Recursive nonparametric regression estimation for independent functional data

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1. Proofs

Throughout this section we use the following notation:

$$\Pi_n = \prod_{j=1}^{n} (1-\gamma_j),$$

$$a_n(\chi) = \Pi_n \sum_{k=1}^{n} \Pi_k^{-1}\gamma_k h_k^{-1} K\left(\frac{\|\chi - \mathcal{X}_k\|}{h_k}\right) Y_k,$$

$$f_n(\chi) = \Pi_n \sum_{k=1}^{n} \Pi_k^{-1}\gamma_k h_k^{-1} K\left(\frac{\|\chi - \mathcal{X}_k\|}{h_k}\right), \tag{11}$$

$$\xi = \lim_{n\to\infty} (n\gamma_n)^{-1}. \tag{12}$$

Let us first state the following technical lemma.

Lemma 1. *Let $(v_n) \in \mathcal{GS}(v^*)$, $(\gamma_n) \in \mathcal{GS}(-\alpha)$, and $m > 0$ such that $m - v^*\xi > 0$ where ξ is defined in (12). We have*

$$\lim_{n\to+\infty} v_n \Pi_n^m \sum_{k=1}^{n} \Pi_k^{-m} \frac{\gamma_k}{v_k} = \frac{1}{m - v^*\xi}.$$

Moreover, for all positive sequence (b_n) such that $\lim_{n\to+\infty} b_n = 0$, and all $\delta \in \mathbb{R}$,

$$\lim_{n\to+\infty} v_n \Pi_n^m \left[\sum_{k=1}^{n} \Pi_k^{-m} \frac{\gamma_k}{v_k} b_k + \delta\right] = 0.$$

Lemma 1 is widely applied throughout the proofs. Let us underline that it is its application, which requires Assumption $(A2)(iii)$ on the limit of $(n\gamma_n)$ as n goes to infinity.

1.1 Proof of Proposition 1

Let us first use the following decomposition

$$\begin{aligned}\mathbb{E}\left[\widehat{r}_n\left(\chi,h\right)\right]-r\left(\chi\right) &= \frac{\mathbb{E}\left[a_n\left(\chi\right)\right]}{\mathbb{E}\left[f_n\left(\chi\right)\right]}-r\left(\chi\right)-\frac{\mathbb{E}\left\{a_n\left(\chi\right)\left[f_n\left(\chi\right)-\mathbb{E}\left(f_n\left(\chi\right)\right)\right]\right\}}{\left\{\mathbb{E}\left[f_n\left(\chi\right)\right]\right\}^2} \\ &\quad +\frac{\mathbb{E}\left\{\widehat{r}_n\left(\chi,h\right)\left[f_n\left(\chi\right)-\mathbb{E}\left(f_n\left(\chi\right)\right)\right]^2\right\}}{\left\{\mathbb{E}\left[f_n\left(\chi\right)\right]\right\}^2} \qquad (13)\end{aligned}$$

Computing the expectation of $f_n\left(\chi\right)$

First, we have

$$\begin{aligned}\mathbb{E}\left[f_n\left(\chi\right)\right] &= \Pi_n\sum_{k=1}^{n}\Pi_k^{-1}\gamma_k h_k^{-1}\mathbb{E}\left[K\left(\frac{\|\chi-\mathcal{X}_k\|}{h_k}\right)\right] \\ &= \Pi_n\sum_{k=1}^{n}\Pi_k^{-1}\gamma_k h_k^{-1}\left\{\int_0^1 K\left(u\right)d\mathbb{P}^{\left(\frac{\|\chi-\mathcal{X}_k\|}{h_k}\right)}\left(u\right)\right\} \\ &= \Pi_n\sum_{k=1}^{n}\Pi_k^{-1}\gamma_k h_k^{-1}\left\{F\left(h_k\right)\left[K\left(1\right)-\int_0^1 K'\left(u\right)\tau_{h_k}\left(u\right)du\right]\right\}.\end{aligned}$$

Then, since we have $\lim_{n\to\infty}\left(n\gamma_n\right)>\mathcal{F}_a$, the application of Lemma 1 gives

$$\mathbb{E}\left[f_n\left(\chi\right)\right]=\frac{1}{1-\left(\mathcal{F}_a-a\right)\xi}M_1h_n^{-1}F\left(h_n\right)\left[1+o\left(1\right)\right]. \qquad (14)$$

Computing the expectation of $a_n\left(\chi\right)$

Now, we have

$$\begin{aligned}\mathbb{E}\left[a_n\left(\chi\right)\right] &= \Pi_n\sum_{k=1}^{n}\Pi_k^{-1}\gamma_k h_k^{-1}\mathbb{E}\left[\left(Y_k-r\left(\chi\right)\right)K\left(\frac{\|\chi-\mathcal{X}_k\|}{h_k}\right)\right] \\ &\quad +r\left(\chi\right)\mathbb{E}\left[f_n\left(\chi\right)\right].\end{aligned}$$

