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BOOTSTRAPPING A SAMPLE QUANTILE WHEN THE DENSITY HAS A JUMP

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Abstract. Albeit the superiority of bootstrapping to jackknifing in estimating the (asymptotic) variance of a sample quantile in the regular case, the bootstrap may encounter technical problems in some non-regular cases. The related methodology for one such important non-regular case is considered here, and the theory is supplemented with numerical studies.

Key words and phrases: Bahadur representation, bootstrap, nonregular density, sample quantile.

1. Introduction

Let $X_1, \ldots, X_n$ be independent and identically distributed random variables with a continuous distribution function (d.f.) F defined on the real line, and let $X_{n:1} < \cdots < X_{n:n}$ be the order statistics associated with the X_i. The population p-quantile is defined by $F^{-1}(p)$, where

$$F^{-1}(u) = \inf\{x : F(x) \geq u\}, \quad u \in (0,1).$$

The sample p-quantile is similarly defined by $F_n^{-1}(p)$, where F_n is the usual sample (or empirical) d.f. For simplicity of presentation, we consider the case of sample median, $\tilde{X}_n = F_n^{-1}(1/2)$, but the results to follow would remain applicable to any p-quantile. Let θ be the population median. Then, in the so called *regular case*, one assumes that F admits a density function f such that $f(x)$ is continuous and positive at $x = \theta$. Then, as n increases,

$$n^{1/2}(\tilde{X}_n - \theta) \sim N(0, \sigma^2), \quad \text{where} \quad \sigma^2 = (2f(\theta))^{-2}. \tag{1.1}$$

Moreover, if we assume that for some $a > 0$ (not necessarily ≥ 1),

$$E_F|X|^a < \infty, \tag{1.2}$$

then, as n increases,

$$nE_F[(\tilde{X}_n - \theta)^2] \to \sigma^2. \tag{1.3}$$

Ghosh et al. (1984) and Babu (1986) have shown that under (1.2) and the positivity and continuity of f at θ, the classical bootstrap method (Efron (1982)) provides a (strongly) consistent estimator of σ^2; the jackknife method is known to be non effective in this particular situation. Our primary interest centers around the performance of the bootstrap method in one important *nonregular* case:

> The d.f. F has a jump-discontinuity of its density f at θ, so that both $f_- = f(\theta-)$ and $f_+ = f(\theta+)$ exist (and are positive), but they are not the same.

Our study is partially motivated by the scale-signed perturbation model:

$$F_\theta(x) = \begin{cases} \Phi((x-\theta)/a), & x < \theta, \\ \Phi((x-\theta)/b), & x > \theta, \end{cases}$$

where $a \neq b$. Then at the population median θ, the right-hand side and left-hand side derivatives of $F(x)$ exist but they are not equal, so there is a jump discontinuity. Ibragimov and Hasminskii (1981) describe other interesting problems where the density has a jump. Chapter 5 of their monograph deals exclusively with the asymptotic theory of likelihood ratio test and the maximum likelihood estimation for such nonregular cases, and that is what we have here.

When θ occurs at a density jump, the asymptotic results in (1.1) and (1.3) fail, but the limiting distribution exists under fairly general regularity conditions (Smirnov (1952)). Hence it may be of some interest to inquire how far classical bootstrapping succeeds in estimating such a limiting distribution and its variance. Two parallel results are given in Section 2. A large-scale simulation result of the bootstrap distribution is given in Section 3. Section 4 gives some concluding remarks.

2. The Variance Estimator and the Bootatrap Distribution

Consider the case where F has a jump-discontinuity of its density f at θ, so that $f_- = f(\theta-)$ and $f_+ = f(\theta+)$ both exist but they are not the same. We assume that

$$f_- > 0, \quad f_+ > 0, \quad \text{but} \quad f_- \neq f_+. \tag{2.1}$$

Then, some simple manipulations lead us to the following limit law:

$$\lim_{n\to\infty} P\{n^{1/2}(\tilde{X}_n - \theta) \leq y\} = \Phi(a_y y), \tag{2.2}$$

where Φ stands for the standard normal d.f. and $a_y = 2f_-$ if $y \leq 0$, $a_y = 2f_+$ if $y \geq 0$. Also, under (1.2) and (2.1),

