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STRONG LAWS FOR WEIGHTED SUMS OF RANDOM ELEMENTS

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Abstract: Strong laws of large numbers for weighted sums of independent random variables are proved for Banach spaces of type p. Chen, Zhu and Fang's (1996) results on real-valued i.i.d. random variables are extended and generalized, and their unsolved problem is answered. Also, necessary conditions for these strong convergences are considered.

Key words and phrases: Banach space of type p, random element, strong law, weighted sum.

1. Introduction

Throughout this paper, $a_i \neq 0$ for $i \geq 1$ and $0 < A_1 \leq A_2 \leq \cdots \to \infty$. Write $u_i = a_i/A_i$ and let $([n,1],[n,2],\ldots,[n,n])$ be a permutation of $(1,\ldots,n)$ such that

$$|u_{[n,1]}| \geq \cdots \geq |u_{[n,n]}|,\ \ [n,i] < [n,j] \ \text{ if } \ i<j \ \text{ and } \ |u_i| = |u_j|.$$

Let $I(\cdot)$ be the indicator function and define

$$V_{n,j} = A_n^{-1} \sum_{i=1}^{n} a_i I(|u_i| \geq |u_{[n,j]}|) \ \text{ for } \ 1 \leq j \leq n \ \text{ and } \ V_n = \max_{1 \leq j \leq n} |V_{n,j}|.$$

Put $N(x) =: \#\{i : A_i/|a_i| \leq x\}$. In 1996, Chen, Zhu and Fang improved the result of Jamison, Orey and Pruitt (1965) as follows:

Theorem A. *Suppose that $e_1, e_2, \ldots$ are i.i.d. real-valued random variables with $Ee_1 = 0$. If $N(n) = O(n)$ and $V_n = O(1)$, then*

$$A_n^{-1} \sum_{i=1}^{n} a_i e_i \to 0 \quad a.s. \tag{1.1}$$

Conversely, if at least one of $N(n) = O(n)$ and $V_n = O(1)$ is not true, then there exists an i.i.d. real-valued sequence $\{e_i\}$ with $Ee_1 = 0$ such that (1.1) *does not hold.*

Theorem B. *Suppose that* $e_1, e_2, \ldots$ *are i.i.d. real-valued random variables with* $Ee_1 = 0$ *and* $E|e_1|^t < \infty$ *for some* $1 < t < 2$*. Then* (1.1) *holds when* $N(n) = O(n^t)$*. Conversely, if* $N(n)$ *is not* $O(n^t)$*, then there exists an i.i.d. real-valued sequence* $\{e_i\}$ *with* $Ee_1 = 0$ *and* $E|e_1|^t < \infty$ *such that* (1.1) *is false.*

Remark 1.1. If $a_1, a_2, \ldots$ are positive constants and $A_n = \sum_{i=1}^n a_i \to \infty$ as $n \to \infty$, then Theorem A reduces to the result of Jamison, Orey and Pruitt (1965).

Remark 1.2. Chen, Zhu and Fang (1996) gave an example to show that Theorem B can't be extended to $t \geq 2$, but they had not given result for this case.

In this paper, our aim is to extend and generalize Theorems A and B to B-valued independent random elements, and to give the result suggested by Remark 1.2. Also, necessary conditions for strong convergences for i.i.d. random variables are investigated.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and B a real separable Banach space with norm $\|\cdot\|$. The Banach space B is called type p $(1 \leq p \leq 2)$ if there exists a $C = C_p > 0$ such that

$$E\|\sum_{i=1}^n X_i\|^p \leq C\sum_{i=1}^n E\|X_i\|^p, \quad n \geq 1,$$

whenever independent B-valued random variables $X_1, \ldots, X_n$ have mean zero and $E\|X_i\|^p < \infty, i = 1, \ldots, n$.

Let $\{X_n, n \geq 1\}$ be a sequence of independent, mean zero, B-valued random elements, $0 < A_i \to \infty$. Under a type p Banach space, the Weak Law of Large Numbers (WLLN) $\sum_{i=1}^n a_i X_i / A_n \xrightarrow{P} 0$ was studied by Adler, Rosalsky and Taylor (1991), but the corresponding Strong Law of Large Numbers (SLLN) $\sum_{i=1}^n a_i X_i / A_n \xrightarrow{a.s.} 0$ does not necessarily hold. For $a_i > 0, i \geq 1$, Howell, Taylor and Woyczynski (1981) investigated the SLLN. In 1992, Fazekas studied rates of convergence in the LLN for B-valued weakly mean-dominated arrays with very general weights, and Fazekas (1985) discussed convergence rates in the Marcinkiewicz and Chung type SLLN for B-valued independent random variables with multidimensional indices. Bozorgnia, Patterson and Taylor (1997) obtained the Chung type SLLN for arrays of rowwise independent random elements. Mikosch and Norvaisa (1987) proved the equivalence between the WLLN and the SLLN.

