Relative Errors in Central Limit Theorem for Student's t Statistic, with Applications

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This supplement gives the proofs of Theorem 1 and Propositions 1-3 in the official publication. Results (formulae) cited are along the lines of the official publication.

5 Proof of Theorem 1

Note that $\sigma_n^2 = \sum_i (X_i - \bar{X})^2 = V_n^2 - S_n^2/n$. It is readily seen from Theorem 2 that, for any $x \geq 0$,

$$\begin{aligned} P(U \geq x) &= P\big(S_n + c \geq x\sqrt{V_n^2 - S_n^2/n}\big) \geq P(S_n + c \geq x\,V_n) \\ &= \{1 - \Phi(2\gamma\, x)\}\,\Psi_{n,\gamma}(x)\,\exp\big(O_1\,\Delta_{n,x}\big)\,\{1 + O_2\,(1+x)\,\rho_n\}\,, \end{aligned} \tag{5.1}$$

which gives the lower bound of (2.1). As for the upper bound, we have,

$$\begin{aligned} P(U \geq x) &= P\big(S_n + c \geq x\sqrt{V_n^2 - S_n^2/n}\big) \\ &\leq P(|S_n| \geq t_0\,V_n) + P\big(S_n + c \geq x\,V_n\sqrt{1 - n^{-1}\,t_0^2}\big), \end{aligned} \tag{5.2}$$

where $t_0 = 12\,\max\big(x, \log n\big)$. Since $e^{-y} \leq 1 - y + 2y^{3/2}$ for $y \geq 0$, it follows easily that, for $0 \leq x \leq \rho_n^{-1}/256$,

$$\begin{aligned} P(V_n \leq 4\sqrt{n}/5) &= P\big(-V_n^2 \geq -16n/25\big) \\ &\leq e^{16x^2}\prod_{j=1}^{n} E\big\{\exp\big(-25\,x^2\,X_j^2/n\big)\big\} \\ &\leq e^{16x^2}\big(1 - 25\,x^2\,n^{-1} + 150\,x^3\,n^{-1}\,\rho_n\big)^n \leq e^{-7\,x^2}. \end{aligned}$$

This result, together with Lemma 6.4 of Jing et al. (2003), imply that,

$$\begin{aligned} P(|S_n| \geq t_0\,V_n) &\leq P\big\{|S_n| \geq \tfrac{1}{6}\,t_0\,(V_n + 4\sqrt{n})\big\} + P(V_n \leq 4\sqrt{n}/5) \\ &\leq 9\,\exp\big[-2\,\big\{\max(x, \log n)\big\}^2\big] \leq 9\,n^{-1/2}\,e^{-3\,x^2/2} \\ &\leq A\,\rho_n\,\{1 - \Phi(2\gamma\,x)\}\,\Psi_{n,\gamma}(x)\,, \end{aligned} \tag{5.3}$$

for $0 \le x \le \rho_n^{-1}/256$ and $|c| \le x\sqrt{n}/5$, where we have used (4.8).

In order to establish the upper bound for $P(S_n + c \ge x V_n \sqrt{1 - n^{-1} t_0^2})$, let $y_0 = x\sqrt{1 - n^{-1} t_0^2}$. Note that $x/2 \le y_0 \le 3x/2$ and $|y_0/x - 1| \le 24\, n^{-1} \max\{x^2, (\log n)^2\}$, for all sufficiently large n. Routine calculations, together with (4.8), imply that $\Delta_{n,y_0} \le 8\,\Delta_{n,x}$, $\Psi_{n,\gamma}(y_0) \le \Psi_{n,\gamma}(x) \exp(A\,\Delta_{n,x})$ and $1 - \Phi(2\gamma y_0) \le \{1 - \Phi(2\gamma x)\} \exp(A\,\Delta_{n,x})$, and hence, using Theorem 2, that

$$\begin{aligned} P\Big(S_n + c \ge x V_n \sqrt{1 - n^{-1} t_0^2}\Big) & \\ &\le \{1 - \Phi(2\,\gamma\, y_0)\}\, \Psi_{n,\gamma}(y_0)\, \exp(A\Delta_{n,y_0})\, \{1 + A\,(1 + y_0)\,\rho_n\} \\ &\le \{1 - \Phi(2\gamma\, x)\}\, \Psi_{n,\gamma}(x)\, \exp(A\Delta_{n,x})\, \{1 + A\,(1 + x)\,\rho_n\}\,, \end{aligned} \tag{5.4}$$

which yields the upper bound in (2.1). The proof Theorem 1 is now complete. □

6 Proof of Proposition 1

Write $\eta_j = 2\,h\,X_j - (h\,X_j)^2 + \xi_j$, where $\xi_j = \theta\, h^4\, X_j^4 I_{|X_j| \le \sqrt{n}\tau}$, and let $\zeta_1, \cdots, \zeta_n$ be independent random variables with ζ_j having distribution function $V_j(u)$ defined by

$$V_j(u) = E\big\{e^{\lambda \eta_j} I(\eta_j \le u)\big\} \big/ E e^{\lambda \eta_j}, \qquad \text{for} \quad j = 1, \cdots, n,$$

