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ADDITIVE FUNCTIONALS OF INFINITE-VARIANCE MOVING AVERAGES

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Abstract: We consider the asymptotic behavior of additive functionals of linear processes with infinite variance innovations. Applying the central limit theory for Markov chains, we establish asymptotic normality for short-range dependent processes. A non-central limit theorem is obtained when the processes are long-range dependent and the innovations are in the domain of attraction of stable laws.

Key words and phrases: Central limit theorem, empirical process, level crossings, linear process, long- and short-range dependence, Markov chain, martingale, stable distribution.

1. Introduction

Let $\{\varepsilon_i\}_{i\in\mathbb{Z}}$ be i.i.d. $c \times 1$ random vectors such that $\mathbb{E}(|\varepsilon_1|^\alpha) < \infty$ for some $0 < \alpha \le 2$ and $\mathbb{E}(\varepsilon_1) = 0$ if $1 \le \alpha \le 2$; let $\{\mathbf{a}_i\}_{i\ge 0}$ be $r \times c$ matrices such that $\sum_{k=1}^{\infty} |\mathbf{a}_k|^\alpha < \infty$, where $r, c \ge 1$ are fixed integers and $|\mathbf{a}| = \big(\sum_{k,l} a_{kl}^2\big)^{1/2}$ for matrix $\mathbf{a} = (a_{kl})$. Then the multiple linear process $X_n = \sum_{i=0}^{\infty} \mathbf{a}_i \varepsilon_{n-i}$ is well-defined (cf Corollary 5.1.3 in Chow and Teicher (1978)). In case $r = c = 1$ and ε_i is stable, then X_n is also stable. The important fractional ARIMA models with infinite variance innovations fall within this framework.

Let K be a measurable function for which $\mathbb{E}[K(X_1)] = 0$. Our goal is to investigate the limiting behavior of $S_n(K) = \sum_{i=1}^{n} K(X_i)$. This problem has received much attention recently; see Hsing (1999), Koul and Surgailis (2001), Surgailis (2002) and references therein for some historical developments. In Hsing (1999), the central limit problem is considered. When K belongs to certain classes of bounded functions, Koul and Surgailis (2001) and Surgailis (2002) obtained non-central limit theorems for univariate processes. As pointed in Koul and Surgailis (2001), the treatment of the asymptotic normality problem in Hsing (1999) is not rigorous. The central limit problem for multiple linear processes with heavy tails remains unsolved. Here we shall establish a central limit theorem under mild conditions.

The method adopted in this paper is based on the limit theory for additive functionals of Markov chains. Gordin and Lifsic (1978) introduced the method and an important generalization was given in Woodroofe (1992). The main idea is to write $K(X_n) = g(\mathbf{X}_n)$ and use the fact that the shift process $\mathbf{X}_n = (\ldots, \varepsilon_{n-1}, \varepsilon_n)$ is a Markov chain. Then the general limit theory developed for the additive functional $S_n(g) = \sum_{i=1}^n g(\xi_i)$ for Markov chains ξ_n is applicable. This theory can be applied to linear processes and enables one to launch a systematic study of their asymptotic behavior. See Wu and Mielniczuk (2001) and Wu (2002, 2003), where certain open problems in linear processes are circumvented. In this article, we apply this technique to processes with infinite variance innovations to establish a central limit theorem for short-range dependent processes and a non-central limit theorem for long-range dependent sequences when the innovations are in the domain of attraction of certain stable laws. Derivations of limit theorems based on such methods seem to be quite simple at a technical level. By comparison, the treatment in Hsing (1999) appears formidable.

This paper is organized as follows. Main results are stated in Section 2 and proved in Section 3. In Section 4, an open problem is proposed.

2. Main Results

Unless otherwise specified, we assume that $\mathbb{E}(|\varepsilon_1|^\alpha) < \infty$ holds for a fixed $0 < \alpha \le 2$ and $\mathbb{E}(\varepsilon_1) = 0$ if $1 \le \alpha \le 2$. For a random vector ξ, let $\|\xi\|_p = [\mathbb{E}(|\xi|^p)]^{1/p}$ and $\|\xi\| = \|\xi\|_2$. Recall the shift process $\mathbf{X}_n = (\ldots, \varepsilon_{n-1}, \varepsilon_n)$ and consider the projection operator $\mathcal{P}_k\xi = \mathbb{E}[\xi|\mathbf{X}_k] - \mathbb{E}[\xi|\mathbf{X}_{k-1}]$, $k \in \mathbb{Z}$. Define the truncated processes $\underline{X}_{n,k} = \sum_{i=-\infty}^k \mathbf{a}_{n-i}\varepsilon_i$ and $\overline{X}_{n,k} = X_n - \underline{X}_{n,k-1}$. As in Ho and Hsing (1997), let

$$K_n(w) = \mathbb{E}[K(w + \overline{X}_{n,1})] \text{ and } K_\infty(w) = \mathbb{E}[K(w + X_n)]. \tag{1}$$