$$\lim_{n\to\infty} E_F\{n^{1/2}(\tilde{X}_n - \theta)\} = (8\pi)^{-1/2}(f_+^{-1} - f_-^{-1}),$$

$$\lim_{n\to\infty} E_F\{n(\tilde{X}_n-\theta)^2\} = 8^{-1}(f_+^{-2}+f_-^{-2}), \tag{2.3}$$

so that the asymptotic variance of $n^{1/2}(\tilde{X}_n-\theta)$ is

$$\sigma^2 = 8^{-1}\{f_+^{-2}+f_-^{-2}-\pi^{-1}(f_+^{-1}-f_-^{-1})^2\}, \tag{2.4}$$

while the asymptotic mean square error is given by (2.3). We are primarily concerned with the following:

How far are (2.2) and (2.4) estimable by the classical bootstrap procedure?

Are there some variants of the classical bootstrap which may perform better?

In these respects, we have both affirmative and negative results.

Let $X_1^*,\ldots, X_m^*$ be m (conditionally) independent r.v.'s drawn with replacement in a simple random sampling scheme from the base sample $X_1,\ldots,X_n$ (or equivalently, the empirical d.f. F_n of $X_1,\ldots,X_n$). Let $\tilde{X}_m^*$ = median $(X_1^*,\ldots,X_m^*)$ be a bootstrap version of $\tilde{X}_n$, and define the bootstrap variance estimator, $s_m^{*^2}$, as the variance of the bootstrap distribution of $m^{1/2}(\tilde{X}_m^*-\tilde{X}_n)$. Allowing the bootstrap sample size m to be dependent on n (i.e., letting $m=m_n$), we intend to study (Theorem 2.1) the stochastic convergence of s_m^{*2} to σ^2 in (2.4). Unlike the regular case, the picture here depends not only on $|f_+^{-1}-f_-^{-1}|$ but also on the ratio m_n/n.

For simplicity of presentation, we take both n and m to be odd integers; the results to follow remain valid even if one or both of them may not be odd. We let $n=2k_n+1\geq 3$ and $m_n=2k_n^*+1\geq 3$, and we may drop the subscript n whenever there is no confusion. This is one of the rare cases where the bootstrap distribution (and its variance) can be calculated analytically. Following Efron (1982, page 77), we see that, given F_n,

$$\begin{aligned} p_{n:i} &= P\{\tilde{X}_m^* = X_{n:i}|F_n\} \\ &= m!(k^*!)^{-2}\int_{(i-1)/n}^{i/n}\{u(1-u)\}^{k^*}du, \quad i=1,\ldots,n. \end{aligned} \tag{2.5}$$

Hence the bootstrap variance estimator of the asymptotic variance (2.4) has an exact expression

$$s_m^{*^2} = \mathrm{Var}\left(m_n^{1/2}\tilde{X}_{m_n}^*|X_1,\ldots,X_n\right) = W_{n1}-W_{n2}^2, \tag{2.6}$$

where

$$W_{n1} = m_nE\left(\tilde{X}_{m_n}^{*^2}|X_1,\ldots,X_n\right) = m_n\sum_{i=1}^n X_{n:i}^2p_{n:i} \tag{2.7}$$

$$W_{n2} = m_n^{1/2} E\left(\tilde{X}^*_{m_n} | X_1, \ldots, X_n\right) = m_n^{1/2} \sum_{i=1}^{n} X_{n:i} p_{n:i}. \tag{2.8}$$

Some routine steps lead to

$$p_{n:k+1\pm r} \sim (2/\pi)^{1/2}(m+2)^{1/2} n^{-1} \exp\{-2(m+2)r^2 n^{-2}\},\ r = 0, 1, \ldots, k. \tag{2.9}$$

Theorem 2.1. *Suppose that* (1.2) *and* (2.1) *hold. Further, suppose that* $m_n \to \infty$ *as* $n \to \infty$.
(i) *If* $m_n/n \to 0$ *as* $n \to \infty$, *then* $s^{*^2}_{m_n} \to \sigma^2$ *in probability.*
(ii) *If* $m_n(\log\log n)/n \to 0$ *as* $n \to \infty$, *then* $s^{*^2}_{m_n} \to \sigma^2$ *almost surely* (a.s.).
(iii) *If* $m_n \sim n$ *as* $n \to \infty$, *then* $s^{*2}_{m_n} - \sigma^2$ *has asymptotically a nondegenerate distribution with mean*

$$(8\pi)^{-1}(1 - 2^{-1/2})(f_+^{-1} - f_-^{-1})^2, \tag{2.10}$$

so that for $f_+ \neq f_-$, $s^{*2}_{m_n}$ *does not stochastically converge to* σ^2 *for almost all* F_n. *The inconsistency of* $s^{*2}_{m_n}$ *remains intact for* $m_n >> n$.