Set $m(\lambda) = E\zeta_1$, $\sigma^2(\lambda) = var(\zeta_1)$, $M_n^2(\lambda) = \sigma^2(\lambda)$,

$$G_n(t) = P\Big\{\frac{\sum_{j=1}^n (\zeta_j - E\zeta_j)}{M_n(\lambda)} \le t\Big\} \quad \text{and} \quad R_n(\lambda) = \frac{x^2 - n\, m(\lambda)}{M_n(\lambda)}.$$

We need the following lemmas before the proof of (4.9).

LEMMA 6.1. *Let $h = x/\sqrt{n}$, $EX = 0$, $EX^2 = 1$ and $E|X|^3 < \infty$. Then, for any $\lambda > 0$, $\theta > 0$ and $x \ge 0$,*

$$\begin{aligned} E e^{\lambda h X - \theta (hX)^2} &= 1 + (\lambda^2/2 - \theta) n^{-1} x^2 + (\lambda^3/6 - \lambda\theta) n^{-3/2} x^3 E X^3 \\ &\quad + A(\lambda, \theta) n^{-1}\, \Delta_{n,x}, \end{aligned} \tag{6.1}$$

where $|A(\lambda, \theta)| \le \max\{e^{\lambda^2/(4\theta)}, (\lambda + \theta)^3/6 + \theta^2/2 + (\lambda + \theta)^4 e^{\lambda}/24, (\lambda + \theta)(1 + \lambda)^2\}$.

Proof. Write $Y = X I_{|X| \le \sqrt{n}\tau}$, where $\tau = 1/(1 + x)$, $\xi = \lambda\, hX - \theta(hX)^2$ and

$$J_1(\lambda, \theta) = E\big(e^{\xi} - 1\big) I_{|X| > \sqrt{n}\tau}, \quad J_2(\lambda, \theta) = E\big(e^{\xi} - 1\big) I_{|X| \le \sqrt{n}\tau}.$$

Noting that $\lambda(hs) - \theta(hs)^2 \le \lambda^2/(4\theta)$ for $s \in R$, we get

$$|J_1(\lambda, \theta)| \le e^{\lambda^2/(4\theta)} P(|X| \ge \sqrt{n}\tau).$$

On the other hand, simple calculation shows that

$$\begin{aligned} E\xi I_{|X|\le\sqrt{n}\tau} &= \lambda h EY - \theta h^2 EY^2, \\ E\xi^2 I_{|X|\le\sqrt{n}\tau} &= \lambda^2 h^2 EY^2 - 2\lambda\theta h^3 EY^3 + \theta^2 h^4 EY^4, \\ E\xi^3 I_{|X|\le\sqrt{n}\tau} &= \lambda^3 h^3 EY^3 + \bar{A}(\lambda,\theta) h^4 EY^4, \end{aligned}$$

where $|\bar{A}(\lambda,\theta)| \le (\lambda+\theta)^3$. By virtue of these estimates and the inequality $|e^s - 1 - s - s^2/2 - s^3/6| \le |s|^4 e^{s\vee 0}/24$ for $s \in R$, it follows easily that

$$\begin{aligned} J_2(\lambda,\theta) &= E\xi I_{|X|\le\sqrt{n}\tau} + \tfrac{1}{2} E\xi^2 I_{|X|\le\sqrt{n}\tau} + \tfrac{1}{6} E\xi^3 I_{|X|\le\sqrt{n}\tau} \\ &\quad + (1/24) O_1 e^{\lambda} E|\xi|^4 I_{|X|\le\sqrt{n}\tau} \\ &= \lambda h EY + (\lambda^2/2 - \theta) h^2 EY^2 + (\lambda^3/6 - \lambda\,\theta)\, h^3 EY^3 + A_1(\lambda,\theta) e^{\lambda} h^4 E|Y|^4 \\ &= (\lambda^2/2 - \theta) n^{-1} x^2 + (\lambda^3/6 - \lambda\theta) n^{-3/2} x^3 EX^3 + A_2(\lambda,\theta) n^{-1} \Delta_{n,x}, \end{aligned}$$

where $|O_1| \le 1$, $|A_1(\lambda,\theta)| \le (\lambda+\theta)^3/6 + \theta^2/2 + (\lambda+\theta)^4 e^{\lambda}/24$ and

$$|A_2(\lambda,\theta)| \le \max\{|A_1(\lambda,\theta)|, (\lambda+\theta)(1+\lambda)^2\}.$$

Combining the bounds on $J_1(\lambda,\theta)$ and $J_2(\lambda,\theta)$, we obtain (6.1). The proof of Lemma 6.1 is complete. □

