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CHARACTERIZATIONS OF THE EXPONENTIAL DISTRIBUTION VIA THE BLOCKING TIME IN A QUEUEING SYSTEM

Gwo Dong Lin

Academia Sinica

Abstract: We prove two characterization theorems for the exponential distribution, which are related to the blocking time in a queueing system with an unreliable server. We also point out that the expected blocking time is always finite even if the mean service time is infinite.

Key words and phrases: Characterization, exponential distribution, blocking time, Laplace-Stieltjes transform.

1. Introduction

Recently, Dimitrov and Khalil (1990) introduce a single-server queueing system with unreliable server and service repetition. In this system, if the service is interrupted, the repair time of the server's failure is assumed to be instantaneous and the service is restarted anew when the server is recovered. The blocking time, namely the total time taken by a customer, can be represented by

$$Z_\lambda = XI(X \le Y) + I(X > Y)(Y + Z_\lambda^*), \tag{1}$$

where I is the indicator function, X is the service time with free interruption, Y is the server failure time and is assumed to obey an exponential distribution $P(Y \le y) = 1 - e^{-\lambda y}$ for $y \ge 0$, Z_λ^* is distributed as Z_λ (denoted by $Z_\lambda^* \overset{\mathrm{d}}{=} Z_\lambda$), and X, Y and Z_λ^* are independent nonnegative random variables.

Using the blocking time Z_λ, Dimitrov and Khalil (1990) obtain two characterizations of the exponential distribution: (a) if $Z_\lambda \overset{\mathrm{d}}{=} X$ for each $\lambda > 0$, then X has an exponential distribution; (b) if $EZ_{\lambda_n} = c$ for each $n \ge 1$, where c is a fixed constant and $\{\lambda_n\}_{n=1}^\infty$ is a sequence of positive and distinct real numbers satisfying $\lim_{n\to\infty} \lambda_n < \infty$, then X has an exponential distribution. Hereafter, the case $X = 0$ (almost surely) is considered as a degenerate exponential random variable.

It is seen that the result (a) above follows immediately from the second result (b) because $EZ_\lambda < \infty$ for each $\lambda > 0$ (see Lemma 3 below). Surprisingly, Van Harn and Steutel (1991) improve the result (a) and prove that if $Z_\lambda \stackrel{d}{=} X$ for *some* $\lambda > 0$, then X has an exponential distribution. On the other hand, the proof of (b) given by Dimitrov and Khalil (1990) is in fact incomplete. In this paper we first improve Dimitrov-Khalil's second result (see Theorem 1 below) and then give another characterization of the exponential distribution via the coefficient of variation of Z_λ (Theorem 2).

2. Lemmas

To improve Dimitrov-Khalil's second result, we need the following lemmas. Let b_λ and g be the Laplace-Stieltjes transforms of Z_λ and X, respectively. Then based on the relation (1), Dimitrov and Khalil (1990) calculate b_λ in terms of g as follows.

Lemma 1.

$$b_\lambda(s) = \frac{(s+\lambda)g(s+\lambda)}{s+\lambda g(s+\lambda)} \quad \textit{for} \quad s \geq 0.$$

For a given $\lambda > 0$, Lemma 1 implies that

$$g(s) = \frac{(s-\lambda)b_\lambda(s-\lambda)}{s-\lambda b_\lambda(s-\lambda)} \quad \text{for all} \quad s > \lambda,$$

so that the distribution of X is uniquely determined by that of Z_λ. Therefore, there exists a one-to-one relation between the distribution of X and the distribution of Z_λ.

Applying the Monotone Convergence Theorem for general measure (see, e.g., Royden (1968, p.227)), we obtain the following Lemma 2, which is useful in calculating the nth moment of a nonnegative random variable; note that we do not exclude the possible case: $EZ^n = \infty$.