Proof. Without any loss of generality, we may set $\theta = 0$. Let $l = l_n = \max\{k : X_{n:k} \leq 0\}$, so that l_n has a binomial$(n, 1/2)$ law. Based on this, we have $n^{-1/2}(2l_n - n) \sim N(0,1)$, as $n \to \infty$, and

$$\overline{\lim}_{n\to 0} |n^{-1/2}(l_n - n/2)| (\log\log n)^{-1/2} = 2^{-1/2}, \quad \text{with prob. 1.} \tag{2.11}$$

Further, let $U_{n:i} = F(X_{n:i}), 1 \leq i \leq n$, and note the $U_{n:i}$ are the uniform order statistics (of a sample of size n). Then the following result follows from Bahadur (1966) with further adaptations from Sen and Ghosh (1971):

$$\max\{|U_{n:i} - U_{n:j} - n^{-1}(i-j)| : 1 \leq i < j \leq i + n^{1/2}\sqrt{\log n} \leq n\} = \mathrm{O}(n^{-3/4}(\log n))$$

a.s. as $n \to \infty$.

Moreover, using (2.5) and (2.9), we obtain the following:

$$\sum_{r\geq 0} p_{n:k+1+r} \sim 1/2, \tag{2.12}$$

$$\sum_{r\geq 0} r p_{n:k+1+r} \sim (8\pi)^{-1} n (m+2)^{-1/2}, \tag{2.13}$$

$$\sum_{r\geq 0} r^2 p_{n:k+1+r} \sim 8^{-1} n^2 (m+2)^{-1}, \tag{2.14}$$

and similar results hold for $k+1+r$ being replaced by $k+1-r, r \geq 0$; a negative sign appears in (2.13). By similar manipulations, we have for every $r^* > 0$,

$$\sum_{0\leq r\leq r^*} r p_{n:k+1+r} \sim (\sqrt{8\pi})^{-1} n (m+2)^{-1/2} (1 - \exp\{2(m+2)(r^*/n)^2\}),$$
$$\sum_{0\leq r\leq r^*} r^2 p_{n:k+1+r} \sim 8^{-1} n^2 (m+2)^{-1} G_{3/2}(2(m+2)(r^*/n)^2),$$

where G_p is the gamma d.f. defined by

$$G_p(x) = [\int_0^x e^{-y} y^{p-1} dy]/\Gamma(p), \text{ for } \quad p > 0, \quad x \in [0, \infty).$$

A similar treatment holds for negative r^*.

Denote by

$$U_n^* = \mathrm{Sign}(l_n - n/2) G_{3/2}((l_n - n/2)^2 2(m+2) n^{-2}).$$

Note that by (2.11) and the fact that G_p is a d.f., we obtain that as $n \to \infty$,

$$m_n/n \to 0 \Rightarrow U_n^* \xrightarrow{P} 0, \tag{2.15}$$

$$m_n(\log\,\log n)/n \to 0 \Rightarrow U_n^* \to 0 \quad \text{a.s},$$

and if $m_n \sim n$, then U_n^* has asymptotically a nondegenerate distribution, symmetric about 0, so that $U_n^* \not\to 0$, in probability/a.s., as $n \to \infty$; the same conclusion holds if $m_n >> n$. Similarly, if we let

$$V_n^* = \exp\{-(l_n - n/2)^2 \, 2(m+2) n^{-2}\},$$

then as $n \to \infty$,

$$\begin{aligned} m_n/n \to 0 &\Rightarrow V_n^* \xrightarrow{P} 1, \\ m_n(\log\,\log n)/n \to 0 &\Rightarrow V_n^* \to 1, \text{ a.s.}, \end{aligned} \tag{2.16}$$

while for $m_n \sim n$ (or $m_n >> n$), $V_n^* - 1$ has asymptotically a nondegenerate distribution.