In the sequel, let $h(x) > 0$ be a slowly varying function as $x \to \infty$; C denotes a finite positive constant which may be different in various places; $\{X_n\} \prec X$ means $\sup_n P(\|X_n\| > x) \leq CP(|X| > x)$, where $x > 0$ and X is some real valued random variable.

Results are stated in Section 2 and proofs are given in Section 3.

2. Main Results

In this section, $\uparrow$ and $\downarrow$ denote non-decreasing and non-increasing, respectively.

Theorem 2.1. *Let B be of type p for some $1 < p \le 2$, and $h(x) \uparrow$ as $x \to \infty$. Suppose that*

$$N(n) = O(nh(n)), \tag{2.1}$$

$$V_n = O(1), \ n \ge 1. \tag{2.2}$$

For each sequence $\{X_n, n \ge 1\}$ of i.i.d. B-valued random elements, if

$$EX_1 = 0 \quad \text{when} \quad \limsup_{n\to\infty} A_n^{-1} |\sum_{i=1}^{n} a_i| > 0 \tag{2.3}$$

and $E\|X_1\|h(\|X_1\|) < \infty$, then

$$A_n^{-1} \sum_{i=1}^{n} a_i X_i \to 0 \quad a.s. \tag{2.4}$$

Conversely, if at least one of (2.1) *and* (2.2) *is not true, then there exists an i.i.d. real-valued sequence $\{X_i\}$ which satisfies $E|X_1|h(|X_1|) < \infty$ and* (2.3) *such that* (2.4) *does not hold.*

Theorem 2.2. *Let $1 < t < 2$, and let B be of type p for some $t < p \le 2$. Suppose that*

$$N(n) = O(n^t h(n)), \ n \ge 1. \tag{2.5}$$

For each sequence $\{X_n\}$ of B-valued independent random elements with $\{X_n\} \prec X$, if (2.3) *is satisfied and $E|X|^t h(|X|) < \infty$, then* (2.4) *holds. Conversely, if* (2.5) *is not true, then there exists an i.i.d. real-valued sequence $\{X_i\}$ which satisfies $E|X_1|^t h(|X_1|) < \infty$ and* (2.3), *but* (2.4) *does not hold.*

Theorem 2.3. *Suppose that $0 \le t < 1$ and that $h(x) \uparrow \infty$ as $x \to \infty$ when $t = 0$. Let $\{X_n\}$ be any sequence of B-valued random elements with $\{X_n\} \prec X$. If* (2.5) *is satisfied and $E|X|^t h(|X|) < \infty$, then* (2.4) *holds.*

Remark 2.1. Because the real space is a Banach space of type 2, taking $h(x) = 1$, Theorems 2.1 and 2.2 extend Theorems A and B to the B-valued setting, respectively. Naturally, the result of Jamison, Orey and Pruitt (1965) is also extended.

Remark 2.2. Let B be of type p for some $1 \le p \le 2$, and let $\{X_n\}$ be a sequence of B-valued independent random elements with $\{X_n\} \prec X$. Under the condition $EN(|X|) < \infty$ and

$$\int_0^\infty t^{p-1} P(|X| > t) \int_t^\infty \frac{N(y)}{y^{p+1}} dy dt < \infty,$$

Howell, Taylor and Woyczynski (1981) proved that for $a_i > 0, i \ge 1$, there exist $c_n \in B, n = 1, \ldots$, such that

$$A_n^{-1} \sum_{i=1}^n a_i X_i - c_n \to 0 \quad a.s.$$

The conditon $EN(|X|) < \infty$ seems superfluous in the above result (as mentioned by Jamison, Orey and Pruitt (1965), without proof.). In fact, for any $t > 0$,

$$\begin{aligned}
&\int_t^\infty \frac{N(y)}{y^{p+1}} dy \\
&= \frac{1}{p} \lim_{A \to \infty} [-\int_t^A N(y) dy^{-p}] = \frac{1}{p} \lim_{A \to \infty} [-N(A)A^{-p} + N(t)t^{-p} + \int_t^A y^{-p} dN(y)] \\
&\ge \frac{1}{p} \lim_{A \to \infty} \sup[-N(A)A^{-p} + N(t)t^{-p} + A^{-p}N(A) - A^{-p}N(t)] = \frac{1}{p} N(t)t^{-p}.
\end{aligned}$$