LEMMA 6.2. *For any $\lambda > 0$, we have*

$$E e^{\lambda\,\eta_1} = 1 + (2\,\lambda^2 - \lambda)\, h^2 + \lambda^2 (4\lambda/3 - 2)\, h^3\, EX^3 + C_1\, n^{-1}\, \Delta_{n,x}, \tag{6.2}$$

$$E\,\eta_1\, e^{\lambda\,\eta_1} = (4\lambda - 1) h^2 + 4\,(\lambda^2 - \lambda)\, h^3\, EX^3 + C_2\, n^{-1}\, \Delta_{n,x}, \tag{6.3}$$

$$E\,\eta_1^2\, e^{\lambda\,\eta_1} = 4h^2 + C_3\, h^3\, E|X|^3, \tag{6.4}$$

$$E|\eta_1|^3 e^{\lambda\,\eta_1} \le (27 + 4A^3)\, e^{\lambda(1+A)}\, h^3\, E|X|^3, \tag{6.5}$$

where C_1, C_2 and C_3 are constants depending only on λ, and C_1, C_2 and C_3 are bounded by an absolute constant A_1 whenever $1/4 \le \lambda \le 3/4$.

Proof. We only prove (6.3). The others are similar and the details are omitted. Write $\eta_1^* = 2\,h\,X_1 - (h\,X_1)^2$. By the same arguments as in proof of Lemma 6.1, we have

$$E\,\eta_1^*\, e^{\lambda\,\eta_1^*} = (4\lambda - 1) h^2 + 4\,(\lambda^2 - \lambda)\, h^3\, EX^3 + O(\lambda)\, n^{-1}\, \Delta_{n,x}, \tag{6.6}$$

where $|O(\lambda)| \le \max\{3(1+e^{\lambda}), \lambda + 14\lambda^2(1+e^{\lambda})\}$. The property (6.3) now follows easily from (6.6) and

$$\begin{aligned} \left|E\,\eta_1\, e^{\lambda\,\eta_1} - E\,\eta_1^*\, e^{\lambda\,\eta_1^*}\right| &\le \left|E\,\eta_1^*\, e^{\lambda\,\eta_1^*}(e^{\lambda\xi_1} - 1)\right| + E\,|\xi_1|\, e^{\lambda\,(\eta_1^* + \xi_1)} \\ &\le A\big\{\lambda \max(e^{\lambda}, e\lambda^{-1}) + 1\big\}\, e^{\lambda(A+1)} x^4 n^{-2} EX^4 I_{|X|\le\sqrt{n}\tau}, \end{aligned}$$

where we have used the facts that $\eta_1^* \le 1$ and

$$|\eta_1^*| e^{\lambda \eta_1^*} \le \lambda^{-1} \sup_{s \le \lambda} |s| e^s \le \lambda^{-1} \max(\lambda e^{\lambda}, e).$$

$\square$

We now turn to the proof of (4.9). By virtue of Lemma 6.2, tedious but simple calculations show that, for any $1/4 \le \lambda \le 3/4$ and $4 \le x \le \rho_n^{-1}/\max\{16, A_1\}$ where A_1 is as in Lemma 6.2,

$$\begin{aligned} Ee^{\lambda\,\eta_j} &= \exp\left\{ (2\,\lambda^2 - \lambda)\,h^2 + \lambda^2 (4\lambda/3 - 2)\,h^3\,EX^3 + O_1^*\,n^{-1}\,\Delta_{n,x} \right\}, & (6.7)\\ m(\lambda) &= E\eta_j e^{\lambda \eta_j} \big/ E e^{\lambda \eta_j} \\ &= (4\lambda - 1) h^2 + 4\,(\lambda^2 - \lambda)\,h^3\,EX^3 + O_2^*\,n^{-1}\,\Delta_{n,x}, & (6.8)\\ \sigma^2(\lambda) &= E\eta_j^2 e^{\lambda \eta_j} \big/ E e^{\lambda \eta_j} - \left(E\eta_j e^{\lambda \eta_j} \big/ E e^{\lambda \eta_j} \right)^2 \\ &= 4h^2 + O_3^*\,n^{-1}\,x^3\,\rho_n, & (6.9)\\ E|\zeta_j|^3 &\le E|\eta_j|^3 e^{\lambda \eta_j} \Big/ E e^{\lambda \eta_j} \le A\,n^{-1}\,x^3\,\rho_n, & (6.10) \end{aligned}$$

where O_1^*, O_2^* and O_3^* are bounded by an absolute constant A_2. We next let λ_0 be the solution of the equation

$$m(\lambda_0) = (x^2 + \delta_{1n})/n. \tag{6.11}$$

By recalling that ζ_1 is a non-generate random variable, we have $m'(\lambda) = \sigma^2(\lambda) > 0$, and hence $m(\lambda)$ is a strict increasing function for $\lambda > 0$. Also note that, by (6.8) and $|\delta_{1n}| \le x^2/2$,