We now return to the proof of the main result. Note that by (2.9), the $p_{n:i}$ converge to 0 (exponentially) as i moves away from $k_n + 1$. As such, we truncate the range of i around $k_n + 1$ and proceed as follows. First, consider the range of $i : l_n < i \le l_n + K n^{1/2} \sqrt{\log n}$, where $K (< \infty)$ is arbitrary (but fixed) and positive. As f is continuous from the right and $f^+ > 0$, we have for all $i : l_n < i \le l_n + K n^{1/2} \sqrt{\log n}$,

$$\begin{aligned} n^{\frac{1}{2}} X_{n:i} &= n^{\frac{1}{2}} (X_{n:i} - \theta) \qquad (\text{at } \theta = 0) \\ &= n^{\frac{1}{2}} \{F(X_{n:i}) - F(\theta)\} \{[F(X_{n:i}) - F(\theta)]/(X_{n:i} - \theta)\}^{-1} \\ &= n^{\frac{1}{2}} \{U_{n:i} - .5\} \{f_+ + \mathrm{o}(1) \text{ a.s.}\}^{-1} \\ &= n^{\frac{1}{2}} \{i/(n+1) - .5 + \mathrm{O}(n^{-.75} \log n) \text{ a.s.}\} \{f_+ + \mathrm{o}(1) \text{ a.s.}\}^{-1}, \end{aligned} \tag{2.17}$$

where in the last step, we have made use of the classical Bahadur (1966) representation of the sample quantile (from the uniform(0,1) distribution). In a similar manner, for all $i : l_n - K n^{1/2} \sqrt{\log n} \le i \le l_n$, as $n \to \infty$,

$$n^{\frac{1}{2}} X_{n:i} = n^{\frac{1}{2}} \{i/(n+1) - .5 + \mathrm{O}(n^{-.75} \log n) \text{ a.s.}\} \{f_- + \mathrm{o}(1) \text{ a.s.}\}^{-1}. \tag{2.18}$$

On the other hand, by (1.2) and (2.9), the contribution of the $X_{n:i}$ in (2.7) - (2.8) for all $i : |i-(n+1)/2| > Kn^{1/2}\sqrt{\log n}$ can be $o(1)$ a.s., as $n \to \infty$. Thus, using (2.5), (2.9), (2.12) - (2.14) along with (2.17) - (2.18), we obtain from (2.7) that as $n \to \infty$,

$$W_{n1} = 8^{-1}\{(f_+^{-2} + f_-^{-2}) + U_n^*(f_-^{-2} - f_+^{-2})\} + \mathrm{o}(1) \quad \text{a.s.} \tag{2.19}$$

In a similar manner, we have for $n \to \infty$,

$$W_{n2} = (8\pi)^{-1/2}(f_+^{-1} - f_-^{-1})V_n^* \mathrm{Sign}(l_n - (n+1)/2) + \mathrm{o}(1) \quad \text{a.s.}, \tag{2.20}$$

so that by (2.19) and (2.20),

$$\begin{aligned} W_{n1} - W_{n2}^2 =& \sigma^2 + 8^{-1}\{U_n^*(f_-^{-2} - f_+^{-2}) + \pi^{-1}(f_+^{-1} - f_-^{-1})^2(1 - V_n^{*2})\} \\ &+ \mathrm{o}(1) \quad \text{a.s.,} \quad \text{as} \quad n \to \infty. \end{aligned} \tag{2.21}$$

The rest of the proof of the theorem follows directly from (2.21) after making use of the convegence results in (2.15) through (2.16).

Let
$$H_{m_n}(y) = P\{m_n^{1/2}(\tilde{X}_{m_n}^* - \tilde{X}_n) \le y | F_n\}$$
be the bootstrap distribution estimator of $P\{n^{1/2}(\tilde{X}_n - \theta) \le y\}$, where $P\{\cdot|F_n\}$ is taken under the conditional law generated by the n^{m_n} (conditionally) equally likely realizations of the X_i^*, $i \le m_n$ (given $X_1, \ldots, X_n$). Then we have the following result similar to Theorem 2.1, (Note that (1.2) is not needed).

