Supplementary Material for "Estimation of smoothness of a stationary Gaussian random field"

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1. Proofs of Theorems in Section 2 of the Main Paper

Throughout the document, we let $(\Omega, \mathcal{F}, P)$ be the probability space where a stationary Gaussian random field $Z(\boldsymbol{s})$ is defined. To self-contained, we state theorems and corollary again in this supplementary material.

Proof of Theorem 2.2 . The proof of the consistency of θ_m is similar with Wu, Lim and Xiao (2013). So we only show (2.10) of the main paper.

We have

$$0 = \frac{d}{d\theta} R_m(c^*, \theta)\Big|_{\theta=\theta_m} = -\log(m) + \frac{\dot{g}_{c^*,\theta_m}(2\pi \boldsymbol{J}/m)}{g_{c^*,\theta_m}(2\pi \boldsymbol{J}/m)} + \log(m)\frac{1}{m^{d-\theta_m}}\frac{\hat{I}_m^{\tau}(2\pi \boldsymbol{J}/m)}{g_{c^*,\theta_m}(2\pi \boldsymbol{J}/m)} - \frac{1}{m^{d-\theta_m}}\frac{\hat{I}_m^{\tau}(2\pi \boldsymbol{J}/m)}{g^2_{c^*,\theta_m}(2\pi \boldsymbol{J}/m)}\dot{g}_{c^*,\theta_m}(2\pi \boldsymbol{J}/m),$$

where $\dot{g_{c,\theta}} = dg_{c,\theta}/d\theta$. We can rewrite $\frac{d}{d\theta}R_m(c^*,\theta)\big|_{\theta=\theta_m}$ as

$$\frac{d}{d\theta} R_m(c^*, \theta)\Big|_{\theta=\theta_m} = -\log(m) + \frac{\dot{g}_{1,\theta_m}(2\pi \boldsymbol{J}/m)}{g_{1,\theta_m}(2\pi \boldsymbol{J}/m)} + \log(m)\frac{c_0}{c^*}m^{\theta_m-\theta_0}\frac{1}{m^{d-\theta_0}}\frac{\hat{I}_m^{\tau}(2\pi \boldsymbol{J}/m)}{g_{c_0,\theta_m}(2\pi \boldsymbol{J}/m)} - \frac{c_0}{c^*}m^{\theta_m-\theta_0}\frac{1}{m^{d-\theta_0}}\frac{\hat{I}_m^{\tau}(2\pi \boldsymbol{J}/m)}{g_{c_0,\theta_m}(2\pi \boldsymbol{J}/m)}\frac{\dot{g}_{1,\theta_m}(2\pi \boldsymbol{J}/m)}{g_{1,\theta_m}(2\pi \boldsymbol{J}/m)}.$$

Note that we have $\left(1/m^{d-\theta_0}\right)\left(\hat{I}_m^{\tau}(2\pi \boldsymbol{J}/m)/g_{c_0,\theta_m}(2\pi \boldsymbol{J}/m)\right) \xrightarrow{p} 1$ since $\left(1/m^{d-\theta_0}\right)\left(\hat{I}_m^{\tau}(2\pi \boldsymbol{J}/m)/g_{c_0,\theta_0}(2\pi \boldsymbol{J}/m)\right) \xrightarrow{p} 1$ by Lim and Stein (2008) and θ_m is consistent. Thus, from the continuity of g, for any $0 < \epsilon < 1$, there exists a positive integer M_ϵ independent of the value of c^* such that

$$P\left\{\left|\frac{1}{m^{d-\theta_0}}\frac{\hat{I}_m^{\tau}(2\pi \boldsymbol{J}/m)}{g_{c_0,\theta_m}(2\pi \boldsymbol{J}/m)} - 1\right| < \epsilon\right\} \geq 1-\epsilon$$

for all $m > M_\epsilon$. Note that

$$\left\{\left|\frac{1}{m^{d-\theta_0}}\frac{\hat{I}_m^{\tau}(2\pi \boldsymbol{J}/m)}{g_{c_0,\theta_m}(2\pi \boldsymbol{J}/m)} - 1\right| < \epsilon\right\} = \left\{1-\epsilon < \frac{1}{m^{d-\theta_0}}\frac{\hat{I}_m^{\tau}(2\pi \boldsymbol{J}/m)}{g_{c_0,\theta_m}(2\pi \boldsymbol{J}/m)} < 1+\epsilon\right\}.$$

Since $\dot{g}(2\pi \boldsymbol{J}/m) < 0$ [Wu (2011)], we have the following inequalities by replacing $\left(1/m^{d-\theta_0}\right)\left(\hat{I}_m^{\tau}(2\pi \boldsymbol{J}/m)/g_{c_0,\theta_m}(2\pi \boldsymbol{J}/m)\right)$ with $1-\epsilon$ and $1+\epsilon$, respectively, in the expression of $\frac{d}{d\theta}R_m(c^*,\theta)\big|_{\theta=\theta_m}$:

$$-\log(m) + \frac{\dot{g}_{1,\theta_m}(2\pi \boldsymbol{J}/m)}{g_{1,\theta_m}(2\pi \boldsymbol{J}/m)} + \log(m)\frac{c_0}{c^*}m^{\theta_m-\theta_0}(1-\epsilon) - \frac{c_0}{c^*}m^{\theta_m-\theta_0}\frac{\dot{g}_{1,\theta_m}(2\pi \boldsymbol{J}/m)}{g_{1,\theta_m}(2\pi \boldsymbol{J}/m)}(1-\epsilon) < \frac{d}{d\theta}R_m(c,\theta)\Big|_{\theta=\theta_m} = 0 \quad (1.1)$$

$$-\log(m) + \frac{\dot{g}_{1,\theta_m}(2\pi \boldsymbol{J}/m)}{g_{1,\theta_m}(2\pi \boldsymbol{J}/m)} + \log(m)\frac{c_0}{c^*}m^{\theta_m-\theta_0}(1+\epsilon) - \frac{c_0}{c^*}m^{\theta_m-\theta_0}\frac{\dot{g}_{1,\theta_m}(2\pi \boldsymbol{J}/m)}{g_{1,\theta_m}(2\pi \boldsymbol{J}/m)}(1+\epsilon) > \frac{d}{d\theta}R_m(c,\theta)\Big|_{\theta=\theta_m} = 0. \quad (1.2)$$

