U, \mathcal{A}_U) \to (\Omega', \mathcal{A}')$ such that $S_U \circ U = U_S \circ S$.*

Proof. Since $(\Omega, \mathcal{A})$ and $(\Omega_U, \mathcal{A}_U)$ are standard Borel spaces, the G-invariant statistic $U_S \circ S$ is $U^{-1}(\mathcal{A}_U)$-measurable. Since $(\Omega', \mathcal{A}')$ is also a standard Borel space, there exists a statistic $S_U : (\Omega_U, \mathcal{A}_U) \to (\Omega', \mathcal{A}')$ such that $U_S \circ S = S_U \circ U$. We prove that S_U is partially θ-sufficient. On the one hand, S_U is θ-oriented because, given $\theta \in \Theta$ and $\phi, \phi' \in \Phi$, we have that $\left(P_{\theta,\phi}^U\right)^{S_U} = \left(P_{\theta,\phi}^S\right)^{U_S} = \left(P_{\theta,\phi'}^S\right)^{U_S} = \left(P_{\theta,\phi'}^U\right)^{S_U}$. On the other hand, S being specific θ-sufficient, we have that, given $\phi \in \Phi$ and $A \in \mathcal{A}_U$, there exists $p_A^\phi : (\Omega_S, \mathcal{A}_S) \to [0,1]$ such that

$$P_{\theta,\phi}\left(U^{-1}(A) \cap S^{-1}(B)\right) = \int_B p_A^\phi \, dP_{\theta,\phi}^S, \quad \forall B \in \mathcal{A}_S, \ \forall \theta \in \Theta.$$

Since $\mathcal{P}_\phi$ is G-invariant, given $g \in G$, there exists a transformation $\overline{g}_\phi$ on Θ such that $P_{\theta,\phi}^g = P_{\overline{g}_\phi(\theta),\phi}$ for all $\theta \in \Theta$. Then, for all $B \in \mathcal{A}_S$ and all $\theta \in \Theta$, we get

$$\begin{aligned}
\int_B p_A^\phi \circ g^S \, dP_{\theta,\phi}^S &= \int_{g^S B} p_A^\phi \, d\left(P_{\theta,\phi}^S\right)^{g^S} = \int_{g^S B} p_A^\phi \, d\left(P_{\theta,\phi}^g\right)^S \\
&= \int_{g^S B} p_A^\phi \, dP_{\overline{g}_\phi(\theta),\phi}^S = P_{\overline{g}_\phi(\theta),\phi}\left(U^{-1}(A) \cap S^{-1}(g^S B)\right) \\
&= P_{\theta,\phi}^g\left(U^{-1}(A) \cap g(S^{-1}(B))\right) \\
&= P_{\theta,\phi}\left((U \circ g)^{-1}(A) \cap S^{-1}(B)\right) \\
&= P_{\theta,\phi}\left(U^{-1}(A) \cap S^{-1}(B)\right).
\end{aligned}$$

Then $p_A^\phi, p_A^\phi \circ g^S \in \bigcap_{\theta\in\Theta} P_{\theta,\phi}\left(U^{-1}(A)|S\right)$. Thus $\{p_A^\phi \neq p_A^\phi \circ g^S\}$ is a $\mathcal{P}_\phi^S$-null event. S being θ-oriented, we have that $p_A^\phi \sim p_A^\phi \circ g^S$. This shows that p_A^ϕ is almost G^S-invariant. By hypothesis, there exists a G^S-invariant statistic $f_A^\phi : (\Omega_S, \mathcal{A}_S) \to [0,1]$, $\mathcal{P}^S$-equivalent to it. Since $(\Omega_S, \mathcal{A}_S)$ and $(\Omega', \mathcal{A}')$ are standard Borel spaces, f_A^ϕ is $U_S^{-1}(\mathcal{A}')$-measurable. Hence there exists a statistic $h_A^\phi : (\Omega', \mathcal{A}') \to [0,1]$ such that $f_A^\phi = h_A^\phi \circ U_S$. Then, given $D \in \mathcal{A}'$ and $\theta \in \Theta$, we have that