Theorem 2.2. *Suppose that* (2.1) *holds and that* $m_n \to \infty$ *as* $n \to \infty$.
(i) *If* $m_n/n \to 0$, *then* $H_{m_n}(y) \to \Phi(a_y y)$ *in probability.*
(ii) *If* $m_n \log \log n/n \to 0$, *then* $H_{m_n}(y) \to \Phi(a_y y)$ a.s.
(iii) *If* $m_n \sim n$, *then* $H_{m_n}(y)$ *has asymptotically a nondegenerate distribution so that* $H_{m_n}(y)$ *is inconsistent.*

Proof. It follows from the arguments in Serfling (1980, pp.78-79) and the continuity of Φ that

$$H_{m_n}(y) = \Phi(2m_n^{1/2}\Delta_{m,n}(y)) + \mathrm{o}(1) \quad \text{a.s.},$$

where $\Delta_{m,n}(y) = F_n(\tilde{X}_n + m_n^{-1/2}y) - 1/2$. In case (i) or (ii),

$$y + m_n^{1/2}(\tilde{X}_n - \theta) = y + \mathrm{o}(1) \quad \text{(in probability or a.s.)},$$

which is positive (negative) if $y > 0$ ($y < 0$) for sufficiently large n. Therefore,

$$\{F(\theta + \tilde{X}_n - \theta + m_n^{-1/2}y) - F(\theta)\}\{\tilde{X}_n - \theta + m_n^{-1/2}y\}^{-1} \longrightarrow 2^{-1}a_y \ \text{(in prob. or a.s.)}.$$

It follows from the proof of Lemma 2.5.4 E of Serfling(1980) that

$$\sup |F_n(\theta + x) - F_n(\theta) - F(\theta + x) + F(\theta)| = O(n^{-3/4}(\log n)^{3/4}) \quad \text{a.s.}, \tag{2.22}$$

where the supremum is taken over $|x| \leq n^{-1/2}(\log n)^{1/2}$. Hence

$$\{F_n(\theta + \tilde{X}_n - \theta + m_n^{-1/2}y) - F_n(\theta)\}\{\tilde{X}_n - \theta + m_n^{-1/2}y\}^{-1} \longrightarrow 2^{-1}a_y$$

(in prob. or a.s.). Therefore

$$\begin{aligned} m_n^{1/2}\Delta_{m,n}(y) &= m_n^{1/2}[F_n(\theta) - 1/2] + [y + m_n^{1/2}(\tilde{X}_n - \theta)]\cdot \\ &\qquad [F_n(\theta + \tilde{X}_n - \theta + m_n^{-1/2}y) - F_n(\theta)][\tilde{X}_n - \theta + m_n^{-1/2}y]^{-1} \\ &= 2^{-1}a_y y + \mathrm{o}(1) \quad \text{(in prob. or a.s.)} \end{aligned}$$

and the result in (i) or (ii) follows. For case (iii), we show that $m_n^{1/2}\Delta_{m,n}(y)$ has asymptotically a nondegenerate distribution when $m_n \sim n$, $f_+ > f_-$ and $y > 0$. From (2.22),

$$\begin{aligned} m_n^{1/2}\Delta_{m,n}(y) &= m_n^{1/2}[F_n(\tilde{X}_n + m_n^{-1/2}y) - F_n(\tilde{X}_n)] + o_p(1) \\ &= m_n^{1/2}[F(\tilde{X}_n + m_n^{-1/2}y) - F(\tilde{X}_n)] + o_p(1) \\ &= g(m_n^{1/2}(\tilde{X}_n - \theta)) + o_p(1), \end{aligned}$$

where

$$g(x) = \begin{cases} f_+ \cdot y, & x > 0, \\ f_- \cdot y, & x \leq 0, x > -y, \\ (f_+ - f_-)x + f_+ \cdot y, & x \leq 0, x \leq -y. \end{cases}$$

Therefore, $m_n^{1/2}\Delta_{m,n}(y)$ converges in law to a random variable having distribution

$$G(t) = \begin{cases} 1, & t > f_+ \cdot y, \\ 1/2, & t < f_+ \cdot y, t > f_- \cdot y, \\ \Phi(a_\xi \cdot \xi), & t < f_+ \cdot y, t \leq f_- \cdot y, \end{cases}$$

where $\xi = (t - f_+ \cdot y)(f_+ - f_-)^{-1}$.