(1.1) and (1.2) can be rewritten as

$$\left(\frac{\dot{g}_{1,\theta_m}(2\pi\boldsymbol{J}/m)}{g_{1,\theta_m}(2\pi\boldsymbol{J}/m)} - \log(m)\right)\left(1 - \frac{c_0}{c^*}m^{\theta_m-\theta_0}(1-\epsilon)\right) < 0. \tag{1.3}$$

$$\left(\frac{\dot{g}_{1,\theta_m}(2\pi\boldsymbol{J}/m)}{g_{1,\theta_m}(2\pi\boldsymbol{J}/m)} - \log(m)\right)\left(1 - \frac{c_0}{c^*}m^{\theta_m-\theta_0}(1+\epsilon)\right) > 0. \tag{1.4}$$

From (1.3) and (1.4), we have

$$\frac{c_0}{c^*}m^{\theta_m-\theta_0}(1-\epsilon) < 1 \quad \text{and} \quad \frac{c_0}{c^*}m^{\theta_m-\theta_0}(1+\epsilon) > 1,$$

since $\left(\dot{g}_{1,\theta_m}(2\pi\boldsymbol{J}/m)/g_{1,\theta_m}(2\pi\boldsymbol{J}/m) - \log(m)\right) < 0$ for large enough m due to the boundedness of g and $\dot{g}$ shown in Wu, Lim and Xiao (2013). By taking the logarithm on both sides of the above inequalities, we obtain (2.10) of the main paper. □

Proof of Corollary 2.1. First, it can be easily shown that when $\max\{0,(d-2)/d\} < \gamma < \frac{d-1}{d}$,

$$\frac{1}{m^{d-\theta_0}}\frac{\hat{I}_m^\tau(2\pi\boldsymbol{J}/m)}{g_{c_0,\theta_0}(2\pi\boldsymbol{J}/m)} \to 1 \quad a.e. \tag{1.5}$$

by the Borel-Cantelli lemma and the Chebyshev's inequality with

$$Var\left(\frac{1}{m^{d-\theta_0}}\frac{\hat{I}_m^\tau(2\pi\boldsymbol{J}/m)}{g_{c_0,\theta_0}(2\pi\boldsymbol{J}/m)}\right) \sim m^{-2\eta}. \tag{1.6}$$

Then, the strong consistency of θ_m can be shown in a similar way as the proof of Theorem 3 in Wu, Lim and Xiao (2013) by using (1.5).

□

Proof of Theorem 2.4. Only the proof of case (i) ($c^* > c_0$) is presented, and that of case (ii) can be shown similarly.

Suppose that the result (i) does not hold, that is, there exists $\delta > 0$ and $M_1 > 0$ such that

$$P(\theta_0 \ > \ \theta_m) > \delta$$

for $m > M_1$. By the consistency of a smoothed periodogram (Lim and Stein (2008)), we have

$$d_m := \frac{\hat{I}_m^\delta(2\pi\boldsymbol{J}/m)}{m^{d-\theta_0} g_{c^*,\theta_0}(2\pi\boldsymbol{J}/m)} \xrightarrow{p} c_0/c^*.$$

Then, there exists a subsequence of $\{m\}$, $\{m_k\}$, such that d_{m_k} converges to c_0/c^* almost surely. By the Egorov's Theorem (Folland, (1999)), there exists $\mathcal{G}_\delta \subset \Omega$ such that d_{m_k} converges to c_0/c^* uniformly on $\mathcal{G}_\delta$ and $P(\mathcal{G}_\delta) > 1 - \delta/2$. Since $c_0/c^* < 1$, there exists M_2 such that $d_{m_k} < 1$ on $\mathcal{G}_\delta$ for $m_k > M_2$ by the uniform convergence.

Let $\Omega_{m_k} = \{\omega \in \Omega : \theta_0 > \theta_{m_k}\}$. On Ω_{m_k}, we have $g_{c^*,\theta_0}(2\pi\boldsymbol{J}/m_k) < g_{c^*,\theta_{m_k}}(2\pi\boldsymbol{J}/m_k)$ since $\dot{g}(2\pi\boldsymbol{J}/m_k) < 0$. Then, we have

$$m_k^{\theta_{m_k}-\theta_0} \frac{g_{c^*,\theta_0}(2\pi\boldsymbol{J}/m_k)}{g_{c^*,\theta_{m_k}}(2\pi\boldsymbol{J}/m_k)} < 1 \text{ on } \Omega_{m_k}.$$

Thus, we can show $R_{m_k}(c^*,\theta_{m_k}) - R_{m_k}(c^*,\theta_0) > 0$ on $\Omega_{m_k} \bigcap \mathcal{G}_\delta$ using the Lemma 2 of Wu, Lim and Xiao (2013). Note that, by construction, $P(\Omega_{m_k} \bigcap \mathcal{G}_\delta) > \delta/2 > 0$ since $P(\Omega_{m_k}) > \delta$ and $P(\mathcal{G}_\delta) > 1 - \delta/2$ for $m_k > M = \max\{M_1, M_2\}$ which contradicts to the fact that θ_{m_k} is the minimizer of $R_{m_k}(c^*,\theta)$. □