Theorem 1 and Corollary 1 assert central and non-central limit theorems for $S_n(K)$, respectively. In Corollary 1 we assume that $\varepsilon_i \in \mathcal{D}(d)$, the domain of attraction of stable law with index $d \in (0, 2)$. Theorem 2 provides an approximation of $S_n(K)$ by the linear functional $\sum_{i=1}^n X_i$. An interesting application is given in Section 2.3, where a limit theorem for level crossings is derived.

2.1. A central limit theorem

For a function f define a local Lipschitz constant by

$$L_f(x) = \sup_{y \neq x: |y-x| \le 1} \frac{|f(y) - f(x)|}{|y - x|}.$$

Theorem 1. *Assume that* $\mathbb{E}[K^2(X_0)] < \infty$ *and that either* (*a*) *there exist* $q > 1$ *and* $\kappa \in \mathbb{N}$ *for which*

$$\sum_{n=1}^\infty |\mathbf{a}_n|^{\frac{\alpha}{2q}} < \infty, \tag{2}$$

$$F_q(\kappa) < \infty, \text{ where } F_q(n) = \mathbb{E}[|L_{K_n}(\underline{X}_{n,0})|^{2p} + |K_n(\underline{X}_{n,0})|^{2p}],\ p = q/(q-1), \quad (3)$$

or (b) there exists $\kappa \in \mathbb{N}$ *for which*

$$F_1(\kappa) < \infty, \text{ where } F_1(n) = \sup_x[|L_{K_n}(x)| + |K_n(x)|] < \infty, \quad (4)$$

$$\sum_{n=1}^{\infty} |\mathbf{a}_n|^{\frac{\alpha}{2}} < \infty. \quad (5)$$

Then $S_n(K)/\sqrt{n} \Rightarrow N(0,\sigma^2)$ *for some* $\sigma < \infty$.

Remark 1. Case (b) can be viewed as a limit of (a) when $q \downarrow 1$. Note that F_q defined in (3) is a non-increasing function in n. To see this, by the smoothing property, $K_{n+1}(x) = \mathbb{E}K_n(x + \mathbf{a}_n\varepsilon_1)$,

$$\begin{aligned} L_{K_{n+1}}(x) &= \sup_{y\neq x:|y-x|\le 1} \frac{|\mathbb{E}[K_n(y+\mathbf{a}_n\varepsilon_1) - K_n(x+\mathbf{a}_n\varepsilon_1)]|}{|y-x|} \\ &\le \mathbb{E}\left\{\sup_{y\neq x:|y-x|\le 1} \frac{|K_n(y+\mathbf{a}_n\varepsilon_1) - K_n(x+\mathbf{a}_n\varepsilon_1)|}{|y-x|}\right\} = \mathbb{E}L_{K_n}(x+\mathbf{a}_n\varepsilon_1) \end{aligned}$$

which, in conjunction with $|K_{n+1}(x)| \le \mathbb{E}|K_n(x + \mathbf{a}_n\varepsilon_1)|$ and the fact that $\underline{X}_{n+1,0} + \mathbf{a}_n\varepsilon_1$ and $\underline{X}_{n,0}$ are identically distributed, implies $F_q(n+1) \le F_q(n)$ by Jensen's inequality. Similarly, $F_1(\cdot)$ defined in (4) is non-increasing.

Remark 2. In Theorem 1, we need not assume $a_n = n^{-\gamma}\ell(n)$ in the univariate case $r = c = 1$, ℓ a slowly varying function, and the innovations ε_i need not be in $\mathcal{D}(d)$. If $r = c = 1$, $a_n = n^{-\gamma}\ell(n)$ and $\varepsilon_i \in \mathcal{D}(d)$, then (5) holds for $2/\gamma < \alpha < d$ provided $d\gamma > 2$, an assumption used in Hsing (1999) to obtain a central limit theorem for $S_n(K)$.

2.2. A non-central limit theorem

Suppose $K'_\infty(0) := (\partial K_\infty(w)/\partial w_1, \ldots, \partial K_\infty(w)/\partial w_r)|_{w=0}$ exists. Let $R_n = S_n(K) - T_n$, where the linear functional $T_n = K'_\infty(0)\sum_{i=1}^n X_i$. Set $A_n(\alpha) = \sum_{i=n}^\infty |\mathbf{a}_i|^\alpha$.