3. Simulation Results

For our numerical study, 100,000 samples of size n were taken from the population whose density function is

$$f(x) = \begin{cases} 1, & -1/2 \leq x < 0, \\ 1/2, & 0 \leq x < 1. \end{cases}$$

To estimate the asymptotic variance (2.4), we compare the bootstrap variance estimators based on three different choices of the $\{m_n\}$ sequences: (i)

$m_n = n$, (ii) $m_n = n(\log n)^{-1}$ and (iii) $m_n = n(\log n)^{-2}$. The n values are so chosen that both n and m_n (to the nearest integer) are odd (see Table 1).

Table 1. Choices of the bootstrap sample size m_n

n	$m_n = n$	$m_n = n(\log n)^{-1}$	$m_n = n(\log n)^{-2}$
49	49	13	3
109	109	23	5
489	489	79	13
999	999	145	21
2499	2499	319	41
4999	4999	587	69

Given n and m_n we first evaluate the set $\{p_{n:i}, i = 1, ..., n\}$, (2.5), and then construct the $n \times n$ matrix $\mathbf{V} = (v_{i,j})$, where

$$v_{i,j} = \begin{cases} p_{n:i}(1 - p_{n:i}), & i = j, \\ -p_{n:i}p_{n:j}, & i \neq j. \end{cases}$$

For each sample $(X_1, \ldots, X_n)$, we obtain the bootstrap variance estimate $s_m^{*^2}(X_1, \ldots, X_n)$ as a quadratic $\tilde{\mathbf{X}}\mathbf{V}\tilde{\mathbf{X}}^T$ of the order statistics $\tilde{\mathbf{X}} = (X_{n:1}, \ldots, X_{n:n})$ (Huang (1991)). Notice that since the *exact* value of the estimator (2.6) is obtainable without the customary second stage (bootstrap) sampling, we are able to afford a very large number of replications. The average of 100,000 such values is then given in Table 2 (columns 3 through 5 for various choices of $\{m_n\}$).

Table 2. The averages of the bootstrap estimates $s_{m_n}^{*^2}$

n	$\mathrm{Var}(n^{1/2}\tilde{X}_n)$	Average of $s_{m_n}^{*^2}$		
		$m = n$	$m = n(\log n)^{-1}$	$m = n(\log n)^{-2}$
49	.5599	.6271	.5217	.3458
109	.5737	.6314	.5528	.4161
489	.5822	.6207	.5812	.5077
999	.5840	.6139	.5855	.5350
2499	.5833	.6091	.5878	.5588
4999	.5814	.6057	.5881	.5694
∞	.5852	.5969	.5852	.5852

For the sake of comparison, the limiting values (as $n \to \infty$) appear at the bottom row. Notice the minute difference between the theoretical asymptotic variance (.5852) and the limiting value (.5969) of the expectation of the classical bootstrap ($m_n = n$). For the classical bootstrap, the asymptotic bias (2.10) is so small that it evaded our detection at an earlier stage of the study.