$$\begin{aligned}
\int_D h_A^\phi \, dP_{\theta,\phi}^{S_U \circ U} &= \int_D h_A^\phi \, dP_{\theta,\phi}^{U_S \circ S} = \int_{U_S^{-1}(D)} h_A^\phi \circ U_S \, dP_{\theta,\phi}^S \\
&= \int_{U_S^{-1}(D)} f_A^\phi \, dP_{\theta,\phi}^S = P_{\theta,\phi}\left(U^{-1}(A) \cap S^{-1}(U_S^{-1}(D))\right) \\
&= P_{\theta,\phi}\left(U^{-1}(A) \cap U^{-1}(S_U^{-1}(D))\right) = P_{\theta,\phi}^U\left(A \cap S_U^{-1}(D)\right).
\end{aligned}$$

Thus $h_A^\phi \in \bigcap_{\theta\in\Theta} P_{\theta,\phi}^U(A|S_U)$, and S_U is specific θ-sufficient.

Now, we turn our attention to the strongly invariant case. The next theorem is the analogue for partial sufficiency of the theorem of Farrell cited in the introduction.

Theorem 7. *Let $(\Omega, \mathcal{A}, \mathcal{P})$ be a strongly G-invariant statistical experiment and suppose that every family $\mathcal{P}_\phi$ is dominated by a σ-finite measure. If $\mathcal{B}$ is a partially θ-sufficient and essentially stable σ-field, then $\mathcal{B} \cap \mathcal{A}_A$ is partially θ-sufficient.*

Proof. Let $\phi \in \Phi$. By the theorem of Farrell, $\mathcal{A}_A^\phi$ is sufficient with respect to $\mathcal{P}_\phi$. Moreover, it follows from Lemma 2 that $\mathcal{B} \cap \mathcal{A}_A^\phi$ is sufficient for $\mathcal{A}_A^\phi$ with respect to $\mathcal{P}_\phi$. Then, by Lemma 1 and the transitivity of sufficiency, $\mathcal{B} \cap \mathcal{A}_A$ is sufficient with respect to $\mathcal{P}_\phi$, i.e., it is specific θ-sufficient. The proof is finished because it is also θ-oriented.

3. An Application in Testing Hypotheses

The next theorem is an application of these results to testing hypotheses. It gives sense to the concept of UMP invariant test after a reduction by partial sufficiency.

Theorem 8. *Suppose that every family $\mathcal{P}_\phi$, and the problem of testing the hypothesis $\Theta_0 \times \Phi$, remains invariant under the action of the group G. Let S be a partially θ-sufficient, stable and equivariant statistic onto $(\Omega_S, \mathcal{A}_S)$ such that, for every almost G^S-invariant real statistic, there exists a $\mathcal{P}^S$-equivalent and G^S-invariant statistic. Then if φ is a UMP G^S-invariant level α test to test the null hypothesis Θ_0 in the image experiment, $\varphi \circ S$ is UMP G-invariant at level α in the original experiment.*

Proof. S being θ-oriented, the family $\mathcal{P}^S$ does not depend on ϕ (and so, the null hypothesis $\Theta_0 \times \Phi$ is reduced to Θ_0). For every $\theta \in \Theta_0$ and $\phi \in \Phi$, we have that $E_{P_{\theta,\phi}}(\varphi) = E_{P_\theta^S}(\varphi \circ S) \leq \alpha$. Thus $\varphi \circ S$ is a G-invariant level α test.