Lemma 2. *Let Z be a nonnegative random variable with Laplace-Stieltjes transform $b(s) = Ee^{-sZ}$ for $s \geq 0$. Then for each integer $n \geq 1$, $EZ^n = \lim_{s\to 0^+}(-1)^n$ $b^{(n)}(s)$.*

Define $h(\lambda) = \frac{1}{\lambda}(\frac{1}{g(\lambda)} - 1)$ for $\lambda > 0$. We now represent the first two moments of Z_λ in terms of the function h.

Lemma 3. *For each $\lambda > 0$, $EZ_\lambda = h(\lambda)$ and $EZ_\lambda^2 = 2\{h^2(\lambda) - h'(\lambda)\}$.*

Proof. Write $b_\lambda(s) = \{1 + sh(s+\lambda)\}^{-1}$ by Lemma 1. Then from Lemma 2 it follows that

$$EZ_\lambda = -\lim_{s\to 0^+} b'_\lambda(s) = h(\lambda)$$

and that

$$EZ_\lambda^2 = \lim_{s\to 0+} b''_\lambda(s) = 2\{h^2(\lambda) - h'(\lambda)\}.$$

Note that for each $\lambda > 0$, the expected blocking time EZ_λ is finite whatever the mean service time EX may be. Specifically, EZ_λ can be smaller than EX (the converse is also possible). For example, if X is the positive random variable with Laplace-Stieltjes transform $g(s) = 1/(1 + s^\alpha)$ for some $\alpha \in (0,1)$, then $EZ_\lambda = \lambda^{\alpha-1} < EX = \infty$ for each $\lambda > 0$. However, if $X = 1$ almost surely, then $EZ_\lambda = (e^\lambda - 1)/\lambda > EX = 1$ for each $\lambda > 0$. The following result is a ramification of the well known Müntz-Szász Theorem (see, e.g., Rudin (1987, pp.313-314)).

Lemma 4. *Let X_1 and X_2 be two nonnegative random variables with Laplace-Stieltjes transforms g_1 and g_2, respectively. Let $\{\lambda_n\}_{n=1}^\infty$ be a sequence of positive and distinct real numbers satisfying one of the following conditions:*
(a) $\lim_{n\to\infty} \lambda_n = \infty$ *and* $\sum_{n=1}^\infty 1/\lambda_n = \infty$;
(b) $\lim_{n\to\infty} \lambda_n = \lambda_0 \in (0,\infty)$; *and*
(c) $\lim_{n\to\infty} \lambda_n = 0$ *and* $\sum_{n=1}^\infty \lambda_n = \infty$.
Further, assume that $g_1(\lambda_n) = g_2(\lambda_n)$ for each $n \geq 1$. Then $X_1 \stackrel{d}{=} X_2$.

Proof. For each $i = 1, 2$, denote the distribution of X_i by F_i. Then

$$g_i(s) = Ee^{-sX_i} = \int_0^\infty e^{-sx} dF_i(x) \quad \text{for} \quad s \geq 0.$$

Define $g_i(z) = \int_0^\infty e^{-zx} dF_i(x)$ in the right-half plane $\Pi = \{z : Rez > 0\}$. Then g_i is bounded and analytic in Π. Set the function

$$h_i(z) = g_i\left(\frac{1+z}{1-z}\right) \quad \text{for} \quad z \in U = \{z : |z| < 1\};$$

then h_i is bounded and analytic in U and so is the function $h = h_1 - h_2$. Letting $\alpha_n = (\lambda_n - 1)/(\lambda_n + 1)$, we have $h(\alpha_n) = 0$ and $\sum_{n=1}^\infty (1 - |\alpha_n|) = \infty$. Therefore, $h(z) = 0$ in U (Rudin (1987, p.312)), or equivalently, $g_1(z) = g_2(z)$ for $z \in \Pi$. Particularly, $g_1(s) = g_2(s)$ for each $s \geq 0$ and hence $F_1 = F_2$. The proof is complete.

As shown below, the condition $\sum_{n=1}^\infty \lambda_n = \infty$ in the case (c) above is redundant if the distribution of X_1 is characterized by the sequence of its moments.

