

DESIGNS OF VARIABLE RESOLUTION ROBUST TO NON-NEGLIGIBLE TWO-FACTOR INTERACTIONS

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Abstract: In some experimental situations, there is a natural grouping of factors. Prior knowledge may indicate that active two-factor interactions only occur within groups of factors. Designs of variable resolution (Lin (2012)) were introduced for such situations. However, main effects from one group in a design of variable resolution may be aliased with non-negligible two-factor interactions from another group. In this article, we introduce robust designs of variable resolution which ensure main effects are robust to non-negligible interactions. Constructions are provided for a number of situations. Some existence results are given as well.

Key words and phrases: Alias, clear effect, fractional factorial design, Hadamard matrix, Kronecker product, orthogonal array, robustness.

1. Introduction

Two-level fractional factorial designs are widely used in many areas of science, engineering, and industry. Often they are used as screening designs to identify important main effects and two-factor interactions in a linear model, assuming three-factor and higher order interactions are negligible. The traditional notions of resolution (Box and Hunter (1961)) and minimum aberration (Fries and Hunter (1980)) treat interactions of the same order as equally likely to be important. However, it is not uncommon that prior knowledge implies some interactions can be safely treated as negligible. Research has been taken up on design choices when some two-factor interactions are negligible and others are non-negligible. See, for example, Franklin and Bailey (1977), Hedayat and Pesotan (1997), Tang and Zhou (2009), and Lekivetz and Tang (2011). If the factors are separated into groups, these non-negligible two-factor interactions may come from factors within disjoint groups, two-factor interactions between factors from different groups being negligible. This can arise when the different groups of factors occur between separate steps in a process, or when some basic knowledge about the underlying physical process provides the groupings. The disjoint groups of factors considered are control factors whose levels can be varied from run to run. The two-factor interactions within groups are non-negligible,

while the two-factor interactions that occur between factors in different groups are safely assumed negligible.

For such a case, Lin (2012) proposed designs of variable resolution, in which the resolution of each group is higher than the overall resolution of the respective design. For example, let A , B , C and D be independent factors making up a 2^4 full factorial design, and consider the 2^{12-8} design d_1 where the 12 factors are partitioned into three groups, $G_1 = \{A, AB, AC, AD\}$, $G_2 = \{B, BC, ABC, D\}$, and $G_3 = \{C, BD, ACD, CD\}$, where, for instance, ABC is the column generated from the product of columns A , B and C . According to Lin (2012), d_1 is a design of variable resolution with the overall resolution III and the resolution of each of its groups G_1 , G_2 and G_3 is V. Now consider the 2^{12-8} design d_2 whose factors are partitioned into the three groups, $G_1 = \{A, AC, AD, ACD\}$, $G_2 = \{B, BC, BD, BCD\}$, $G_3 = \{AB, ABC, ABD, ABCD\}$. The overall resolution of design d_2 is III and the resolution of each of its groups is IV. Intuitively, a higher resolution for each group would be preferred and thus d_1 appears preferable to d_2 . However, if the estimation of main effects is the primary goal, design d_2 would be more appealing than d_1 . In d_2 , the main effects are orthogonal to all two-factor interactions occurring within all groups, while d_1 has aliasing between main effects and two-factor interactions within other groups.

In this article, we introduce and construct a new class of designs, called *robust designs of variable resolution*, to eliminate the contamination of non-negligible two-factor interactions on the estimation of main effects, where the non-negligible two-factor interactions occur between factors within each of the groups of factors. The remainder of this article is organized as follows. Section 2 provides background knowledge, notation and definitions. Section 3 presents several construction methods for robust designs of variable resolution. Section 4 provides existence results of such designs. Section 5 concludes the article with discussion.