$$m(1/4) \le x^2/(2n) \le m(\lambda_0) \le 3x^2/(2n) \le m(3/4),$$

for $4 \le x \le \rho_n^{-1}/A_0$, with $A_0 = \max\{16, A_1, 2A_2\}$. These facts, together with (6.8) again, imply that λ_0 is the unique solution of (6.11), $1/4 \le \lambda_0 \le 3/4$ and

$$|\lambda_0 - 1/2 + (\lambda_0^2 - \lambda_0) h EX^3 - \delta_{1n}/(4x^2)| \le A_2\,\Delta_{n,x}/(4x^2), \tag{6.12}$$

for $4 \le x \le \rho_n^{-1}/A_0$. By using (6.12) and the fact that

$$(E|X|^3)^2 \le 2(E|X|^3 I_{|X| \ge \sqrt{n}\tau})^2 + 2E|X|^4 I_{|X| \le \sqrt{n}\tau}, \tag{6.13}$$

we have that for $4 \le x \le \rho_n^{-1}/A_0$,

$$\begin{aligned} \left| \lambda_0 - \lambda_1 - \beta\,h\,EX^3 \right| &\le A_2\,\Delta_{n,x}/(4x^2) + 3|\lambda_0 - \lambda_1|\,h\,E|X|^3 \\ &\le A_2\,\Delta_{n,x}/(2x^2) + 3\,(h\,E|X|^3)^2 \\ &\le (A_2 + 3)\,\Delta_{n,x}/x^2, & (6.14) \end{aligned}$$

where $\lambda_1 = \frac{1}{2} + \delta_{1n}/(4x^2)$ and $\beta = \lambda_1 - \lambda_1^2$. Therefore, by using (6.7) and (6.13) and recalling $|\delta_{1n}| \le x^2/2$ [also $\delta_{1n} = 2(2\lambda_1 - 1)x^2$], $3/8 \le \lambda_1 \le 5/8$ and $\beta \le \lambda_1$, tedious but simple calculations show that, for $4 \le x \le \rho_n^{-1}/A_0$,

$$\begin{aligned} e^{-\lambda_0(x^2+\delta_{1n})} \prod_{j=1}^n E e^{\lambda_0 \eta_j} &= \exp\Big[-(\lambda_1 + \beta\, x\, EX^3/\sqrt{n})\delta_{1n} + 2\lambda_1(\lambda_1 - 1)x^2 \\ &\quad + \big\{2(2\lambda_1 - 1)\beta + \lambda_1^2(4\lambda_1/3 - 2)\big\} x^3 EX^3/\sqrt{n}\Big] \exp\big(O_1 \Delta_{n,x}\big), \\ &= \exp(-2\lambda_1^2 x^2)\, \Psi_{n,\lambda_1}\, \exp\big\{O_1 \Delta_{n,x}\big\}, \end{aligned} \tag{6.15}$$

where O_1 is bounded by an absolute constant. By (6.9), we also have

$$|M_n^2(\lambda_0)/x^2 - 4| \le A_2\, x\, \rho_n, \qquad 3.5x^4 \le M_n^2(\lambda_0) \le 4.5x^2, \tag{6.16}$$

for $4 \le x \le \rho_n^{-1}/A_0$.

We are now ready to prove (4.9). By the conjugate method and (6.15), we have that, for $4 \le x \le \rho_n^{-1}/A_0$,

$$\begin{aligned} P\big(2\,h\,S_n - h^2 V_n^2 + \theta\, h^4 Q_n \ge x^2 + \delta_{1n}\big) &= P\Big(\sum_{j=1}^n \eta_j \ge x^2 + \delta_{1n}\Big) \\ = \prod_{j=1}^n E e^{\lambda_0 \eta_j} &\int_{x^2+\delta_{1n}}^\infty e^{-\lambda_0 u} dP\Big(\sum_{j=1}^n \zeta_j \le u\Big), \\ = \prod_{j=1}^n E e^{\lambda_0 \eta_j}\, &e^{-\lambda_0(x^2+\delta_{1n})} \Big[\int_0^\infty e^{-\lambda_0 M_n(\lambda_0) v} d\big\{G_n(v) - \Phi(v)\big\} \\ &\quad + \int_0^\infty e^{-\lambda_0 M_n(\lambda_0)\, v} d\Phi(v)\Big] \\ := \exp(-2\lambda_1^2 x^2)\, &\Psi_{n,\lambda_1}\, \exp\big(O_1 \Delta_{n,x}\big)\, \big\{I_1(\lambda_0) + I_2(\lambda_0)\big\}. \end{aligned} \tag{6.17}$$

To estimate $I_2(\lambda_0)$, write $\psi(x) = \big\{1 - \Phi(x)\big\}/\Phi'(x) = e^{x^2/2}\int_x^\infty e^{-y^2/2} dy$. Clearly, for $y \ge 3/2$,

$$\frac{1}{2y} \le \psi(y) \le \frac{1}{y} \quad \text{and} \quad |\psi'(y)| = |y\psi(y) - 1| \le y^{-2}.$$