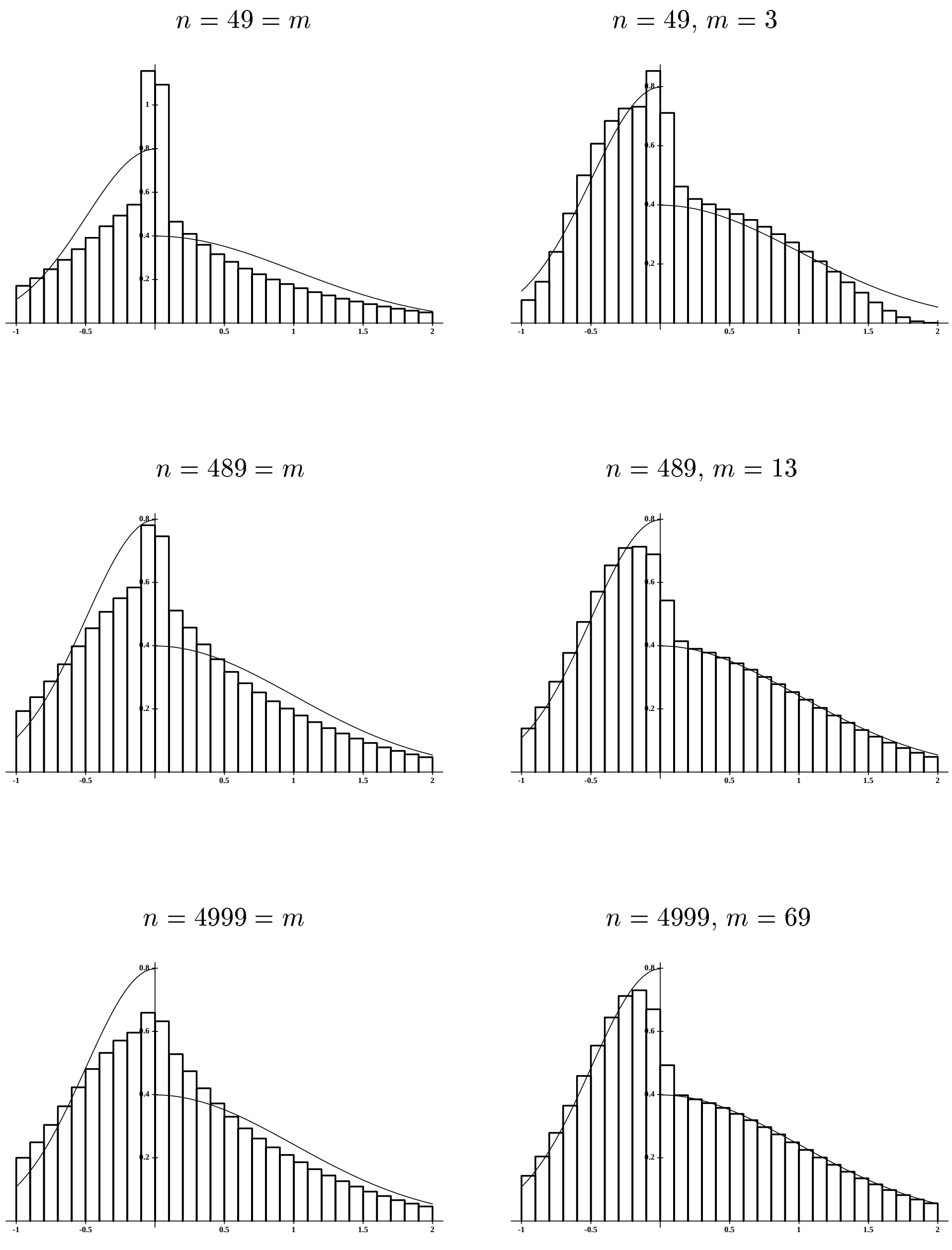

Figure 1. Bootstrap distribution of $m_n^{1/2}(\tilde{X}^*_{m_n} - \tilde{X}_n)$

A better view of the inconsistency of the classical bootstrap is provided by the bootstrap distribution of $m_n^{1/2}(\tilde{X}^*_{m_n} - \tilde{X}_n)$. We show in Figure 1 six histograms, three on the left for the classical, $\{m_n = n\}$, and three on the right for the modified, $\{m_n = n(\log n)^{-2}\}$; at increasing sample sizes $n = 49, 489$ and 4999. The interval $(-1.0, 2.0)$ is divided into 30 subintervals, and for each sample $(X_1, \ldots, X_n)$ we distribute the mass $\{p_{n:i}, i = 1, \ldots, n\}$ as follows: $p_{n:i}$ to the subinterval wherein the value $m^{-1/2}(X_{n:i} - \tilde{X}_{m_n})$ lies, $i = 1, \ldots, n$. After 100,000 replications the total mass at each interval is then divided by 100,000, and is compared against the superimposed limiting density curve (2.2). It is seen that only the modified bootstrap (the three histograms shown on the right hand side) shows good convergence to the limiting density (See Figure 1).

4. Remarks

The idea of choosing the bootstrap sample size m_n different from n is not new. It was studied by Bickel and Freedman (1981) and by Swanepoel (1986). Our finding does extend and complement the results of Ghosh *et al* (1984). It also provides a dual to the other popular nonparametric variance estimator, the jackknife. It is known (see, for instance, Wu (1986)) that by carefully choosing d a delete-d jackknife estimator overcomes some of the deficiencies of the ordinary jackknife. Interestingly, here in the bootstrap we also find it advantageous to use a smaller resample size m. Unlike the delete-d jackknife, however, which suffers from a combinatoric explosion of computation with increasing d, the bootstrap is just the opposite. The smaller the resample size m the easier it is to resample and to compute.

Several other *nonregular* cases, including a V-shaped and a U-shaped density f, where $f(\theta) = 0$, have been studied in Huang, Sen and Shao (1992).

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