2. Concepts and Notation

A two-level factorial design of n runs for p factors can be represented by $d = (d_1, \dots, d_p)$ where $d_j = (d_{1j}, \dots, d_{nj})^T$ is the j th column of d and $d_{ij} = \pm 1$ for $i = 1, \dots, n$ and $j = 1, \dots, p$. Given any m columns, $s = \{d_{j_1}, \dots, d_{j_m}\}$, of design d , their J -characteristic is

$$J(s) = J(d_{j_1}, \dots, d_{j_m}) = \left| \sum_{i=1}^n d_{ij_1} \cdots d_{ij_m} \right|. \quad (2.1)$$

The *resolution*, r , of a design is the smallest integer such that $\max_{|s|=r} J(s) > 0$. For convenience, we denote a two-level design with n runs, p factors and resolution r as $D(n, p, r)$. For regular designs whose columns are either independent or fully aliased, $r = 3, 4$, and 5 correspond to designs of resolution III, IV and V,

respectively. For notational simplicity, we also use the Roman numeral notation for resolution of non-regular designs whose columns can be independent, fully aliased or partially aliased, although non-regular designs are quantified by generalized resolution (Deng and Tang (1999); Grömping and Xu (2014)). A design is called *column-orthogonal* if any of its two distinct columns has J -characteristic 0. Column-orthogonal designs are different from designs of resolution III in that each column in the former does not need to have equal frequency of each level. Thus, designs of resolution III are column-orthogonal, but column-orthogonal designs may not be of resolution III. Other examples of column-orthogonal designs include Hadamard matrices (Hedayat, Sloane, and Stufken (1999)), and designs of resolution IV and higher.

A design $D(n, p, r)$ is said to be of variable resolution if its columns can be partitioned into k groups, where the i th group, D_i , having p_i factors, is a $D(n, p_i, r_i)$, where $r < r_i \leq p_i + 1$ for $i = 1, \dots, k - 1$, and $r \leq r_k \leq p_k + 1$ (Lin (2012)). Thus, each group has higher resolution than the resolution of the whole design, while allowing for the possibility that the last group has the same resolution as the entire design. Such a design is denoted by $D\{n, (p_1, \dots, p_k), (r_1, \dots, r_k); r\}$.

To introduce robust designs of variable resolution, recall the framework of quantifying the contamination of non-negligible two-factor interactions on the estimation of main effects (Deng and Tang (1999)). Suppose the true model is

$$Y = 1_n\beta_0 + X_{11}\beta_{11} + \dots + X_{k1}\beta_{k1} + X_{12}\beta_{12} + \dots + X_{k2}\beta_{k2} + \epsilon, \tag{2.2}$$

where Y is the response vector of n observations, 1_n is the column of n 1's, β_0 is the regression coefficient for the grand mean, X_{i1} is the design matrix for the main effects of the p_i factors in group i , β_{i1} represents the vector of the corresponding regression coefficients, X_{i2} is the model matrix corresponding to the $p_i(p_i - 1)/2$ two-factor interactions within group i , with β_{i2} the corresponding vector of regression coefficients, and ϵ is random error with mean 0 and constant variance. For a fitted model involving only the main effects, the least squares estimate of $\beta_1 = (\beta_{11}^T, \dots, \beta_{k1}^T)^T$ is given by $\hat{\beta}_1 = (\hat{\beta}_{11}^T, \dots, \hat{\beta}_{k1}^T)^T = (X_1^T X_1)^{-1} X_1^T Y$, where $X_1 = (X_{11}, \dots, X_{k1})$ is the design matrix and $X_{i1} = D_i$ for $i = 1, \dots, k$. Under (2.2), we have $E(\hat{\beta}_1) = \beta_1 + \sum_{i=1}^k n^{-1} X_1^T X_{i2} \beta_{i2}$. Lin (2012) showed that to minimize the contamination of non-negligible two-factor interactions on the estimation of main effects, it is equivalent to minimize

$$C_3 = \sum_{i=1}^k \|n^{-1} X_1^T X_{i2}\|^2 = \sum_{i=1}^k [3B_{(i,3)} + \sum_{i' \neq i} B_{(i,1),(i',2)}], \tag{2.3}$$

where

$$B_{(i,3)} = n^{-2} \sum_{c_1, c_2, c_3 \in D_i} J(c_1, c_2, c_3)^2, \tag{2.4}$$

$$B_{(i,1),(i',2)} = n^{-2} \sum_{c_1 \in D_i, c_2, c_3 \in D_{i'}} J(c_1, c_2, c_3)^2. \quad (2.5)$$