Condition 1. For sufficiently large n, $K_n(\cdot)$ is differentiable and there exist $q \ge 1$, $1 \le \nu < \alpha/q$ such that

$$\|K_{n-1}(\underline{X}_{n,1}) - K_n(\underline{X}_{n,0}) - K'_\infty(0)\mathbf{a}_{n-1}\varepsilon_1\|_\nu = \mathcal{O}[|\mathbf{a}_{n-1}|^{\frac{\alpha}{q\nu}} + |\mathbf{a}_{n-1}|A_n^{\frac{1}{q\nu}}(\alpha)]. \quad (6)$$

Intuitively, (6) asserts a first order expansion of Taylor's type.

Theorem 2. *Assume Condition 1 and* $|\mathbf{a}_n| = n^{-\gamma}\ell(n)$, *where* $1 > \gamma > 1/\alpha$. *Then for all* $\beta' < \beta_0 = \min\{1, \gamma\alpha/(q\nu), \gamma + (\gamma\alpha - 1)/(q\nu)\}$,

$$\|R_n\|_\nu = \mathcal{O}[n^{1-\beta'+1/\nu}]. \quad (7)$$

To state Corollary 1, we assume that $\varepsilon_i \in \mathcal{D}(d)$. Namely, there exists a slowly varying function ℓ_ε such that $\sum_{i=1}^n \varepsilon_i/[n^{\frac{1}{d}}\ell_\varepsilon(n)] \Rightarrow Z$, where Z is a stable law with index d.

Corollary 1. *Let $r = c = 1$. Assume Condition 1, $a_n = n^{-\gamma}\ell(n)$ and $-\beta_0 + 1/\nu < -\gamma + 1/d$. Then $S_n(K)/D_n \Rightarrow K'_\infty(0)Z$, where*

$$D_n = \frac{n^{1-\gamma+\frac{1}{d}}\ell(n)\ell_\varepsilon(n)}{1-\gamma}\left\{\int_0^\infty [t^{1-\gamma} - (\max(t-1,0))^{1-\gamma}]^d dt\right\}^{\frac{1}{d}}.$$

Proof. By Theorem 2, $R_n = o_P(D_n)$. Hence the Corollary follows from the classical result that $\sum_{i=1}^n X_i/D_n \Rightarrow Z$ (Avram and Taqqu (1986)).

Proposition 1. *Assume that there exist $\nu \in [1,\alpha)$, $q \in [1, \alpha/\nu)$ and $n \in \mathbb{N}$ such that*

$$F_q(n) := \mathbb{E}[|L_{K'_n}(\underline{X}_{n,0})|^{\nu p} + |K'_n(\underline{X}_{n,0})|^{\nu p} + |K_n(\underline{X}_{n,0})|^{\nu p} + |K_n(\underline{X}_{n,1})|^{\nu p}] < \infty, \tag{8}$$

where $p = q/(q-1)$. Here $F_1(n) < \infty$ is interpreted as $\sup_x |L_{K'_n}(x)| + |K'_n(x)| + |K_n(x)|] < \infty$. Then Condition 1 holds.

By Remark 1, $F_q(\cdot)$ is non-increasing. As illustrated in Example 2, Corollary 1 goes beyond earlier ones by allowing unbounded functionals K; see Koul and Surgailis (2001) and Surgailis (2002) where boundedness are required.

Example 1. Let $\alpha = d - y^2$, $\nu = d - y$ and $q = 1 + cy$, where $c < (\gamma d - 1)/(\gamma d^2)$ is a nonnegative number. Then as $y \downarrow 0$, $(-\gamma + 1/d) - (-\beta_0 + 1/\nu) = (-\gamma c + \gamma/d - 1/d^2)y + \mathcal{O}(y^2) > 0$ since $-\gamma c + \gamma/d - 1/d^2 > 0$. Corollary 1 is applicable if $y > 0$ is sufficiently small.

Example 2. Let $a_0 = 1$, $r = c = 1$ and let ε_i be i.i.d. symmetric-α-stable random variables with index $d \in (1,2)$ and characteristic function $\phi(t) = \mathbb{E}[\exp(t\sqrt{-1}\varepsilon_1)]$ $= \exp(-|t|^d)$ for $t \in \mathbb{R}$. Assume that there exists $C > 0$ such that for all $u \in \mathbb{R}$,