Taylor's expansion of ϕ around 0 ensures that

$$\begin{aligned}
\mathbb{E}\left[\left(Y_k - r\left(\chi\right)\right)K\left(\frac{\|\chi - \mathcal{X}_k\|}{h_k}\right)\right] &= \mathbb{E}\left[\left(r\left(\mathcal{X}_k\right) - r\left(\chi\right)\right)K\left(\frac{\|\chi - \mathcal{X}_k\|}{h_k}\right)\right] \\
&= \mathbb{E}\left[\phi\left(\|\chi - \mathcal{X}_k\|\right)K\left(\frac{\|\chi - \mathcal{X}_k\|}{h_k}\right)\right] \\
&= \int_0^1 \phi\left(h_k u\right)K\left(u\right)d\mathbb{P}^{\left(\frac{\|\chi-\mathcal{X}_k\|}{h_k}\right)}\left(u\right) \\
&= h_k\phi'\left(0\right)\int_0^1 uK\left(u\right)d\mathbb{P}^{\left(\frac{\|\chi-\mathcal{X}_k\|}{h_k}\right)}\left(u\right) + o\left(h_k\right).
\end{aligned}$$

Moreover, it follows from the proof of Lemma 2 in Ferraty et al. (2007), the assumption $(A2)$ and Fubini's Theorem

$$\int_0^1 uK\left(u\right)d\mathbb{P}^{\left(\frac{\|\chi-\mathcal{X}_k\|}{h_k}\right)}\left(u\right) = F\left(h_k\right)\left[K\left(1\right) - \int_0^1 \left(uK\left(u\right)\right)'\tau_{h_k}\left(u\right)du\right].$$

Then, since we have $\lim_{n\to\infty}\left(n\gamma_n\right) > \mathcal{F}_a$, the application of Lemma 1 gives

$$\begin{aligned}
\mathbb{E}\left[a_n\left(\chi\right)\right] &= \left\{\frac{1}{1-\mathcal{F}_a\xi}\phi'\left(0\right)M_0 + r\left(\chi\right)\frac{1}{1-\left(\mathcal{F}_a - a\right)\xi}M_1 h_n^{-1}\right\} \\
&\quad F\left(h_n\right)\left[1 + o\left(1\right)\right]. \qquad (15)
\end{aligned}$$

The combination of (14) and (15) gives

$$\frac{\mathbb{E}\left[a_n\left(\chi\right)\right]}{\mathbb{E}\left[f_n\left(\chi\right)\right]} - r\left(\chi\right) = h_n\phi'\left(0\right)\frac{1-\left(\mathcal{F}_a - a\right)\xi}{1-\mathcal{F}_a\xi}\frac{M_0}{M_1}\left[1 + o(1)\right]. \qquad (16)$$

Computing the variance of $f_n(\chi)$

First, we have

$$
\begin{aligned}
Var\left[f_n(\chi)\right] &= \Pi_n^2 \sum_{k=1}^{n} \Pi_k^{-2} \gamma_k^2 h_k^{-2} Var\left[K\left(\frac{\|\chi-\mathcal{X}_k\|}{h_k}\right)\right], \\
&= \Pi_n^2 \sum_{k=1}^{n} \Pi_k^{-2} \gamma_k^2 h_k^{-2} \left\{\mathbb{E}\left[K^2\left(\frac{\|\chi-\mathcal{X}_k\|}{h_k}\right)\right]\right. \\
&\quad \left. -\,\mathbb{E}\left[K\left(\frac{\|\chi-\mathcal{X}_k\|}{h_k}\right)\right]^2\right\} \\
&= \Pi_n^2 \sum_{k=1}^{n} \Pi_k^{-2} \gamma_k^2 h_k^{-2} \left\{\int_0^1 K^2(u)\, d\mathbb{P}^{\left(\frac{\|\chi-\mathcal{X}_k\|}{h_k}\right)}(u)\right. \\
&\quad \left. -\left[\int_0^1 K(u)\, d\mathbb{P}^{\left(\frac{\|\chi-\mathcal{X}_k\|}{h_k}\right)}(u)\right]^2\right\} \\
&= \Pi_n^2 \sum_{k=1}^{n} \Pi_k^{-2} \gamma_k^2 h_k^{-2} F(h_k) \left\{\left[K^2(1) - \int_0^1 \left(K^2(u)\right)' \tau_{h_k}(u)\, du\right]\right. \\
&\quad \left. -F(h_k)\left[K(1) - \int_0^1 K'(u)\, \tau_{h_k}(u)\, du\right]^2\right\}.
\end{aligned}
$$

Then, since we have $\lim_{n\to\infty}(n\gamma_n) > (\mathcal{F}_a+\alpha)/2 - a$, the application of Lemma 1 gives

$$
\begin{aligned}
&Var\left[f_n(\chi)\right] \\
&\quad = \frac{1}{2-(\mathcal{F}_a+\alpha-2a)\,\xi} \frac{\gamma_n}{h_n^2} F(h_n)\, M_2\left[1+o(1)\right]. \qquad (17)
\end{aligned}
$$