By hypothesis, $\mathcal{B} := S^{-1}(\mathcal{A}_S)$ is a partially θ-sufficient and stable σ-field such that $\mathcal{B} \cap \mathcal{A}_I \sim \mathcal{B} \cap \mathcal{A}_A$. Let ψ be an invariant level α test in the original experiment. Then, given $\phi \in \Phi$, Corollary 4 shows that there exists a G-invariant test $p_\psi^\phi \in \cap_{\theta \in \Theta} E_{P_{\theta,\phi}}(\psi|\mathcal{B})$. If q_ψ^ϕ is a test in the image experiment such that $p_\psi^\phi = q_\psi^\phi \circ S$, q_ψ^ϕ is G^S-invariant and, for $\theta \in \Theta_0$, $E_{P_\theta^S}(q_\psi^\phi) = E_{P_{\theta,\phi}}(p_\psi^\phi) = E_{P_{\theta,\phi}}(\psi) \leq \alpha$. Since φ is a UMP invariant at level α, we have that, for every $\theta \notin \Theta_0$, $E_{P_{\theta,\phi}}(\psi) = E_{P_\theta^S}(q_\psi^\phi) \leq E_{P_\theta^S}(\varphi) = E_{P_{\theta,\phi}}(\varphi \circ S)$.

4. Examples

To illustrate the previous results, we give three examples. First we give a useful result.

Throughout this section, we only shall consider statistical experiments of the form $(\Omega_1 \times \Omega_2, \mathcal{A}_1 \times \mathcal{A}_2, \{P_\theta \otimes \mathcal{L}_\phi : \theta \in \Theta,\ \phi \in \Phi\})$, where, for $\phi \in \Phi$, $\mathcal{L}_\phi : (\Omega, \mathcal{A}_1) \rightarrowtail (\Omega, \mathcal{A}_2)$ is a stochastic kernel (i.e., $\mathcal{L}_\phi : \Omega_1 \times \mathcal{A}_2 \to [0,1]$ is a map such that, for every $\omega_1 \in \Omega_1$, $\mathcal{L}_\phi(\omega_1, \cdot)$ is a probability measure on $\mathcal{A}_2$ and, for every $A_2 \in \mathcal{A}_2$, $\mathcal{L}_\phi(\cdot, A_2)$ is an $\mathcal{A}_1$-measurable map) and $P_\theta \otimes \mathcal{L}_\phi$ denotes the unique probability measure on the product σ-field such that $(P_\theta \otimes \mathcal{L}_\phi)(A_1 \times A_2) = \int_{A_1} \mathcal{L}_\phi(\omega_1, A_2) dP_\theta(\omega_1)$ for every $A_i \in \mathcal{A}_i$, $i = 1, 2$.

In the examples below, it is useful to characterize the invariance of every subfamily $\mathcal{P}_\phi$ in terms of the stochastic kernels $\mathcal{L}_\phi$. In the next, we write $\mathbb{G}$ for the group of all the bimeasurable transformations g on the product space $(\Omega_1\times\Omega_2,\mathcal{A}_1\times\mathcal{A}_2)$ that can be represented as $g(\omega_1,\omega_2)=(g_1(\omega_1),g_2(\omega_2))$, where g_1 and g_2 are bimeasurable transformations on $(\Omega_1,\mathcal{A}_1)$ and $(\Omega_2,\mathcal{A}_2)$, resp. For such a transformation g and a stochastic kernel $\mathcal{L}_\phi : (\Omega,\mathcal{A}_1)\rightarrowtail(\Omega,\mathcal{A}_2)$, we obtain a new stochastic kernel $\mathcal{L}^g : (\Omega,\mathcal{A}_1)\rightarrowtail(\Omega,\mathcal{A}_2)$ setting $\mathcal{L}^g(\omega_1,A_2) := \mathcal{L}(g_1^{-1}(\omega_1),g_2^{-1}(A_2))$. Let G be a subgroup of $\mathbb{G}$. The stochastic kernel $\mathcal{L}$ is said to be G-invariant if $\mathcal{L}^g=\mathcal{L}$ for all $g\in G$. $\mathcal{L}$ is said to be weakly almost G-invariant when, for every $g\in G$ and every $A_2\in\mathcal{A}_2$, there exists a $\{P_\theta\colon\theta\in\Theta\}$-null event $N_{g,A_2}\in\mathcal{A}_1$ such that $\mathcal{L}^g(\omega_1,A_2)=\mathcal{L}(\omega_1,A_2)$ for all $\omega_1\in\Omega_1\setminus N_{g,A_2}$. If the null events N_{g,A_2} can be chosen not depending on A_2, $\mathcal{L}$ is said to be almost G-invariant.