These estimates, together with the facts that $3x/4 \le \lambda_0 M_n(\lambda_0) \le 5x/4$ by (6.16) and $1/4 \le \lambda_0 \le 3/4$, and

$$|\lambda_0 M_n(\lambda_0) - 2\lambda_1\, x| \le M_n(\lambda_0)|\lambda_0 - \lambda_1| + \lambda_1\, |M_n(h_0) - 2x| \le A\, x^2\, \rho_n,$$

by (6.14) and (6.16), imply that for $4 \le x \le \rho_n^{-1}/A_0$,

$$\begin{aligned} I_2(\lambda_0) &= \psi\{\lambda_0 M_n(\lambda_0)\}/\sqrt{2\pi} \\ &= \Big[\psi(2\lambda_1 x) + \psi'(\theta^*)\left\{\lambda_0 M_n(\lambda_0) - 2\lambda_1\, x\right\}\Big]/\sqrt{2\pi}, \qquad \text{[where } \theta^* \in (3x/4, 5x/4)] \\ &= \frac{\psi(2\lambda_1\, x)}{\sqrt{2\pi}}\left(1 + O_2\, x\, \rho_n\right) \\ &= e^{2\lambda_1^2 x^2}\left\{1 - \Phi(2\lambda_1 x)\right\}\left(1 + O_2\, x\, \rho_n\,\right), \end{aligned} \tag{6.18}$$

where $|O_2| \le A$. As for $I_1(\lambda_0)$, by (6.10) and (6.16), integration by parts and the Berry-Esseen theorem, we get

$$|I_1(\lambda_0)| \le 2 \sup_v |G_n(v) - \Phi(v)| \le 4 M_n^{-3}(\lambda_0) \sum_{j=1}^n E|\zeta_j|^3 \le A\, \rho_n.$$

This implies that, for $x \ge 4$,

$$I_1(\lambda_0) \;=\; O_3\, x\, \rho_n\, e^{2\lambda_1^2\, x^2}\left\{1 - \Phi(2\lambda_1 x)\right\}, \tag{6.19}$$

since $7/16 \le \lambda_1 \le 9/16$, where $|O_3| \le A$. Taking the estimates (6.18) and (6.19) into (6.17), we obtain the required (4.9). The proof of Proposition 1 is now complete. $\square$

7 Proof of Proposition 2

We only prove (4.10). The property (4.11) follows from (4.10) and the similar arguments as in the proof of (5.4).

We first assume $4 \le x \le \rho_n^{-1/2}/4$. Let $\Omega_n = (1 - x^{-1}/2, 1 + x^{-1}/2)$ and $\lambda_1 = \frac{1}{2}\{1 + \delta_{1n}/(2x^2)\}$, where $\delta_{1n} = -2 + 2h\delta_{2n}$. Note that $|\delta_{1n}| \le x^2/2$ whenever $|\delta_{2n}| \le x\sqrt{n}/4$ and $|\lambda_1 - \lambda_2| \le 1/x^2$, where $\lambda_2 = \frac{1}{2}\left\{1 + \delta_{2n}/(x\sqrt{n})\right\}$. It is readily seen from (4.5) and (4.7) with $s = 1$ and Proposition 1 with $\theta = 0$ that

$$\begin{aligned} P\left(S_n \ge xV_n + \delta_{2n},\ \frac{V_n}{\sqrt{n}} \in \Omega_n\right) &\le P\big(2h\, S_n - h^2\, V_n^2 \ge x^2 - 2 + 2\, h\delta_{2n}\big) \\ &\le \left\{1 - \Phi(2\lambda_2\, x)\right\} \Psi_{n,\lambda_2}(x)\, \exp\left\{A\,(\Delta_{n,x} + 1)\right\}. \end{aligned}$$

So it suffices to show that, for $4 \le x \le \rho_n^{-1/2}/4$,

$$I_j \;\le\; \left\{1 - \Phi(2\lambda_2\, x)\right\} \Psi_{n,\lambda_2}(x)\, \exp\left\{A\,(\Delta_{n,x} + 1)\right\}, \qquad j = 1 \text{ and } 2, \tag{7.1}$$

where

$$I_1 = P\left\{S_n \ge xV_n + \delta_{2n}, V_n^2 \ge n(1+x^{-1}/2)\right\},$$
$$I_2 = P\left\{S_n \ge xV_n + \delta_{2n}, V_n^2 \le n\big(1-x^{-1}/2\big)\right\}.$$