Computing the variance of $a_n(\chi)$

First, we have

$$\begin{aligned}
Var\left[a_n(\chi)\right] &= \Pi_n^2\sum_{k=1}^n \Pi_k^{-2}\gamma_k^2 h_k^{-2} Var\left[Y_k K\left(\frac{\|\chi-\mathcal{X}_k\|}{h_k}\right)\right],\\
&= \Pi_n^2\sum_{k=1}^n \Pi_k^{-2}\gamma_k^2 h_k^{-2}\left\{\mathbb{E}\left[Y_k^2 K^2\left(\frac{\|\chi-\mathcal{X}_k\|}{h_k}\right)\right]\right.\\
&\quad\left. -\ \mathbb{E}\left[Y_k K\left(\frac{\|\chi-\mathcal{X}_k\|}{h_k}\right)\right]^2\right\}\\
&= \Pi_n^2\sum_{k=1}^n \Pi_k^{-2}\gamma_k^2 h_k^{-2}\left\{\left(r^2(\chi)+\sigma_\varepsilon^2(\chi)\right)\mathbb{E}\left[K^2\left(\frac{\|\chi-\mathcal{X}_k\|}{h_k}\right)\right]\right.\\
&\quad -\left(\mathbb{E}\left[\left(Y_k-r(\chi)\right)K\left(\frac{\|\chi-\mathcal{X}_k\|}{h_k}\right)\right]\right.\\
&\quad\left.\left. +\ r(\chi)\,\mathbb{E}\left[K\left(\frac{\|\chi-\mathcal{X}_k\|}{h_k}\right)\right]\right)^2\right\}\\
&= \Pi_n^2\sum_{k=1}^n \Pi_k^{-2}\gamma_k^2 h_k^{-2} F(h_k)\left(r^2(\chi)+\sigma_\varepsilon^2(\chi)\right)\\
&\quad \left[K^2(1)-\int_0^1\left(K^2\right)'(u)\,\tau_{h_k}(u)\,du\right]\\
&\quad -\Pi_n^2\sum_{k=1}^n \Pi_k^{-2}\gamma_k^2 h_k^{-2} F(h_k)^2\left\{h_k\phi'(0)\right.\\
&\quad \left[K(1)-\int_0^1\left(uK(u)\right)'\tau_{h_k}(u)\,du\right]\\
&\quad \left. +r(\chi)\left[K(1)-\int_0^1 K'(u)\,\tau_{h_k}(u)\,du\right]\right\}^2.
\end{aligned}$$

Then, since we have $\lim_{n\to\infty}(n\gamma_n) > (\mathcal{F}_a+\alpha)/2-a$, the application of Lemma 1 gives

$$\begin{aligned}
Var\left[a_n(\chi)\right] &= \frac{1}{2-(\mathcal{F}_a+\alpha-2a)\,\xi}\left(r^2(\chi)+\sigma_\varepsilon^2(\chi)\right)\\
&\quad \frac{\gamma_n}{h_n^2}F(h_n)\,M_2\left[1+o(1)\right].
\end{aligned} \tag{18}$$

Computing the covariance between $a_n(\chi)$ and $f_n(\chi)$

First, we have

$$\begin{aligned}\mathbb{E}\left[a_n(\chi) f_n(\chi)\right] &= \Pi_n^2 \sum_{k=1}^{n} \Pi_k^{-2} \gamma_k^2 h_k^{-2} \mathbb{E}\left[Y_k K^2\left(\frac{\|\chi - \mathcal{X}_k\|}{h_k}\right)\right] \\ &+\Pi_n^2 \sum_{\substack{k,k'=1\\ k\neq k'}}^{n} \Pi_k^{-1} \Pi_{k'}^{-1} \gamma_k \gamma_{k'} h_k^{-1} \mathbb{E}\left[Y_k K\left(\frac{\|\chi - \mathcal{X}_k\|}{h_k}\right)\right] \\ &\times h_{k'}^{-1} \mathbb{E}\left[Y_{k'} K\left(\frac{\|\chi - \mathcal{X}_{k'}\|}{h_{k'}}\right)\right]. \end{aligned} \tag{19}$$

Moreover, we have

$$\begin{aligned}\mathbb{E}\left[Y_k K^2\left(\frac{\|\chi - \mathcal{X}_k\|}{h_k}\right)\right] &= \mathbb{E}\left[\left(Y_k - r(\chi)\right) K^2\left(\frac{\|\chi - \mathcal{X}_k\|}{h_k}\right)\right] \\ &+r(\chi)\, \mathbb{E}\left[K^2\left(\frac{\|\chi - \mathcal{X}_k\|}{h_k}\right)\right].\end{aligned}$$