In this framework, we have the following result.

Proposition 9. *Given $\phi\in\Phi$, G leaves invariant the family $\mathcal{P}_\phi$ if, and only if, the stochastic kernel $\mathcal{L}_\phi$ is weakly almost G-invariant.*

Proof. Let $g=(g_1,g_2)\in G$ and $\theta\in\Theta$. For $A_1\in\mathcal{A}_1$ and $A_2\in\mathcal{A}_2$, we have that

$$\begin{aligned}(P_\theta\otimes\mathcal{L}_\phi)^g(A_1\times A_2) &= (P_\theta\otimes\mathcal{L}_\phi)\left(g_1^{-1}(A_1)\times g_2^{-1}(A_2)\right)\\ &= \int_{g_1^{-1}A_1}\mathcal{L}_\phi\left(\omega_1,g_2^{-1}(A_2)\right)dP_\theta(\omega_1)\\ &= \int_{A_1}\mathcal{L}_\phi\left(g_1^{-1}(\omega_1),g_2^{-1}(A_2)\right)dP_\theta^{g_1}(\omega_1)\\ &= \int_{A_1}\mathcal{L}_\phi^g(\omega_1,A_2)\;dP_\theta^{g_1}.\end{aligned}$$

If $\mathcal{P}_\phi$ is G-invariant, then $(P_\theta\otimes\mathcal{L}_\phi)^g = P_\theta^{g_1}\otimes\mathcal{L}_\phi$. Hence $\{\mathcal{L}_\phi^g(\cdot,A_2)\neq\mathcal{L}_\phi(\cdot,A_2)\}$ is $P_\theta^{g_1}$-null, for every $\theta\in\Theta$, which shows that $\mathcal{L}$ is weakly almost G-invariant.

To prove the converse, note that the hypothesis shows that the probability measures $(P_\theta\otimes\mathcal{L}_\phi)^g$ and $P_\theta^{g_1}\otimes\mathcal{L}_\phi$ coincide on the measurable rectangles, and Dynkin's Theorem finishes the proof.

Example 1. Consider the statistical experiment $(\mathbb{R}^2,\mathcal{R}^2,\{P_\sigma\otimes\mathcal{L}_\phi\colon\sigma,\phi>0\})$, where $\mathcal{R}^2$ denotes the Borel σ-field in $\mathbb{R}^2$, $P_\sigma=N(0,\sigma^2)$ and $\mathcal{L}_\phi(x,\cdot)=N(0,x^2\phi^2)$. Let $G:=\{g_a\colon a>0\}$, where $g:(x,y)\in\mathbb{R}^2\mapsto g_a(x,y):=(h_a(x),h_a(y))$ and $h_a:x\in\mathbb{R}\mapsto h_a(x):=ax$. G is a group of transformations that leaves invariant the experiment above. Given $\phi>0$, we have that $\mathcal{L}_\phi^{g_a}(x,A)=\mathcal{L}_\phi\left(a^{-1}x,a^{-1}A\right)=\left[N\left(0,a^{-2}x^2\phi^2\right)\right]^{h_a}(A)=N(0,x^2\phi^2)(A)=\mathcal{L}_\phi(x,\ A)$, and so $\mathcal{L}_\phi$ is G-invariant. Then, $\mathcal{P}_\phi$ remains invariant under the action of G. The first projection S is a partially σ-sufficient, stable and equivariant statistic onto $\mathbb{R}$. If C is the unit circumference in $\mathbb{R}^2$, the statistic U from $\mathbb{R}^2$ onto $C \cup \{(0,0)\}$ defined as