To estimate I_2, write $B_1 = \{(s,t) : s \ge x\sqrt{t} + 2\lambda_2 h\delta_{2n}, 0 \le t \le 4\lambda_2^2(x^2 - x/2)\}$. By noting that $\sqrt{1-x^{-1}/2} \ge 1 - x^{-1}/4 - x^{-2}/4$ since $x \ge 4$, it follows easily from Lemma 6.1 with $\lambda = 2\lambda_2$ and $\theta = 4\lambda_2^2$, and then (4.6) with $t_0 = \lambda_2$ that, for $4 \le x \le \rho_n^{-1/2}/4$,

$$\begin{aligned}
I_2 &= P\big\{(2\lambda_2 hS_n, 4\lambda_2^2h^2V_n^2) \in B_1\big\} \\
&\le E\exp\big(2\lambda_2 hS_n - 8\lambda_2^2\,h^2V_n^2\big)\exp\big\{-\inf_{(s,t)\in B_1}(s-2\,t)\big\} \\
&\le E\exp\big(2\lambda_2 hS_n - 8\lambda_2^2\,h^2V_n^2\big)\exp\big\{-2\lambda_2 h\,\delta_{2n} - 2\lambda_2 x\sqrt{x^2-x/2} + 8\lambda_2^2\big(x^2-x/2\big)\big\} \\
&\le e\,\exp\big\{-2\lambda_2 h\,\delta_{2n} + 2(\lambda_2^2-\lambda_2)x^2 - (8/3)\lambda_2^3x^3EX^3/\sqrt{n} + \lambda_2(1/2-4\lambda_2)x + A\Delta_{n,x}\big\} \\
&\le e\,\exp\big(-2\lambda_2^2x^2\big)\,\Psi_{n,\lambda_2}(x)\,\exp\big\{2\lambda_2^2(1-2\lambda_2)x^3\rho_n - x/4 + A\Delta_{n,x}\big\} \\
&\le e\,\exp\big(-2\lambda_2^2x^2\big)\Psi_{n,\lambda_2}(x)\exp\big(-x/8 + A\Delta_{n,x}\big) \\
&\le 8\sqrt{2\pi}e\,\big\{1-\Phi(2\lambda_2 x)\big\}\,\Psi_{n,\lambda_2}(x)\exp(A\,\Delta_{n,x}), \qquad (7.2)
\end{aligned}$$

where we have used the facts that $h\delta_{2n} = (2\lambda_2 - 1)x^2$ and $1/3 \le \lambda_2 \le 2/3$.

As for I_1, we have

$$\begin{aligned}
I_1 &\le P\left\{S_n \ge xV_n + h\delta_{2n}, n\big(1+x^{-1}/2\big) \le V_n^2 \le 9n\right\} \\
&\quad + P(S_n \ge xV_n + h\delta_{2n}, V_n^2 \ge 9n) \\
&= I_1^{(1)} + I_1^{(2)}, \qquad \textit{say}. \qquad (7.3)
\end{aligned}$$

Similarly to the proof of (7.2), by letting $B_2 = \{(s,t) : s \ge x\sqrt{t} + 2\lambda_2 h\delta_{2n}, 4\lambda_2^2(x^2 + x^{-1}/2) \le t \le 36\,\lambda_2^2x^2\}$, we get, for $4 \le x \le \rho_n^{-1/2}/4$,

$$\begin{aligned}
I_1^{(1)} &= P\big\{(2\lambda_2 hS_n, 4\lambda_2^2h^2V_n^2) \in B_2\big\} \\
&\le E\exp\big(2\lambda_2 hS_n - 2\lambda_2^2\,h^2V_n^2/3\big)\exp\big\{-\inf_{(s,t)\in B_2}(s - t/6)\big\} \\
&\le E\exp\big(2\lambda_2 hS_n - 2\lambda_2^2\,h^2V_n^2/3\big)\exp\big\{-2\lambda_2 h\,\delta_{2n} - 2\lambda_2 x\sqrt{x^2+x/2} + 2\lambda_2^2\big(x^2+x/2\big)/3\big\} \\
&\le e\,\exp\big(-2\lambda_2^2x^2\big)\,\Psi_{n,\lambda_2}(x)\,\exp\big\{2\lambda_2^2(1-\lambda_2/3)x^3\rho_n - x/6 + A\Delta_{n,x}\big\} \\
&\le 4\sqrt{2\pi}e\,\big\{1-\Phi(2\lambda_2 x)\big\}\,\Psi_{n,\lambda_2}(x)\exp(A\,\Delta_{n,x}),
\end{aligned}$$

where we have used the fact that $\sqrt{1+x^{-1}/2} \ge 1 + x^{-1}/4 - x^{-2}/16$. On the other hand, by letting $\hat{S}_n = \sum_{j=1}^n X_j I_{|X_j| \le 25\sqrt{n}/x}$, as in the proof of Lemma 3 of Shao (1999) with minor modifications, we have that,

$$
\begin{aligned}
I_1^{(2)} &\le P(\hat S_n \ge xV_n/2 + h\delta_{2n}, V_n^2 \ge 9n) + P\Big(\sum_{j=1}^n X_j I_{|X_j|\ge 25\sqrt{n}/x} \ge x\,V_n/2\Big) \\
&\le P(\hat S_n \ge 3\,x\sqrt{n}/2 + h\delta_{2n}) + P\Big(\sum_{j=1}^n I_{|X_j|\ge 25\sqrt{n}/x} \ge x^2\,/4\Big) \\
&\le e^{-h\delta_{2n}-x^2} + e^{-2x^2} \\
&\le A\big\{1-\Phi(2\lambda_2 x)\big\}\,\Psi_{n,\lambda_2}(x)\exp(A\,\Delta_{n,x}).
\end{aligned}
$$