Lemma 5. *Let X_1, X_2, g_1 and g_2 be the same as those in Lemma 4. Let $\{\lambda_n\}_{n=1}^\infty$ be a sequence of positive real numbers converging to zero. Assume further that X_1 has finite moments of all orders, that the distribution of X_1 is uniquely determined by its moments $\{EX_1^n\}_{n=1}^\infty$, and that $g_1(\lambda_n) = g_2(\lambda_n)$ for each $n \geq 1$. Then $X_1 \stackrel{d}{=} X_2$.*

Proof. Since $g_1(0) = g_2(0) = 1$ and $g_1(\lambda_n) = g_2(\lambda_n)$ for each $n \geq 1$, there exists a $\lambda_n' \in (0, \lambda_n)$ (for each $n \geq 1$) such that $g_1'(\lambda_n') = g_2'(\lambda_n')$. Also, $\lim_{n\to\infty} \lambda_n' = 0$ because $\lim_{n\to\infty} \lambda_n = 0$, and hence, by Lemma 2,

$$EX_2 = -\lim_{n\to\infty} g_2'(\lambda_n') = -\lim_{n\to\infty} g_1'(\lambda_n') = EX_1.$$

Consequently, $g_2'(0) = g_1'(0)$. By induction, it can be proved that $EX_2^n = EX_1^n$ for each integer $n \geq 1$. Therefore $X_1 \stackrel{d}{=} X_2$ due to the assumption that the sequence of moments $\{EX_1^n\}_{n=1}^{\infty}$ uniquely determines the distribution of X_1. This completes the proof.

3. Characterization Theorems

We are ready to improve Dimitrov-Khalil's second result.

Theorem 1. *Let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of positive and distinct real numbers satisfying $\lim_{n\to\infty} \lambda_n = \lambda_0 \in [0, \infty]$ and let $\sum_{n=1}^{\infty} 1/\lambda_n = \infty$ if $\lambda_0 = \infty$. Assume further that for each $n \geq 1$, $EZ_{\lambda_n} = c$, where c is a fixed constant. Then X is exponentially distributed.*

Proof. It follows from Lemma 3 and the assumption $EZ_{\lambda_n} = c$ that

$$h(\lambda_n) = \frac{1}{\lambda_n}\left(\frac{1}{g(\lambda_n)} - 1\right) = c \quad \text{for each} \quad n \geq 1.$$

Therefore, $g(\lambda_n) = 1/(1 + c\lambda_n)$ for each $n \geq 1$. Since the function $g_*(\lambda) = 1/(1+c\lambda)$, $\lambda \geq 0$, is the Laplace-Stieltjes transform of an exponential distribution, which is characterized by its moments, the desired result follows from Lemmas 4 and 5. The proof is complete.

Finally, we prove the following characterization of the exponential distribution via the coefficient of variation of Z_λ, which is defined by $\nu_\lambda = \{EZ_\lambda^2 - (EZ_\lambda)^2\}^{1/2}/EZ_\lambda$. Note that $\nu_\lambda = 1$ if and only if $EZ_\lambda^2 = 2(EZ_\lambda)^2$.

Theorem 2. *Let $\lambda_0 > 0$ and assume that for each $\lambda \in (0, \lambda_0)$, the coefficient of variation of Z_λ is equal to one. Then X is exponentially distributed.*

Proof. Using Lemma 3 and the condition on the coefficient of variation of Z_λ, we have $h'(\lambda) = 0$ for each $\lambda \in (0, \lambda_0)$. Therefore, h is a constant function on the interval $(0, \lambda_0)$, say $h(\lambda) = c$, so that $g(\lambda) = 1/(1 + c\lambda)$ for $\lambda \in (0, \lambda_0)$. By Lemma 4, we conclude that X is exponentially distributed. This completes the proof.

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Institute of Statistical Science, Academia Sinica, Taipei 11529, Taiwan.

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