$$U(x,y) = \begin{cases} \frac{1}{\sqrt{x^2+y^2}} \quad (x,y) & \text{if } (x,y) \neq (0,0) \\ \qquad (0,0) & \text{if } (x,y) = (0,0) \end{cases}$$

is maximal invariant. Moreover, the statistic U_S from $\mathbb{R}$ onto $\{-1,0,1\}$ defined as

$$U_S(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is maximal invariant in the image experiment of S. Then, by Theorem 6, the map $S_U : C \cup \{(0,0)\} \to \{-1,0,+1\}$ defined by

$$S_U(u,v) = \begin{cases} 1 & \text{if } u > 0 \\ -1 & \text{if } u < 0 \\ 0 & \text{if } u = 0 \end{cases}$$

is partially σ-sufficient in the image experiment of U.

Example 2. Consider the statistical experiment $(\mathbb{R}^n \times \mathbb{R}^+, \mathcal{R}^n \times \mathcal{R}^+, \{P_\theta \otimes \mathcal{L}_\phi : \theta, \phi > 0\})$, where $P_\theta = N_n(0, \theta^2 \mathbb{I}_n)$, $\mathbb{I}_n$ is the indentity matrix of order n, and $\mathcal{L}_\phi(x,\cdot)$ is the probability distribution $\chi_1^2\left(\|x\|^2\right)^{h_{\phi^2}}$ of the statistic $h_{\phi^2} : x \in \mathbb{R}^n \mapsto \phi^2 x$ with respect to the noncentral chi-squared distribution $\chi_1^2\left(\|x\|^2\right)$ with 1 degree of freedom and noncentrality parameter $\|x\|^2$. Let $G := \{g_\Lambda : \Lambda \in O_n\}$, where O_n denotes the group of orthogonal transformations on $\mathbb{R}^n$ and $g_\Lambda(x,y) := (\Lambda x, y)$, for $(x,y) \in \mathbb{R}^n \times \mathbb{R}^+$ and $\Lambda \in O_n$. Since the stochastic kernels $\mathcal{L}_\phi$ are invariant, G leaves invariant every family $\mathcal{P}_\phi$.

The first projection $S(x,y) := x$ is a partially θ-sufficient, equivariant and stable statistic onto $\mathbb{R}^n$, $U_S : x \in \mathbb{R}^n \mapsto \|x\|^2$ is a maximal G^S-invariant statistic, and $U : (x,y) \in \mathbb{R}^n \times \mathbb{R}^+ \mapsto (\|x\|_2^2, y)$ is a maximal G-invariant statistic. It follows from Theorem 6 that $S_U : (u,y) \in \mathbb{R}^+ \times \mathbb{R}^+ \mapsto u$ is a partially θ-sufficient statistic on the image experiment of U.

Since $\mathcal{B} := S^{-1}(\mathcal{R}^n) = \mathcal{R}^n \times \{\emptyset, \mathbb{R}^+\}$, it is a partially θ-sufficient and stable σ-field that satisfies A(ii). We also have that $\mathcal{A}_I = U^{-1}(\mathbb{R}^+ \times \mathbb{R}^+)$. Then, it follows from Corollary 4 that $\mathcal{B} \cap \mathcal{A}_I$ is partially θ-sufficient for $\mathcal{A}_I$. Moreover, it can be easily checked that every probability measure $P_\theta \otimes \mathcal{L}_\phi$ is G-invariant; Theorem 7 shows that $\mathcal{B} \cap \mathcal{A}_I$ is a partially θ-sufficient σ-field for the whole experiment.