Taking the estimates for $I_1^{(1)}$ and $I_1^{(2)}$ into (7.3), we obtain that

$$
I_1 \le (4e+1)\big\{1-\Phi(2\lambda_2 x)\big\}\Psi_{n,\lambda_2}\exp(A\Delta_{n,x}).
$$

This proves (7.1) for $j=1$ and 2, hence completes the proof of Proposition 2 for $4 \le x \le \rho_n^{-1/2}/4$.

We next assume $\rho_n^{-1/2}/4 \le x \le \rho_n^{-1}/A_0$, where A_0 is as in Proposition 1 and $A_0 \ge 256$. In this case, let $\Omega_n = (1-4\Delta_{n,x}^{1/2}/x, 1+4\Delta_{n,x}^{1/2}/x)$ and $\lambda_1 = 1/2 + \delta_{1n}/4x^2$ where $\delta_{1n} = -16\Delta_{n,x} + 2h\delta_{2n}$. Note that $\Delta_{n,x} \le x^2/128$ and recall $|\delta_{2n}| \le x\sqrt{n}/4$. We obtain $|\lambda_1 - \lambda_2| \le 4\Delta_{n,x}/x^2$ and $|\delta_{1n}| \le x^2/2$. By virtue of these facts, it follows from (4.5) and (4.7) with $s=0$ and Proposition 1 with $\theta = 0$ that, for $4 \le x \le \rho_n^{-1}/A_0$,

$$
\begin{aligned}
P\left(S_n \ge xV_n + \delta_{2n},\ \frac{V_n}{\sqrt{n}} \in \Omega_n\right) &\le P\big(2h\,S_n - h^2\,V_n^2 \ge x^2 - 16\Delta_{n,x} + 2\,h\delta_{2n}\big) \\
&\le \big\{1-\Phi(2\lambda_1\,x)\big\}\,\Psi_{n,\lambda_1}(x)\,\exp\big(A\,\Delta_{n,x}\big)(1+Ax\rho_n) \\
&\le A\big\{1-\Phi(2\lambda_2\,x)\big\}\,\Psi_{n,\lambda_2}(x)\,\exp\big(A\,\Delta_{n,x}\big).
\end{aligned}
$$

Now we only need to show that, for $\rho_n^{-1/2}/4 \le x \le \rho_n^{-1}/A_0$,

$$
I_j^* \;\le\; \big\{1-\Phi(2\lambda_2\,x)\big\}\,\Psi_{n,\lambda_2}(x)\,\exp\big(A\,\Delta_{n,x}\big), \qquad j=1 \text{ and } 2, \tag{7.4}
$$

where

$$
\begin{aligned}
I_1^* &= P\big\{S_n \ge xV_n + \delta_{2n}, V_n^2 \ge n(1+4\Delta_{n,x}^{1/2}/x)\big\}, \\
I_2^* &= P\big\{S_n \ge xV_n + \delta_{2n}, V_n^2 \le n\big(1-4\Delta_{n,x}^{1/2}/x\big)\big\}.
\end{aligned}
$$

The proof of (7.4) is similar to (7.1). So we only give a outline for $j=2$. Indeed, by letting $B_1^* = \{(s,t) : s \ge x\sqrt{t} + 2\lambda_2 h\delta_{2n}, 0 \le t \le 4\lambda_2^2(x^2 - 4x\Delta_{n,x}^{1/2})\}$, as in the proof of

(7.2), we obtain

$$\begin{aligned}
I_2^* &\le E\exp\Big(2\lambda_2 h S_n - 8\lambda_2^2 h^2 V_n^2\Big)\exp\Big\{-2\lambda_2 h\,\delta_{2n} \\
&\qquad -2\lambda_2 x\sqrt{x^2-4x\Delta_{n,x}^{1/2}} + 8\lambda_2^2\big(x^2-4x\Delta_{n,x}^{1/2}\big)\Big\} \\
&\le e\,\exp\big\{-2\lambda_2 h\,\delta_{2n} + 2(\lambda_2^2-\lambda_2)x^2 - (8/3)\lambda_2^3 x^3 EX^3/\sqrt{n} \\
&\qquad +(-32\lambda_2^2+4\lambda_2)x\Delta_{n,x}^{1/2} + A\Delta_{n,x}\big\} \\
&\le e\,\exp\big(-2\lambda_2^2x^2\big)\Psi_{n,\lambda_2}(x)\exp\Big(2x^3\rho_n - 3x\,\Delta_{n,x}^{1/2} + A\Delta_{n,x}\Big) \\
&\le 32e^2\,\big\{1-\Phi(2\lambda_2 x)\big\}\,\Psi_{n,\lambda_2}(x)\exp(A\,\Delta_{n,x}), \qquad (7.5)
\end{aligned}$$