Taylor's expansion of ϕ around 0 ensures that

$$\begin{aligned}\mathbb{E}\left[\left(Y_k - r(\chi)\right) K^2\left(\frac{\|\chi - \mathcal{X}_k\|}{h_k}\right)\right] &= \mathbb{E}\left[\left(r(\mathcal{X}_k) - r(\chi)\right) K^2\left(\frac{\|\chi - \mathcal{X}_k\|}{h_k}\right)\right] \\ &= \mathbb{E}\left[\phi\left(\|\chi - \mathcal{X}_k\|\right) K^2\left(\frac{\|\chi - \mathcal{X}_k\|}{h_k}\right)\right] \\ &= \int_0^1 \phi(h_k u) K^2(u)\, d\mathbb{P}^{\left(\frac{\|\chi - \mathcal{X}_k\|}{h_k}\right)}(u) \\ &= h_k \phi'(0) \int_0^1 u K^2(u)\, d\mathbb{P}^{\left(\frac{\|\chi - \mathcal{X}_k\|}{h_k}\right)}(u) + o(h_k).\end{aligned}$$

Moreover, it follows from the proof of Lemma 2 in Ferraty et al. (2007), the assumption $(A2)$ and Fubini's Theorem

$$\int_0^1 u K^2(u)\, d\mathbb{P}^{\left(\frac{\|\chi - \mathcal{X}_k\|}{h_k}\right)}(u) = F(h_k)\left[K^2(1) - \int_0^1 \left(u K^2(u)\right)' \tau_{h_k}(u)\, du\right],$$

Then, we get

$$\begin{aligned}&\mathbb{E}\left[Y_k K^2\left(\frac{\|\chi - \mathcal{X}_k\|}{h_k}\right)\right] \\ &\quad= h_k \phi'(0) F(h_k)\left[K^2(1) - \int_0^1 \left(u K^2(u)\right)' \tau_{h_k}(u)\, du\right] \\ &\quad+r(\chi) F(h_k)\left[K^2(1) - \int_0^1 \left(K^2(u)\right)' \tau_{h_k}(u)\, du\right].\end{aligned} \tag{110}$$

Then, in view of (19), (110) and since we have $\lim_{n\to\infty}(n\gamma_n) > (\mathcal{F}_a + \alpha - 2a)$, the application of Lemma 1 gives

$$\begin{aligned} &Cov\left(a_n\left(\chi\right), f_n\left(\chi\right)\right) \\ &= \frac{1}{2-\left(\mathcal{F}_a+\alpha-2a\right)\xi}\frac{\gamma_n}{h_n^2}F\left(h_n\right)M_2 r\left(\chi\right)\left[1+o\left(1\right)\right]. \end{aligned} \tag{111}$$

Computing the expectation of $\widehat{r}_n\left(\chi,h\right)$

First, it follows from (17) and (18), that

$$\mathbb{E}\left\{a_n\left(\chi\right)\left[f_n\left(\chi\right)-\mathbb{E}\left(f_n\left(\chi\right)\right)\right]\right\} = O\left(\frac{\gamma_n}{F\left(h_n\right)}\right). \tag{112}$$

$$\mathbb{E}\left\{\widehat{r}_n\left(\chi,h\right)\left[f_n\left(\chi\right)-\mathbb{E}\left(f_n\left(\chi\right)\right)\right]^2\right\} = O\left(\frac{\gamma_n}{F\left(h_n\right)}\right). \tag{113}$$

Then (2.8) follows from (13), (16), (112) and (113).

Computing the variance of $\widehat{r}_n\left(\chi,h\right)$

We have

$$\begin{aligned} Var\left[\widehat{r}_n\left(\chi,h\right)\right] \quad &\simeq \quad \frac{Var\left[a_n\left(\chi\right)\right]}{\left\{\mathbb{E}\left[f_n\left(\chi\right)\right]\right\}^2} - 4\frac{\mathbb{E}\left[a_n\left(\chi\right)\right]Cov\left(a_n\left(\chi\right), f_n\left(\chi\right)\right)}{\left\{\mathbb{E}\left[f_n\left(\chi\right)\right]\right\}^3} \\ &\quad +3Var\left[f_n\left(\chi\right)\right]\frac{\left\{\mathbb{E}\left[a_n\left(\chi\right)\right]\right\}^2}{\left\{\mathbb{E}\left[f_n\left(\chi\right)\right]\right\}^4}. \end{aligned} \tag{114}$$

Then, the combination of (14), (15), (17), (18), (111) and (114), ensures that

$$Var\left[\widehat{r}_n\left(\chi,h\right)\right] \quad = \quad \sigma_\varepsilon^2\left(\chi\right)\frac{M_2}{M_1^2}\frac{\left(1-\left(\mathcal{F}_a-a\right)\xi\right)^2}{\left(2-\left(\mathcal{F}_a+\alpha-2a\right)\xi\right)}\frac{\gamma_n}{F\left(h_n\right)}\left[1+o\left(1\right)\right].$$

1.2 Proof of Theorem 1

Let us at first assume that, if $a \geq \left(\alpha+\mathcal{F}_a\right)/2$, then

$$\begin{aligned} &\sqrt{\gamma_n^{-1}F\left(h_n\right)}\left(\widehat{r}_n\left(\chi,h\right)-\mathbb{E}\left[\widehat{r}_n\left(\chi,h\right)\right]\right) \\ &\quad \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \sigma_\varepsilon^2\left(\chi\right)\frac{\left(1-\left(\mathcal{F}_a-a\right)\xi\right)^2}{\left(2-\left(\mathcal{F}_a+\alpha-2a\right)\xi\right)}\frac{M_2}{M_1^2}\right). \end{aligned} \tag{115}$$