$$|K(u)| \leq C(1 + |u|)^{\alpha/(\nu p)}. \tag{9}$$

Then (8) holds with $n = 1$. Notice that (9) allows unbounded and discontinuous functions if $p < \infty$. To see that (9) implies (8), let f be the density of ε_i. By Theorem 2.4.2 in Ibragimov and Linnik (1971), $f(t) \sim c_d|t|^{-1-d}$ as $|t| \to \infty$, where $c_d = \pi^{-1}\Gamma(1+d)\sin(\pi d/2)$ and Γ is the Gamma function. It is easily seen that a similar argument also implies

$$|f'(t)| \sim c_d(1+d)|t|^{-2-d} \text{ and } |f''(t)| \sim c_d(1+d)(2+d)|t|^{-3-d} \tag{10}$$

as $|t| \to \infty$. Observe that $K_1(x) = \int_{\mathbb{R}} K(w+x)f(x)dx = \int_{\mathbb{R}} K(u)f(u-x)du$ and $K_1'(x) = -\int_{\mathbb{R}} K(u)f'(u-x)du$. Let $g(x;1) = \sup_{|y|\le 1} |g(x+y)|$ be the maximal function of g. Using $1+|x+v| \le (1+|x|)(1+|v|)$, we have

$$L_{K_1'}(x) \le \int_{\mathbb{R}} |K(u)| \sup_{|y-x|\le 1} \frac{|f'(u-y)-f'(u-x)|}{|x-y|} du \le \int_{\mathbb{R}} |K(x+v)||f''(v;1)|dv$$
$$\le C(1+|x|)^{\alpha/(\nu p)} \int_{\mathbb{R}} (1+|v|)^{\alpha/(\nu p)} |f''(v;1)|dv \le C_1(1+|x|)^{\alpha/(\nu p)}$$

in view of (10), where C_1 is a constant. Similarly, $|K_1'(x)| \le C_2(1+|x|)^{\alpha/(\nu p)}$ and $|K_1(x)| \le C_3(1+|x|)^{\alpha/(\nu p)}$ hold for some constants C_2 and C_3. Thus (8) holds since $\alpha < d$.

2.3. Level crossing analysis

Consider the one dimensional process $X_n = \sum_{i=0}^{\infty} a_i \varepsilon_{n-i}$, where $\varepsilon_i \in \mathcal{D}(d)$ and $a_n = n^{-\gamma}\ell(n)$ with $\gamma > 1/\alpha$ and $1 < d < 2$. Corollary 1 gives the asymptotic distribution for the instantaneous filter $S_n(K) = \sum_{i=1}^n K(X_i)$. As a simple non-instantaneous filter, we discuss the level crossing statistics $N_n = \sum_{t=1}^n \mathbf{1}_{(X_{t-1}-l)(X_t-l)\le 0}$ and $M_n^{(\tau)} = \sum_{t=1}^n \mathbf{1}_{(\Delta^\tau X_t)(\Delta^\tau X_{t-1})\le 0}$, where $\Delta X_t = X_t - X_{t-1}$ is the difference operator and $\Delta^\tau X_t = \Delta \ldots \Delta X_t$ for $\tau \in \mathbb{N}$. Note that N_n counts how many times the process $\{X_t\}_{t=1}^n$ crosses level l, and $M_n^{(\tau)}$ is the τth order crossing count. In particular, $M_n^{(1)}$ is the number of local extremes of the process $\{X_t\}_{t=1}^n$. In certain engineering problems, it is popular to conduct analysis based on crossing counts; see Kedem (1994) for more details. Limit theorems for N_n and $M_n^{(\tau)}$ are useful for related statistical inference. A central limit theorem for N_n is derived in Wu (2002), where $l = 0$ and $\{X_n\}$ is a short-range dependent linear process with finite variance.

For $r \ge 1$, $(X_n, X_{n-1}, \ldots, X_{n-r+1})'$ can be viewed as a multiple linear process $Y_n = \sum_{i=0}^{\infty} \mathbf{a}_i \varepsilon_{n-i}$, where $\mathbf{a}_i = (a_i, a_{i-1}, \ldots, a_{i-r+1})'$ and $a_i = 0$ for $i < 0$. Let $F_n(u,v)$ be the joint distribution function of $(\overline{X}_{n,0}, \overline{X}_{n-1,0})$ and $f_n(u,v)$ its density; let $F(u)$ and $F(u,v)$ be the distribution functions of X_n and $(X_n, X_{n-1})'$, respectively. Write $f^{(10)}(u,v) = \partial f(u,v)/\partial u$, $f^{(01)}(u,v) = \partial f(u,v)/\partial v$ and $f(u) = dF(u)/du$. Recall Corollary 1 for the definition of D_n.