where we have used the fact that $2x^3\rho_n - 3x\,\Delta_{n,x}^{1/2} \le -x/32$, since, by (6.13),

$$x/16 \le x^3\,\rho_n = n^{-1/2}x^3E|X|^3 \le \sqrt{2}\big(\Delta_{n,x}^2 + x^2\Delta_{n,x}\big)^{1/2} \le 2x\,\Delta_{n,x}^{1/2},$$

whenever $x \ge \rho_n^{-1/2}/4$ and $\Delta_{n,x} \le x^2/128$. This gives the proof of Proposition 2 for $\rho_n^{-1/2}/4 \le x \le \rho_n^{-1}/A_0$. The proof of Proposition 2 is now complete. □

8 Proof of Proposition 3

The proof of this proposition is similar to that of Theorem 2.2 in Wang (2005). We only provide a outline for the difference. Define notations $I^-(y)$, $J^-(y)$ and $\mathcal{L}_n(y)$ as in Lemmas 5.4 and 5.5 of Wang (2005). It follows (5.17) and Lemmas 5.4–5.6 in Wang (2005) that there exists an absolute constant A_0 such that, for $4 \le x \le \rho_n^{-1}/A_0$ and $y_0 = x + \delta_{3n}$,

$$\begin{aligned}
P\Big(T_n + \Delta_{n,n} \ge x + \delta_{3n}\Big) &\le \tfrac{1}{2}\big\{I^-(y_0) + 1 - J^-(y_0)\big\} \\
&\le 1-\Phi(y_0) + \mathcal{L}_n(y_0) + A\big\{\rho_n e^{-y_0^2/2} + (x\rho_n)^{3/2}\big\} \\
&\le 1-\Phi(y_0) + \frac{EX^3}{\sqrt{2\pi n}}\Big(\frac{y_0^2}{6} - \frac{y_0 x}{2}\Big)e^{-y_0^2/2} \\
&\qquad + A\big\{(\rho_n + \Delta_{n,x}/x)\,e^{-y_0^2/2} + (x\rho_n)^{3/2}\big\}. \qquad (8.1)
\end{aligned}$$

Recalling $2x/3 \le y_0 \le 4x/3$ and using (4.8), we have, for $x \ge 4$ and $k = 1, 2$,

$$\Big|y_0^{k+1}\big\{1-\Phi(y_0)\big\} - \frac{y_0^k}{\sqrt{2\pi}}e^{-y_0^2/2}\Big| \le \frac{x^{k-2}}{\sqrt{2\pi}}e^{-y_0^2/2}$$

and $e^{-y_0^2/2} \le \frac{4}{3}\sqrt{2\pi}\, x\,\big\{1-\Phi(y_0)\big\}$. By virtue of these estimates and (8.1),

$$
\begin{aligned}
&P\Big(T_n + \Delta_{n,n} \ge x + \delta_{3n}\Big) \\
&\quad \le \ \big\{1-\Phi(y_0)\big\}\big\{1 + \frac{EX^3}{\sqrt{n}}\big(\frac{y_0^3}{6} - \frac{y_0^2 x}{2}\big) + A\big(x\rho_n + \Delta_{n,x}\big)\big\} + \ A\,(x\rho_n)^{3/2} \\
&\quad = \ \big\{1-\Phi(2\lambda_3\, x)\big\}\big\{1 + \lambda_3^2(4\lambda_3/3 - 2)\, x^3\, EX^3/\sqrt{n} \\
&\qquad\qquad\qquad\qquad + A\big(x\rho_n + \Delta_{n,x}\big)\big\} + \ A\,(x\rho_n)^{3/2} \\
&\quad \le \ \big\{1-\Phi(2\lambda_3\, x)\big\}\Psi_{n,\lambda_3}(x)\big\{1 + A(x\rho_n + \Delta_{n,x})\big\} + \ A\,(x\rho_n)^{3/2} \\
&\quad \le \ \big\{1-\Phi(2\lambda_3\, x)\big\}\Psi_{n,\lambda_3}(x)\exp\big(A\Delta_{n,x}\big)\big(1 + A\,x\rho_n\big) + \ A\,(x\rho_n)^{3/2},
\end{aligned}
$$

where $\lambda_3 = \frac{1}{2}\,(1 + \delta_{2n}/x)$. This proves Proposition 3. $\square$