In the case when $\gamma_n^{-1} h_n^2 F(h_n) \to c$, Part 1 of Theorem 1 follows from the combination of (2.8) and (115). In the case $\gamma_n^{-1} h_n^2 F(h_n) \to \infty$, (2.10) implies that

$$h_n^{-2} \left(\widehat{r}_n(\chi, h) - \mathbb{E}\left(\widehat{r}_n(\chi, h)\right)\right) \xrightarrow{\mathbb{P}} 0,$$

and the application of (2.8) gives Part 2 of Theorem 1.

We now prove (115). In view of (11), we have

$$\widehat{r}_n(\chi, h) - \mathbb{E}\left[\widehat{r}_n(\chi, h)\right] = \frac{a_n(\chi)}{f_n(\chi)} - \frac{\mathbb{E}\left[a_n(\chi)\right]}{\mathbb{E}\left[f_n(\chi)\right]} + o\left(\sqrt{\frac{\gamma_n}{F(h_n)}}\right).$$

Moreover, since we have

$$\begin{aligned} &\frac{a_n(\chi)}{f_n(\chi)} - \frac{\mathbb{E}\left[a_n(\chi)\right]}{\mathbb{E}\left[f_n(\chi)\right]} \\ &\quad = \frac{1}{f_n(\chi)\,\mathbb{E}\left[f_n(\chi)\right]} \left\{\mathbb{E}\left[f_n(\chi)\right]\left(a_n(\chi) - \mathbb{E}\left[a_n(\chi)\right]\right) - \mathbb{E}\left[a_n(\chi)\right]\left(f_n(\chi) - \mathbb{E}\left[f_n(\chi)\right]\right)\right\}. \end{aligned}$$

Using Slutsky's theorem and (2.8) and (2.9), we get (115).

1.3 Proof of Theorem 2

Computing the expectation of $\psi_{n,1}(\chi, h, g)$

First, we have

$$\begin{aligned} \mathbb{E}\left[\psi_{n,1}(\chi, h, g)\right] &= \Pi_n^2 \sum_{k=1}^n \Pi_k^{-2} \gamma_k^2 h_k^{-1} g_k^{-1} \mathbb{E}\left[K\left(\frac{\|\chi - \mathcal{X}_k\|}{h_k}\right) K\left(\frac{\|\chi - \mathcal{X}_k\|}{g_k}\right) Y_k\right] \\ &= \Pi_n^2 \sum_{k=1}^n \Pi_k^{-2} \gamma_k^2 h_k^{-1} g_k^{-1} \mathbb{E}\left[\left(Y_k - r(\chi)\right) K\left(\frac{\|\chi - \mathcal{X}_k\|}{h_k}\right) K\left(\frac{\|\chi - \mathcal{X}_k\|}{g_k}\right)\right] \\ &\quad + r(\chi)\, \Pi_n^2 \sum_{k=1}^n \Pi_k^{-2} \gamma_k^2 h_k^{-1} g_k^{-1} \mathbb{E}\left[K\left(\frac{\|\chi - \mathcal{X}_k\|}{h_k}\right) K\left(\frac{\|\chi - \mathcal{X}_k\|}{g_k}\right)\right]. \end{aligned}$$

Taylor's expansion of ϕ around 0 ensures that

$$\begin{aligned}
&\mathbb{E}\left[\left(Y_k - r\left(\chi\right)\right) K\left(\frac{\|\chi - \mathcal{X}_k\|}{h_k}\right) K\left(\frac{\|\chi - \mathcal{X}_k\|}{g_k}\right)\right] \\
&\quad = \mathbb{E}\left[\left(r\left(\mathcal{X}_k\right) - r\left(\chi\right)\right) K\left(\frac{\|\chi - \mathcal{X}_k\|}{h_k}\right) K\left(\frac{\|\chi - \mathcal{X}_k\|}{g_k}\right)\right] \\
&\quad = \mathbb{E}\left[\phi\left(\|\chi - \mathcal{X}_k\|\right) K\left(\frac{\|\chi - \mathcal{X}_k\|}{h_k}\right) K\left(\frac{\|\chi - \mathcal{X}_k\|}{g_k}\right)\right] \\
&\quad = \int_0^1 \phi\left(h_k u\right) K\left(u\right) K\left(\frac{h_k}{g_k} u\right) d\mathbb{P}^{\left(\frac{\|\chi - \mathcal{X}_k\|}{h_k}\right)}\left(u\right) \\
&\quad = h_k \phi'\left(0\right) \int_0^1 u K\left(u\right) K\left(\frac{h_k}{g_k} u\right) d\mathbb{P}^{\left(\frac{\|\chi - \mathcal{X}_k\|}{h_k}\right)}\left(u\right) + o\left(h_k\right) \\
&\quad = h_k \phi'\left(0\right) F\left(h_k\right)\left[K\left(1\right) K\left(\frac{h_k}{g_k}\right) - \int_0^1 \left(u K\left(u\right) K\left(\frac{h_k}{g_k} u\right)\right)' \tau_{h_k}\left(u\right) du\right] + o\left(h_k\right)
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}\left[K\left(\frac{\|\chi - \mathcal{X}_k\|}{h_k}\right) K\left(\frac{\|\chi - \mathcal{X}_k\|}{g_k}\right)\right] \\
&\quad = \int_0^1 K\left(u\right) K\left(\frac{h_k}{g_k} u\right) d\mathbb{P}^{\left(\frac{\|\chi - \mathcal{X}_k\|}{h_k}\right)}\left(u\right) \\
&\quad = F\left(h_k\right)\left[K\left(1\right) K\left(\frac{h_k}{g_k}\right) - \int_0^1 \left(K\left(u\right) K\left(\frac{h_k}{g_k} u\right)\right)' \tau_{h_k}\left(u\right) du\right].
\end{aligned}$$