Hence

$$\begin{aligned}
EN(|X|) &= \int_0^\infty t^p t^{-p} N(t) dP(|X| \le t) \le p \int_0^\infty t^p \int_t^\infty \frac{N(y)}{y^{p+1}} dy dP(|X| \le t) \\
&= p \int_0^\infty \frac{N(y)}{y^{p+1}} dy [-t^p P(|X| > t)|_0^y + p \int_0^y t^{p-1} P(|X| > t) dt] \\
&\le p^2 \int_0^\infty t^{p-1} P(|X| > t) \int_t^\infty \frac{N(y)}{y^{p+1}} dy dt < \infty.
\end{aligned}$$

Theorem 2.4. *Suppose that $0 \le t < 2$ and that $h(x) \uparrow$ when $t = 1, h(x) \uparrow \infty$ as $x \to \infty$ when $t = 0$. Assume that*

$$n^t h(n) = O(N(n)) \tag{2.6}$$

and that for i.i.d. B-valued sequence $\{X_i\}$, (2.4) is satisfied. Then $E\|X_1\|^t h(\|X_1\|) < \infty$. Furthermore, if B is of type p for some $1 \le t < p \le 2$, $N(n) = O(n^t h(n))$ and $V_n = O(1)$ when $t = 1$, then for $1 \le t < 2$, (2.3) holds.

For $t \ge 2$, we have

Theorem 2.5. *Let $\{X_i\}$ be a sequence of real valued independent random variables with $\{X_n\} \prec X$. Suppose that for $t \ge 2$, (2.5) is satisfied. If (2.3) holds* *and* $E|X|^t h(|X|) < \infty$, *then, without changing the distribution of* $\{X_n\}$, *we can redefine* $\{X_n\}$ *on a richer probability space, together with a sequence of independent normal random variables* $\{Y_n,\ n \geq 1\}$ *with* $Y_n \stackrel{\mathcal{D}}{=} N(0,\ VarX_1^2 I(|X_1| \leq A_n/|a_n|))$, *such that*

$$\sum_{i=1}^{n} a_i X_i - \sum_{i=1}^{n} a_i Y_i = o(A_n) \quad a.s. \tag{2.7}$$

3. Proof of Main Result

By the Three Series Theorem of B-valued independent random elements (cf. Wu and Wang (1990, p.154)), it is easy to obtain

Lemma 1. *Let* $\{X_i, i \geq 1\}$ *be a sequence of independent random variables in a Banach space of type p for some* $1 < p \leq 2$ *and* $EX_n = 0$. *Suppose that* $\Psi(t)$ *is a positive, even and continuous function such that* $\frac{\Psi(|t|)}{|t|} \uparrow$ and $\frac{\Psi(|t|)}{|t|^p} \downarrow$ as $|t| \to \infty$. *If* $\sum_{n=1}^{\infty} \frac{E\Psi(\|X_n\|)}{\Psi(A_n)} < \infty$, *then* $A_n^{-1} \sum_{i=1}^{n} X_i \to 0, a.s.$

The proofs of the converse parts in Theorems 2.1 and 2.2 are similar to those in Jamison, Orey and Pruitt (1965) and Chen, Zhu and Fang (1996).

Proof of Theorem 2.1. Let $Y_i = X_i I(\|X_i\| \leq A_i/|a_i|),\ Z_i = X_i - Y_i,\ U_n = \sum_{i=1}^{n} a_i Y_i,\ V_n = \sum_{i=1}^{n} a_i Z_i$. Then

$$A_n^{-1} \sum_{i=1}^{n} a_i X_i = A_n^{-1} U_n + A_n^{-1} V_n. \tag{3.1}$$

From (2.1) we get $A_i/|a_i| \to \infty$ as $i \to \infty$, and (2.1) and $E\|X_1\| h(\|X_1\|) < \infty$ imply

$$\sum_{i=1}^{\infty} P(\|X_i\| > A_i/|a_i|) < \infty. \tag{3.2}$$

By the Borel-Cantelli Lemma, $\sum_{i=1}^{\infty} A_i^{-1} \|a_i Z_i\| < \infty, a.s.$, which implies

$$A_n^{-1} V_n \to 0 \quad a.s. \tag{3.3}$$

Next, we prove

$$A_n^{-1} \|EU_n\| \to 0 \ \text{ as } \ n \to \infty. \tag{3.4}$$

Note that

$$A_n^{-1} \|EU_n\| \leq A_n^{-1} \| \sum_{i=1}^{n} a_i EX_i \| + A_n^{-1} \| \sum_{i=1}^{n} a_i EX_i I(\|X_i\| > A_i/|a_i|) \| =: I_n + II_n. \tag{3.5}$$