Example 3. Consider the statistical experiment

$$\left((\mathbb{R}^2)^n, (\mathcal{R}^2)^n, \left\{ N_2\left(0, \begin{bmatrix} \theta^2 & \psi \\ \psi & \xi^2 \end{bmatrix}\right)^n : \theta, \xi > 0, \psi \in \mathbb{R} \right\} \right)$$

corresponding to a size n sample of a bivariate normal distribution with mean 0 and unknown covariance matrix. Setting $\beta = \psi/\theta^2$ and $\sigma^2 = \xi^2 - \psi^2/\theta^2$, the statistical experiment can be written in the form

$$\left((\mathbb{R}^2)^n, (\mathcal{R}^2)^n, \left\{ N_2 \left(0, \begin{bmatrix} \theta^2 & \beta\theta^2 \\ \beta\theta^2 & \sigma^2 + \beta^2\theta^2 \end{bmatrix} \right)^n : \theta, \sigma > 0, \beta \in \mathbb{R} \right\} \right).$$

We write the points in $(\mathbb{R}^2)^n$ and the identity map on $(\mathbb{R}^2)^n$, respectively, as

$$(x, y) = \begin{pmatrix} x_1 & y_1 \\ \vdots & \vdots \\ x_n & y_n \end{pmatrix}, \qquad (X, Y) = \begin{pmatrix} X_1 & Y_1 \\ \vdots & \vdots \\ X_n & Y_n \end{pmatrix}.$$

We also write $X = (X_1, \ldots, X_n)^t$ and $Y = (Y_1, \ldots, Y_n)^t$. So the marginal distribution of X and the conditional distribution of Y given $X = x$ with respect to

$$N_2 \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \theta^2 & \beta\theta^2 \\ \beta\theta^2 & \sigma^2 + \beta^2\theta^2 \end{pmatrix} \right)^n$$

are $P_\theta := N_n\left(0, \theta^2 \mathbb{I}_n\right)$ and $\mathcal{L}_{(\beta,\sigma)}(x, \cdot) := N_n\left(\beta x, \sigma^2 \mathbb{I}_n\right)$, respectively. Writing $\phi = (\beta, \sigma^2)$, the probability measures of the family above can be disintegrated as $P_\theta \otimes \mathcal{L}_\phi$.

Let $G := \{g_\Lambda : \Lambda \in O_n\}$, where O_n is the group of orthogonal matrices of order n and $g_\Lambda(x, y) := (\Lambda x, \Lambda y)$, for $(x, y) \in (\mathbb{R}^2)^n$ and $\Lambda \in O_n$. For $x \in \mathbb{R}^n$ and $A \in \mathcal{R}^n$, we have that $[\mathcal{L}_\phi(x, A)]^{g_\Lambda} = \mathcal{L}_\phi\left(\Lambda^t x, \Lambda^t A\right) = \left[N_n\left(\beta\Lambda^t x, \sigma^2 \mathbb{I}_n\right)\right]^\Lambda (A) = N_n\left(\beta x, \sigma^2 \mathbb{I}_n\right)(A) = \mathcal{L}_\phi\left(x, A\right)$. So, G leaves invariant each family $\mathcal{P}_\phi$.

The map S which assigns to a $n \times 2$ matrix its first column is a partially θ-sufficient, equivariant and stable statistic onto $\mathbb{R}^n$. The statistic U which assigns to the $n \times 2$ matrix (x, y) the symmetric and nonnegative definite matrix

$$U(x, y) = \begin{pmatrix} \|x\|^2 & x'y \\ x'y & \|y\|^2 \end{pmatrix},$$

is maximal G-invariant, and $U_S : x \longmapsto \|x\|^2$ is a maximal G^S-invariant statistic on the image experiment of S. The rest of the assumptions of Theorem 6 also are satisfied. Hence it follows from this theorem that the statistic S_U defined by

$$\begin{pmatrix} U_{11} & U_{12} \\ U_{12} & U_{22} \end{pmatrix} \overset{S_U}{\longmapsto} U_{11}$$

is partially θ-sufficient.

Acknowledgement

The authors has been partially supported by the Spanish Ministerio de Cienciay Tecnología under the project BFM2002-01217. This work was supported by the Junta de Extremadura (Spain) under the project IPR99A016.

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(Received April 2002; accepted February 2003)