Then, since we have $\lim_{n\to\infty}\left(n\gamma_n\right) > \left(\mathcal{F}_a + \alpha - a - g\right)/2$, the application of Lemma 1 gives

$$\begin{aligned}
\mathbb{E}\left[\psi_{n,1}\left(\chi, h, g\right)\right] \quad = \quad & \frac{1}{2 - \left(\mathcal{F}_a + \alpha - a - g\right)\xi} \frac{\gamma_n}{g_n} \frac{F\left(h_n\right)}{h_n} r\left(\chi\right) K\left(0\right) M_1 \\
& \left[1 + o\left(1\right)\right].
\end{aligned} \tag{116}$$

Computing the expectation of $\psi_{n,2}\left(\chi, h, g\right)$

First, we have

$$\begin{aligned}
\mathbb{E}\left[\psi_{n,2}\left(\chi, h, g\right)\right] \quad = \quad & \Pi_n^2 \sum_{\substack{k,k'=1 \\ k\neq k'}}^{n} \Pi_k^{-1} \Pi_{k'}^{-1} \gamma_k \gamma_{k'} h_k^{-1} g_{k'}^{-1} \mathbb{E}\left[K\left(\frac{\|\chi - \mathcal{X}_k\|}{h_k}\right)\right] \\
& \mathbb{E}\left[K\left(\frac{\|\chi - \mathcal{X}_{k'}\|}{g_{k'}}\right) Y_{k'}\right].
\end{aligned}$$

Moreover, we have

$$\mathbb{E}\left[K\left(\frac{\|\chi-\mathcal{X}_k\|}{h_k}\right)\right] = F\left(h_k\right)\left[K\left(1\right)-\int_0^1 K'\left(u\right)\tau_{h_k}\left(u\right)du\right]$$

and

$$\begin{aligned}\mathbb{E}\left[K\left(\frac{\|\chi-\mathcal{X}_k\|}{g_k}\right)Y_k\right] &= g_k F\left(g_k\right)\phi'\left(0\right)\left[K\left(1\right)-\int_0^1 \left(uK\left(u\right)\right)'\tau_{h_k}\left(u\right)du\right] \\ &\quad +r\left(\chi\right)F\left(g_k\right)\left[K\left(1\right)-\int_0^1 K'\left(u\right)\tau_{g_k}\left(u\right)du\right].\end{aligned}$$

Then, since we have $\lim_{n\to\infty}\left(n\gamma_n\right)>\left(\mathcal{F}_a-a\right)$, the application of Lemma 1 gives

$$\begin{aligned}\mathbb{E}\left[\psi_{n,2}\left(\chi,h,g\right)\right] &= \left[\frac{1}{1-\left(\mathcal{F}_a-a\right)\xi}\right]\left[\frac{1}{1-\left(\mathcal{F}_g-g\right)\xi}\right]r\left(\chi\right) \\ &\quad \frac{F\left(g_n\right)}{g_n}\frac{F\left(h_n\right)}{h_n}M_1^2\left[1+o\left(1\right)\right].\end{aligned} \tag{117}$$

Then, the combination of (116) and (117) ensures that

$$\begin{aligned}\mathbb{E}\left[\phi_n\left(\chi,h,g\right)\right] &= \left\{\frac{1-\left(\mathcal{F}_a-a\right)\xi}{2-\left(\mathcal{F}_a+\alpha-a-g\right)\xi}\frac{\gamma_n}{h_n}\frac{F\left(h_n\right)}{F\left(g_n\right)}r\left(\chi\right)K\left(0\right)M_1\right. \\ &\quad \left.+\left[\frac{1}{1-\left(\mathcal{F}_a-a\right)\xi}\right]r\left(\chi\right)\frac{F\left(h_n\right)}{h_n}M_1\right\} \\ &\quad \left[1+o\left(1\right)\right].\end{aligned} \tag{118}$$