When $\lim_{n\to\infty} A_n^{-1}|\sum_{i=1}^n a_i| = 0$, we get $I_n \to 0$ from $E\|X_1\| < \infty$.
Note that when $\limsup_{n\to\infty} A_n^{-1}|\sum_{i=1}^n a_i| > 0$, $EX_n = 0$. Therefore, to prove (3.4), we need only prove that $II_n \to 0$. Let $E_{n,j} = (\frac{1}{|u_{[n,j]}|}, \frac{1}{|u_{[n,j+1]}|}]$, $E_{n,n} = (\frac{1}{|u_{[n,n]}|}, \infty)$.

$$\begin{aligned}
II_n &= A_n^{-1}\|\sum_{i=1}^n a_{[n,i]} EX_1 I(\|X_1\| > |u_{[n,i]}|^{-1})\| \\
&= A_n^{-1}\|\sum_{i=1}^n a_{[n,i]} \sum_{j=i}^n EX_1 I(\|X_1\| \in E_{n,j})\| \\
&= A_n^{-1}\|\sum_{j=1}^n EX_1 I(\|X_1\| \in E_{n,j}) \sum_{i=1}^j a_{[n,i]}\| \\
&= A_n^{-1}\|\sum_{j=1}^n EX_1 I(\|X_1\| \in E_{n,j}) V_{n,j}\| \le (\sum_{j=1}^{h-1} + \sum_{j=h}^n)|V_{n,j}|E\|X_1\|I(\|X_1\| \in E_{n,j}) \\
&=: III_n + IIII_n,
\end{aligned}$$

where h is a fixed integer with $2 \le h \le n$. Set $u^* = \max_{i\ge 1}\{|u_i|\}$, note that $|V_{n,j}| = A_n^{-1}|\sum_{i=1}^j a_{[n,i]}| \le ju^* A_{\max_{1\le i\le n}[n,i]}/A_n$. Since $a_i/A_i \to 0$ as $i \to \infty$, there exists a positive integer H (depending on h but not depending on n) such that $\max_{1\le i<h}[n,i] \le H$ and $III_n \le C\sum_{j=1}^{h-1}|V_{n,j}| \le Cu^*A_H(\sum_{j=1}^{h-1} j)/A_n \le Cu^*A_H h^2/A_n \to 0$ as $n \to \infty$. It is easy to see from (2.2) that $IIII_n \le CE\|X_1\|I(\|X_1\| > \frac{1}{|u_{[n,h]}|})$. Since $u_i \to 0$, $\forall\epsilon > 0, |u_{[n,h]}| < \epsilon$ by taking h large enough and $n \ge h$. Furthermore we get $IIII_n < \epsilon$ from $E\|X_1\| < \infty$. This proves (3.4). Thus, to prove $A_n^{-1}U_n \to 0, a.s.$, it suffices to show from Lemma 1 that $J_1 = \sum_{k=1}^\infty A_k^{-p}|a_k|^p E\|X_k\|^p I(\|X_k\| \le A_k/|a_k|) < \infty$. In fact, by (2.1),

$$\begin{aligned}
J_1 &\le C\sum_{j=1}^\infty \sum_{j-1<A_k/|a_k|\le j} j^{-p}E\|X_1\|^p I(\|X_1\| \le j) \\
&= C\sum_{j=1}^\infty [N(j) - N(j-1)]j^{-p}\sum_{n=1}^j E\|X_1\|^p I(n-1 < \|X_1\| \le n) \\
&\le CE\|X_1\|^t h(\|X_1\|) < \infty.
\end{aligned}$$

Proof of Theorem 2.2. Let $Y_i,\ Z_i,\ U_n,\ V_n$ be as in Theorem 2.1. By Theorem 2.1, it suffices to show that

$$A_n^{-1}\|EU_n\| \to 0, \text{ as } n \to \infty. \tag{3.6}$$

From (2.5) and $E|X|^t h(|X|) < \infty$ we get

$$\sum_{i=1}^\infty A_i^{-1}|a_i|E\|X_i\|I(\|X_i\| > A_i/|a_i|) < \infty. \tag{3.7}$$

By the property of $h(x)$, when $\lim_{n\to\infty} A_n^{-1}|\sum_{i=1}^n a_i| = 0$, $E|X|^t h(|X|) < \infty$ implies

$$A_n^{-1}|\sum_{i=1}^n a_i|E\|X_1\| \to 0, \text{ as } n \to \infty. \tag{3.8}$$

Note that when $\limsup_{n\to\infty} A_n^{-1}|\sum_{i=1}^n a_i| > 0$, $EX_n = 0$. Therefore, from (3.5), this verifies (3.6) by (3.7) and (3.8).