Corollary 2. *Let $\gamma > 1/\alpha$ and $1 < d < 2$.* (i) *If there exists $\beta' > 1$ such that $|a_n - a_{n+1}| = \mathcal{O}(n^{-\beta'})$, then for all $\tau \ge 1$, $[M_n^{(\tau)} - \mathbb{E}M_n^{(\tau)}]/\sqrt{n} \Rightarrow N(0, \sigma_\tau^2)$ for some $\sigma_\tau^2 < \infty$.* (ii) *If $\gamma > 2/\alpha$, then $(N_n - \mathbb{E}N_n)/\sqrt{n} \Rightarrow N(0,\sigma^2)$.* (iii) *If $\gamma < 1$, then*

$$\frac{N_n - \mathbb{E}N_n}{D_n} \Rightarrow 2[f^{(01)}(l,l) + f^{(10)}(l,l) - f(l)]Z. \tag{11}$$

Proof. Consider (iii) first. Let $f_n^{(10)}(u,v) = \partial f_n(u,v)/\partial u$ and $f_n^{(01)}(u,v) = \partial f_n(u,v)/\partial v$. By properties of the characteristic function of $\varepsilon_i \in \mathcal{D}(d)$, it follows from the inversion formula that there exist $n_0 \in \mathbb{N}$ and $C < \infty$ for which

$$\sup_{n\geq n_0} \sup_{u,v\in\mathbb{R}} [f_n(u,v) + |f_n^{(01)}(u,v)| + |f_n^{(10)}(u,v)|] < C. \tag{12}$$

By Proposition 1, Theorem 2 can be applied to the functional $K(x_1,x_2) = \mathbf{1}_{(x_1-l)(x_2-l)\leq 0} - \mathbb{E}\mathbf{1}_{(X_0-l)(X_1-l)\leq 0}$. Observe that $K_\infty(x_1,x_2) = F(l-x_1) + F(l-x_2) - 2F(l-x_1, l-x_2) - 2[F(l) - F(l,l)]$. Then $K'_\infty(0) = (-f(l) + 2f^{(10)}(l,l), -f(l) + 2f^{(01)}(l,l))'$. Using the same argument as in Corollary 1, (11) follows from $K'_\infty(0)\sum_{i=1}^n Y_i/D_n \Rightarrow 2[f^{(01)}(l,l) + f^{(10)}(l,l) - f(l)]Z$.

The cases (i) and (ii) similarly follow from Theorem 1 by establishing inequalities like (12). We omit the details since they only involve elementary computations based on the inversion formula between density functions and characteristic functions.

3. Proofs

By the smoothing property, $\mathbb{E}[K(X_n)|\mathbf{X}_k] = K_{n-k}(\underline{X}_{n,k})$ holds for $k \leq n$. Let $\{\varepsilon'_i,\, i \in \mathbb{Z}\}$ be an independent copy of $\{\varepsilon_i,\, i\in\mathbb{Z}\}$ and $\underline{X}'_{n,1} = \mathbf{a}_{n-1}\varepsilon'_1 + \underline{X}_{n,0}$. Then $\mathbb{E}[K_{n-1}(\underline{X}'_{n,1})|\mathbf{X}_1] = K_n(\underline{X}_{n,0})$ almost surely. In the proofs below, we will use these claims. We omit the proof of the following lemma.

Lemma 1. *Assume* $\mathbb{E}(|Z|^\alpha) < \infty$. *Then for* $q_1 \geq \alpha \geq q_2 \geq 0$, *as* $M \to \infty$,

$$\mathbb{E}[|Z/M|^{q_1}\mathbf{1}_{|Z|\leq M}] + \mathbb{E}[|Z/M|^{q_2}\mathbf{1}_{|Z|>M}] = \mathcal{O}(1/M^\alpha).$$

Proof of Theorem 1. First we show that, under (a),

$$\sum_{n=1}^{\infty} \|\mathcal{P}_1 K(X_n)\| < \infty. \tag{13}$$

By Cauchy's inequality,

$$\begin{aligned}\|\mathcal{P}_1 K(X_n)\| &= \|\mathbb{E}[K_{n-1}(\underline{X}'_{n,1}) - K_{n-1}(\underline{X}_{n,1})|\mathbf{X}_1]\| \leq \|K_{n-1}(\underline{X}'_{n,1}) - K_{n-1}(\underline{X}_{n,1})\| \\ &\leq \|[K_{n-1}(\underline{X}'_{n,1}) - K_{n-1}(\underline{X}_{n,1})] \times \mathbf{1}_{|\underline{X}'_{n,1}-\underline{X}_{n,1}|\leq 1}\| \\ &\quad + \|[K_{n-1}(\underline{X}'_{n,1}) - K_{n-1}(\underline{X}_{n,1})] \times \mathbf{1}_{|\underline{X}'_{n,1}-\underline{X}_{n,1}|>1}\|.\end{aligned}$$