Then,

$$\begin{aligned}\mathbb{E}\left[\frac{\phi_n\left(\chi,h,g\right)}{\widehat{f}_n\left(\chi\right)}-r\left(\chi\right)\right] &= \frac{\left(1-\left(\mathcal{F}_a-a\right)\xi\right)\left(1-\left(\mathcal{F}_g-g\right)\xi\right)}{2-\left(\mathcal{F}_a+\alpha-a-g\right)\xi}\frac{\gamma_n}{F\left(g_n\right)} \\ &\quad \times\frac{K\left(0\right)}{M_1}r\left(\chi\right)\left[1+o\left(1\right)\right].\end{aligned} \tag{119}$$

Moreover, it follows from (2.15)

$$\mathbb{E}\left[\widehat{b}_n\left(\chi,h,g\right)\right]=\mathbb{E}\left[\frac{\phi_n\left(\chi,h,g\right)}{\widehat{f}_n\left(\chi\right)}-r\left(\chi\right)\right]-\mathbb{E}\left[\widehat{r}_n\left(\chi,g\right)-r\left(\chi\right)\right],$$

then,

$$\begin{aligned}\mathbb{E}\left[\widehat{b}_n\left(\chi,h,g\right)\right] &= \frac{1-\left(\mathcal{F}_g-g\right)\xi}{1-\mathcal{F}_g\xi}\left\{\frac{\left(1-\left(\mathcal{F}_a-a\right)\xi\right)\left(1-\mathcal{F}_g\xi\right)}{2-\left(\mathcal{F}_a+\alpha-a-g\right)\xi}\frac{\gamma_n}{F\left(g_n\right)}\frac{K\left(0\right)}{M_1}r\left(\chi\right)\right. \\ &\quad \left.-\,g_n\phi'\left(0\right)\frac{M_0}{M_1}\right\}\end{aligned}$$

Computing the variance of $\widehat{b}_n(\chi,h,g)$

First, we have

$$\begin{aligned}
&\mathbb{E}\left[\left(\widehat{b}_n(\chi,h,g)-\widehat{b}_n(\chi,h)\right)^2\right]\\
&=\mathbb{E}\left[\left(\frac{\phi_n(\chi,h,g)}{\widehat{f}_n(\chi)}-r(\chi)\right)^2\right]+\mathbb{E}\left[(\widehat{r}_n(\chi,g)-r(\chi))^2+(\widehat{r}_n(\chi,h)-r(\chi))^2\right]\\
&-2\mathbb{E}\left[(\widehat{r}_n(\chi,g)-r(\chi))\left(\frac{\phi_n(\chi,h,g)}{\widehat{f}_n(\chi)}-r(\chi)\right)\right]\\
&-2\mathbb{E}\left[(\widehat{r}_n(\chi,h)-r(\chi))\left(\frac{\phi_n(\chi,h,g)}{\widehat{f}_n(\chi)}-r(\chi)\right)\right]\\
&+2\mathbb{E}\left[(\widehat{r}_n(\chi,g)-r(\chi))(\widehat{r}_n(\chi,h)-r(\chi))\right]. \qquad (120)
\end{aligned}$$

Computing the expectation of $\psi_{n,1}^2(\chi,h,g)$

We have

$$\begin{aligned}
\mathbb{E}\left[\psi_{n,1}^2(\chi,h,g)\right] &= \Pi_n^4\sum_{k=1}^n\Pi_k^{-4}\gamma_k^4h_k^{-2}g_k^{-2}\mathbb{E}\left[K^2\left(\frac{\|\chi-\mathcal{X}_k\|}{h_k}\right)K^2\left(\frac{\|\chi-\mathcal{X}_k\|}{g_k}\right)Y_k^2\right]\\
&+\Pi_n^4\sum_{\substack{k,k'=1\\k\neq k'}}^n\Pi_k^{-2}\Pi_{k'}^{-2}\gamma_k^2\gamma_{k'}^2h_k^{-1}h_{k'}^{-1}g_k^{-1}g_{k'}^{-1}\\
&\times\mathbb{E}\left[K\left(\frac{\|\chi-\mathcal{X}_k\|}{h_k}\right)K\left(\frac{\|\chi-\mathcal{X}_k\|}{g_k}\right)Y_k\right]\\
&\times\mathbb{E}\left[K\left(\frac{\|\chi-\mathcal{X}_{k'}\|}{h_{k'}}\right)K\left(\frac{\|\chi-\mathcal{X}_{k'}\|}{g_k}\right)Y_{k'}\right].
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&\mathbb{E}\left[K^2\left(\frac{\|\chi-\mathcal{X}_k\|}{h_k}\right)K^2\left(\frac{\|\chi-\mathcal{X}_k\|}{g_k}\right)Y_k^2\right]\\
&=\left(r^2(\chi)+\sigma_\varepsilon^2(\chi)\right)\mathbb{E}\left[K^2\left(\frac{\|\chi-\mathcal{X}_k\|}{h_k}\right)K^2\left(\frac{\|\chi-\mathcal{X}_k\|}{g_k}\right)\right]\\
&=\left(r^2(\chi)+\sigma_\varepsilon^2(\chi)\right)\int_0^1K^2(u)K^2\left(\frac{h_k}{g_k}u\right)d\mathbb{P}^{\left(\frac{\|\chi-\mathcal{X}_k\|}{h_k}\right)}(u)\\
&=F(h_k)\left(r^2(\chi)+\sigma_\varepsilon^2(\chi)\right)\left[K^2(1)K^2\left(\frac{h_k}{g_k}\right)-\int_0^1\left(K^2(u)K^2\left(\frac{h_k}{g_k}u\right)\right)'\tau_{h_k}(u)\,du\right].
\end{aligned}$$