Proof of Theorem 2.3. From the assumption we get

$$\sum_{i=1}^{\infty} P(\|X_i\| > A_i/|a_i|) < \infty \quad \text{and} \quad \sum_{i=1}^{\infty} \frac{|a_i|E\|X_i\|I(\|X_i\| \le A_i/|a_i|)}{A_i} < \infty.$$

Hence $\sum_{i=1}^{\infty} \frac{\|a_i X_i\|}{A_i} < \infty, a.s.$ Therefore by the Kronecker Lemma, $A_n^{-1}\sum_{i=1}^n a_i X_i \to 0 \ \ a.s.$

Proof of Theorem 2.4. Note that $A_n^{-1}\sum_{i=1}^n a_i X_i \to 0$ $a.s.$ implies

$$\sum_{n=1}^{\infty} P(\|X_1\| \ge \epsilon A_n/|a_n|) < \infty \quad \text{for all } \epsilon > 0. \tag{3.9}$$

This and (2.6) imply that

$$E\|X_1\|^t h(\|X_1\|) < \infty. \tag{3.10}$$

Furthermore, if B is of type p $(1 \le t < p \le 2)$, $N(n) = O(n^t h(n))$ and $V_n = O(1)$ when $t = 1$. Then by Theorems 2.1 and 2.2, we obtain from (3.10) that

$$A_n^{-1}\sum_{i=1}^n a_i(X_i - EX_i) \to 0 \ \ a.s. \ \text{ as } \ n \to \infty. \tag{3.11}$$

From Assumption (2.4) and (3.11) we get

$$A_n^{-1}\sum_{i=1}^n a_i EX_i \to 0 \ \text{ as } \ n \to \infty. \tag{3.12}$$

If $\limsup_{n\to\infty} A_n^{-1}|\sum_{i=1}^n a_i| = a > 0$, there exists a sequence of positive integes $\{n_k\}$ such that $n_k \uparrow \infty$ as $k \to \infty$ and

$$\lim_{k\to\infty} A_{n_k}^{-1}|\sum_{i=1}^{n_k} a_i| = a > 0. \tag{3.13}$$

Obviously, (3.12) and (3.13) imply $EX_1 = 0$.

Proof of Theorem 2.5. Note that

$$\begin{aligned}\sum_{i=1}^{n} a_i X_i &= \sum_{i=1}^{n} a_i[X_i I(|X_i| \le A_i/|a_i|) - EX_i I(|X_i| \le A_i/|a_i|)] \\ &\quad + \sum_{i=1}^{n} a_i EX_i I(|X_i| \le A_i/|a_i|) + \sum_{i=1}^{n} a_i X_i I(|X_i| > A_i/|a_i|) \\ &=: I_n^{(1)} + I_n^{(2)} + I_n^{(3)}. \end{aligned} \tag{3.14}$$

From the proof of Theorem 2.2 we know

$$A_n^{-1} I_n^{(2)} \to 0 \quad \text{and} \quad A_n^{-1} I_n^{(3)} \xrightarrow{a.s.} 0 \text{ as } n \to \infty. \tag{3.15}$$

As in the proof of $J_1 < \infty$, for $p > t$ we have

$$\sum_{i=1}^{\infty} \frac{E|a_i[X_i I(\|X_i\| \le A_i/|a_i|) - EX_i I(\|X_i\| \le A_i/|a_i|)]|^p}{A_i^p} < \infty.$$

Hence, the sequences $\{A_i\}$ and $\{a_i[X_i I(\|X_i\| \le A_i/|a_i|) - EX_i I(\|X_i\| \le A_i/|a_i|)],\ i \ge 1\}$ satisfy the assumptions of Theorem 1.3 of Shao (1995). Therefore, without changing the distribution of $\{X_n\}$, we can redefine $\{X_n\}$ on a richer probability space, together with a sequence of independent normal random variables $\{Y_n,\ n \ge 1\}$ with $Y_n \stackrel{\mathcal{D}}{=} N(0,\ VarX_1^2 I(|X_1| \le A_n/|a_n|))$, such that

$$|I_n^{(1)} - \sum_{i=1}^{n} a_i Y_i| = o(A_n) \quad a.s. \tag{3.16}$$

This completes the proof of (2.7) by (3.14)-(3.16).

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