Let $\Delta = \underline{X}'_{n,1} - \underline{X}_{n,1} = \mathbf{a}_{n-1}(\varepsilon'_1 - \varepsilon_1)$. Then by Hölder's inequality and Lemma 1,

$$\begin{aligned}&\|[K_{n-1}(\underline{X}'_{n,1}) - K_{n-1}(\underline{X}_{n,1})] \times \mathbf{1}_{|\Delta|\leq 1}\|^2 \leq \mathbb{E}\{L^2_{K_{n-1}}(\underline{X}_{n,1}) \times |\Delta|^2 \mathbf{1}_{|\Delta|\leq 1}\} \\ &\leq \|L^2_{K_{n-1}}(\underline{X}_{n,1})\|_p \times \||\Delta|^2\mathbf{1}_{|\Delta|\leq 1}\|_q = \mathcal{O}[\mathbb{E}(|\Delta|^\alpha \mathbf{1}_{|\Delta|\leq 1})]^{\frac{1}{q}} = \mathcal{O}(|\mathbf{a}_{n-1}|^{\frac{\alpha}{q}}).\end{aligned}$$

Similarly, $\|K_{n-1}(\underline{X}_{n,1}) \times \mathbf{1}_{|\Delta|>1}\|^2 \leq \|K^2_{n-1}(\underline{X}_{n,1})\|_p \times \|\mathbf{1}_{|\Delta|>1}\|_q = \mathcal{O}(|\mathbf{a}_{n-1}|^{\frac{\alpha}{q}})$. Thus (13) follows from (2), and $\sum_{n=1}^{\infty} \mathcal{P}_1 K(X_n)$ converges to ξ (say) in $\mathcal{L}^2$. By Theorem 1 in Woodroofe (1992), $\{S_n(K) - \mathbb{E}[S_n(K)|\mathbf{X}_0]\}/\sqrt{n} \Rightarrow N(0, \|\xi\|^2)$. To conclude the proof, it remains to establish $\|\mathbb{E}[S_n(K)|\mathbf{X}_0]\|^2 = o(n)$. To this end notice that $\mathcal{P}_{-j}S_n(K)$, $j = 0, -1, \ldots$, are orthogonal and

$$\|\mathbb{E}[S_n(K)|\mathbf{X}_0]\|^2 = \sum_{j=0}^{\infty} \|\mathcal{P}_{-j}S_n(K)\|^2 \leq \sum_{j=0}^{\infty} \left[\sum_{i=1}^{n} \|\mathcal{P}_{-j}K(X_i)\|\right]^2$$
$$= \mathcal{O}\left[\sum_{j=0}^{\infty}\sum_{i=1}^{n} \|\mathcal{P}_1 K(X_{i+j+1})\|\right] = o(n)$$

by (13). Since arguments for (b) are similar, we omit the details.

Proof of Theorem 2. Let $V_i = K(X_i) - K'_{\infty}(0)X_i$. Clearly $\mathcal{P}_j R_n$, $j = \ldots, -1, 0, \ldots, n$, are martingale differences. Then by Bahr-Esseen's (cf. Avram and Taqqu, 1986) and Minkowski's inequalities,

$$\|R_n\|_\nu^\nu \leq 2\sum_{j=-\infty}^{n} \|\mathcal{P}_j R_n\|_\nu^\nu \leq 2\sum_{j=-\infty}^{n}\left[\sum_{i=1}^{n}\|\mathcal{P}_j V_i\|_\nu\right]^\nu = 2\sum_{j=-\infty}^{n}\left[\sum_{i=1}^{n}\|\mathcal{P}_1 V_{i-j+1}\|_\nu\right]^\nu.$$

Write $\theta_n = |\mathbf{a}_{n-1}|^{\frac{\alpha}{q\nu}} + |\mathbf{a}_{n-1}|A_n^{\frac{1}{q\nu}}(\alpha)$ and $\Theta_n = \sum_{k=0}^{n}\theta_n$. By Karamata's theorem, $\Theta_n = \mathcal{O}(n^{1-\beta'})$ and $\sum_{j=n}^{\infty}\theta_j^\nu = \mathcal{O}(n^{1-\nu\beta'})$. Observe that $\mathcal{P}_k V_i = 0$ for $k > i$. Then (6) implies

$$\sum_{j=1}^{n}\left[\sum_{i=1}^{n}\|\mathcal{P}_1 V_{i-j+1}\|_\nu\right]^\nu = \sum_{j=1}^{n}\left[\sum_{i=j}^{n}\|\mathcal{P}_1 V_{i-j+1}\|_\nu\right]^\nu \leq n\left[\sum_{i=1}^{n}\|\mathcal{P}_1 V_i\|_\nu\right]^\nu = \mathcal{O}(n\Theta_n^\nu).$$