Then, since we have $\lim_{n\to\infty}(n\gamma_n)>(\mathcal{F}_a+3\alpha-2a-2g)$, the application of Lemma 1 gives

$$\Pi_n^4 \sum_{k=1}^n \Pi_k^{-4} \gamma_k^4 h_k^{-2} g_k^{-2} \mathbb{E}\left[K^2\left(\frac{\|\chi-\mathcal{X}_k\|}{h_k}\right) K^2\left(\frac{\|\chi-\mathcal{X}_k\|}{g_k}\right) Y_k^2\right]$$
$$= \frac{1}{4-(\mathcal{F}_a+3\alpha-2a-2g)} \frac{\gamma_n^3 F(h_n)}{h_n^2 g_n^2} K^2(0)\left(r^2(\chi)+\sigma_\varepsilon^2(\chi)\right) M_2\,[1+o(1)].$$

Moreover, we have

$$\mathbb{E}\left[K\left(\frac{\|\chi-\mathcal{X}_k\|}{h_k}\right) K\left(\frac{\|\chi-\mathcal{X}_k\|}{g_k}\right) Y_k\right]$$
$$= h_k F(h_k)\,\phi'(0)\left[K(1)\,K\left(\frac{h_k}{g_k}\right) - \int_0^1 \left(uK(u)\,K\left(\frac{h_k}{g_k}u\right)\right)' \tau_{h_k}(u)\,du\right] + o(h_k)$$
$$+ r(\chi)\,F(h_k)\left[K(1)\,K\left(\frac{h_k}{g_k}\right) - \int_0^1 \left(K(u)\,K\left(\frac{h_k}{g_k}u\right)\right)' \tau_{h_k}(u)\,du\right].$$

Then,

$$\Pi_n^4 \sum_{\substack{k,k'=1\\ k\neq k'}}^n \Pi_k^{-2}\Pi_{k'}^{-2}\gamma_k^2\gamma_{k'}^2 h_k^{-1}h_{k'}^{-1}g_k^{-1}g_{k'}^{-1}\mathbb{E}\left[K\left(\frac{\|\chi-\mathcal{X}_k\|}{h_k}\right) K\left(\frac{\|\chi-\mathcal{X}_k\|}{g_k}\right) Y_k\right]$$
$$\mathbb{E}\left[K\left(\frac{\|\chi-\mathcal{X}_{k'}\|}{h_{k'}}\right) K\left(\frac{\|\chi-\mathcal{X}_{k'}\|}{g_k}\right) Y_{k'}\right]$$
$$= \left\{\left[\frac{1}{2-(\mathcal{F}_a+\alpha-a-g)\,\xi}\right]^2 \frac{\gamma_n^2 F^2(h_n)}{g_n^2 h_n^2} - \frac{1}{4-(2\mathcal{F}_a+3\alpha-2g)\,\xi}\frac{\gamma_n^3 F^2(h_n)}{g_n^2}\right\}$$
$$r^2(\chi)\,K^2(0)\,M_1^2.$$

Then,

$$\begin{aligned}\mathbb{E}\left[\psi_{n,1}^2(\chi,h,g)\right] &= \frac{1}{4-(\mathcal{F}_a+3\alpha-2a-2g)}\frac{\gamma_n^3 F(h_n)}{h_n^2 g_n^2} K^2(0)\left(r^2(\chi)+\sigma_\varepsilon^2(\chi)\right) M_2 \\ &\quad + \left[\frac{1}{2-(\mathcal{F}_a+\alpha-a-g)\,\xi}\right]^2 \frac{\gamma_n^2 F^2(h_n)}{g_n^2 h_n^2} \\ &\quad \times r^2(\chi)\,K^2(0)\,M_1^2. \end{aligned} \tag{121}$$

Moreover, we have

$$\begin{aligned}\mathbb{E}\left[\psi_{n,2}^2(\chi,h,g)\right] &= \left[\frac{1}{2-(\mathcal{F}_a+\alpha-2a)}\right]\left[\frac{1}{2-(\alpha-2g)}\right]\frac{\gamma_n^2 F(h_n)}{h_n^2 g_n^2} \\ &\quad \times K^2(0)\left(r^2(\chi)+\sigma_\varepsilon^2(\chi)\right) M_2. \end{aligned} \tag{122}$$

Then, the combination of (120), (121), (122), (119), Proposition 1 and some classical computations gives Theorem 2.