On the other hand, since $\ell(n)$ is slowly varying,

$$\sum_{j=-\infty}^{0}\left[\sum_{i=1}^{n}\|\mathcal{P}_1 V_{i-j+1}\|_\nu\right]^\nu = \left[\sum_{j=-n+1}^{0} + \sum_{j=-\infty}^{-n}\right](\Theta_{n-j+1} - \Theta_{1-j})^\nu$$
$$= \mathcal{O}(n\Theta_{2n}^\nu) + n^\nu \sum_{j=n}^{\infty}\mathcal{O}(\theta_j^\nu),$$

which clearly implies (7).

Proof of Proposition 1. Let $U = K_n(\underline{X}_{n,1}) - K_n(\underline{X}_{n,0}) - K'_n(\underline{X}_{n,0})\mathbf{a}_{n-1}\varepsilon_1$. Then

$$\frac{1}{2^\nu}\mathbb{E}(|U|^\nu) \leq \mathbb{E}(|U\mathbf{1}_{|\mathbf{a}_{n-1}\varepsilon_1|\leq 1}|^\nu) + \mathbb{E}(|U\mathbf{1}_{|\mathbf{a}_{n-1}\varepsilon_1|>1}|^\nu)$$
$$\leq \mathbb{E}(|L_{K'_n}(\underline{X}_{n,0})|\mathbf{a}_{n-1}\varepsilon_1|^2\mathbf{1}_{|\mathbf{a}_{n-1}\varepsilon_1|\leq 1}|^\nu) + 3^\nu\mathbb{E}[|K_n(\underline{X}_{n,1})\mathbf{1}_{|\mathbf{a}_{n-1}\varepsilon_1|>1}|^\nu]$$
$$+ 3^\nu\mathbb{E}[|K_n(\underline{X}_{n,0})\mathbf{1}_{|\mathbf{a}_{n-1}\varepsilon_1|>1}|^\nu] + 3^\nu\mathbb{E}[|K'_n(\underline{X}_{n,0})||\mathbf{a}_{n-1}\varepsilon_1|\mathbf{1}_{|\mathbf{a}_{n-1}\varepsilon_1|>1}|^\nu].$$

By Lemma 1 and (8), the first, third and fourth terms in the proceeding display are of order $\mathcal{O}(|\mathbf{a}_{n-1}|^{\alpha})$. The second one, by Hölder's inequality, has order $\||K_n(\underline{X}_{n,1})|^{\nu}\|_p \times \|\mathbf{1}_{|\mathbf{a}_{n-1}\varepsilon_1|>1}\|_q = \mathcal{O}(|\mathbf{a}_{n-1}|^{\frac{\alpha}{q}})$. So, $\|U\|_{\nu} = \mathcal{O}(|\mathbf{a}_{n-1}|^{\frac{\alpha}{\nu q}})$. Let $V = K_{n-1}(\underline{X}_{n,1} + \mathbf{a}_{n-1}\varepsilon'_1) - K_{n-1}(\underline{X}_{n,1}) - K'_{n-1}(\underline{X}_{n,1})\mathbf{a}_{n-1}\varepsilon'_1$. Similarly $\|V\|_{\nu} = \mathcal{O}(|\mathbf{a}_{n-1}|^{\frac{\alpha}{\nu q}})$, which implies $\|K_n(\underline{X}_{n,1}) - K_{n-1}(\underline{X}_{n,1})\|_{\nu} = \mathcal{O}(|\mathbf{a}_{n-1}|^{\frac{\alpha}{\nu q}})$ in view of the fact that $K_n(x) - K_{n-1}(x) = \mathbb{E}[K_{n-1}(x + \mathbf{a}_{n-1}\varepsilon'_1) - K_{n-1}(x) - \mathbf{a}_{n-1}\varepsilon'_1 K'_{n-1}(x)]$. Hence $\|K_{n-1}(\underline{X}_{n,1}) - K_n(\underline{X}_{n,0}) - K'_n(\underline{X}_{n,0})\mathbf{a}_{n-1}\varepsilon_1\|_{\nu} = \mathcal{O}(|\mathbf{a}_{n-1}|^{\frac{\alpha}{\nu q}})$. To conclude the proof, it suffices to verify that $\|K'_n(\underline{X}_{n,0}) - K'_{\infty}(0)\|_{\nu} = \mathcal{O}[A_n^{1/(\nu q)}(\alpha)]$. Since $K_{\infty}(\iota) = \mathbb{E}K_n(\iota + \underline{X}_{n,0})$,

$$\Big|\frac{K_{\infty}(\iota) - K_{\infty}(0)}{\iota} - \mathbb{E}K'_n(\underline{X}_{n,0})\Big| \le \mathbb{E}\Big|\frac{K_n(\iota + \underline{X}_{n,0}) - K_n(\underline{X}_{n,0})}{\iota} - K'_n(\underline{X}_{n,0})\Big|$$
$$\le |\iota|\mathbb{E}|L_{K_n}(\underline{X}_{n,0})|,$$

implying that $\mathbb{E}K'_n(\underline{X}_{n,0}) = K'_{\infty}(0)$ for sufficiently large n by letting $\iota \downarrow 0$. Let $\underline{X}^*_{n,0} = \sum_{i=n}^{\infty} \mathbf{a}_i \varepsilon'_{n-i}$ and $\Delta = \underline{X}_{n,0} - \underline{X}^*_{n,0}$. Then $\mathbb{E}(|\Delta|^{\alpha}) = \mathcal{O}[A_n(\alpha)]$. By Jensen's inequality,

$$\begin{aligned}
&\|K'_n(\underline{X}_{n,0}) - K'_{\infty}(0)\|_{\nu} = \|\mathbb{E}[K'_n(\underline{X}_{n,0}) - K'_n(\underline{X}^*_{n,0})|\mathbf{X}_0]\|_{\nu}\\
&\le \|K'_n(\underline{X}_{n,0}) - K'_n(\underline{X}^*_{n,0})\|_{\nu}\\
&\le \|[K'_n(\underline{X}_{n,0}) - K'_n(\underline{X}^*_{n,0})] \times \mathbf{1}_{|\Delta|\le 1}\|_{\nu} + \|[K'_n(\underline{X}_{n,0}) - K'_n(\underline{X}^*_{n,0})] \times \mathbf{1}_{|\Delta|>1}\|_{\nu}
\end{aligned}$$

which, by (8) and Lemma 1, is of order $\mathcal{O}[A_n^{\frac{1}{\nu q}}(\alpha)]$ by using the same arguments as in the proof of Theorem 1.

4. An Open Problem

Theorem 2 can be viewed as a first order expansion of $S_n(K)$; namely it is approximated by a linear functional $\sum_{i=1}^{n} X_i$. Do higher order expansions exist? The issue is well understood when the innovations have finite fourth moment, see Ho and Hsing (1997). Such expansions would enable one to obtain finer results than Corollary 1 in the degenerate case $K'_{\infty}(0) = 0$.

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References

Avram, F. and Taqqu, M. S. (1986), Weak convergence of moving averages with infinite variance. In *Dependence in Probability and Statistics: A Survey of Recent Results* (Edited by Eberlein and Taqqu), 399-416. Birkhauser, Boston.

Chow, Y. S. and Teicher, H. (1988). *Probability Theory*. Springer Verlag, New York.

Gordin, M. I. and Lifsic, B. (1978). The central limit theorem for stationary Markov processes. *Doklady* **19**, 392-394.

Ho, H.-C. and Hsing, T. (1997). Limit theorems for functionals of moving averages. *Ann. Probab.* **25**, 1636-1669.

Hsing, T. (1999). On the asymptotic distributions of partial sums of functionals of infinite-variance moving averages. *Ann. Probab.* **27**, 1579-1599.

Ibragimov, I. A. and Yu. V. Linnik (1971). *Independent and Stationary Sequences of Random Variables.* Wolters-Noordhoff, Groningen.

Kedem, B. (1994). *Time Series Analysis by Higher Order Crossings*. IEEE Press, Piscataway, New Jersey.

Koul, H. and Surgailis, D. (2001). Asymptotics of empirical processes of long memory moving averages with infinite variance. *Stochastic Process. Appl.* **91**, 309-336.

Surgailis, D. (2002). Stable limits of empirical processes of moving averages with infinite variance. *Stochastic Process. Appl.* **100**, 255-274.

Woodroofe, M. (1992). A central limit theorem for functions of a Markov chain with applications to shifts. *Stochastic Process. Appl.* **41**, 33-44.

Wu, W. B. and J. Mielniczuk (2002). Kernel density estimation for linear processes. *Ann. Statist.* **30**, 1441-1459.

Wu, W. B. (2002). Central limit theorems for functionals of linear processes and their applications. *Statist. Sinica* **12**, 635-649.

Wu, W. B. (2003). Empirical Processes of Long-memory Sequences. To appear, Bernoulli.

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