The $B_{(i,3)}$ in (2.4) measures the aliasing between main effects and two-factor interactions within the i th group, while the $B_{(i,1),(i',2)}$ in (2.5) measures the aliasing between main effects of factors in the i th group and the two-factor interactions occurring within the i' th group. For a design of variable resolution with $r_i \geq 4$ and $r \geq 3$, $B_{(i,3)}$ in (2.4) is equal to 0 for $i = 1, \dots, k$. Note that because of $r_i \geq 4$, we assume $p_i \geq 3$ for $i = 1, \dots, k$ throughout. If such a design of variable resolution has $B_{(i,1),(i',2)}$ in (2.5) equal to 0 for $i = 1, \dots, k$, it is called *robust design of variable resolution*, a robust design of variable resolution has C_3 in (2.3) equal to 0. Such a design is denoted by $RD\{n, (p_1, \dots, p_k), (r_1, \dots, r_k); r\}$. If the overall resolution of a design is IV or higher, it is a robust design of variable resolution. Thus, we focus on constructing robust designs of variable resolution with overall resolution III.

3. Construction of Robust Designs of Variable Resolution

This section provides several constructions for two-level robust designs of variable resolution using Kronecker products and clear two-factor interactions.

3.1 Construction via Kronecker products

This section introduces Constructions 1 and 2 to construct two classes of robust designs of variable resolution using designs of resolution III. Construction 1 provides designs with groups of equal resolution, while Construction 2 has more groups, but higher resolution within some groups. Both constructions use Kronecker products. Let $x = (x_1, \dots, x_{n_1})^T$ and $y = (y_1, \dots, y_{n_2})^T$. The Kronecker product of x and y is

$$x \otimes y = (x_1 y_1, \dots, x_1 y_{n_2}, \dots, x_{n_1} y_1, \dots, x_{n_1} y_{n_2})^T.$$

Construction 1. Let $A = (a_1, \dots, a_p)$ be a $D(n_1, p, 3)$ where $p \geq 2$. For $i = 1, \dots, p$, let B_i be a column-orthogonal design with n_2 rows and q_i columns, where $q_i \geq 3$, and let $D_i = a_i \otimes B_i$.

Proposition 1. *Design $d = (D_1, \dots, D_p)$ in Construction 1 is a $RD\{n_1 n_2, (q_1, \dots, q_p), (4, \dots, 4); 3\}$.*

Proof. Proposition 1 follows directly by the definition of a robust design of variable resolution and the property of J -characteristics of Kronecker products of columns (Tang (2006));

$$J(a_1 \otimes b_1, \dots, a_p \otimes b_p) = J(a_1, \dots, a_p)J(b_1, \dots, b_p), \quad (3.1)$$

where $a_j = (a_{1j}, \dots, a_{n_1j})^T$ and $b_j = (b_{1j}, \dots, b_{n_2j})^T$ for $j = 1, \dots, p$.

We focus on robust designs of variable resolution with the overall resolution III, because if the overall resolution is IV or higher, it is already a robust design of variable resolution. In Construction 1, if columns of B_i are from a design, say B , where any three columns have a J -characteristic of 0, the resulting design $d = (D_1, \dots, D_p)$ is of resolution IV. The run size of a two-level robust design of variable resolution provided by Construction 1 is a multiple of 16 because n_1 and n_2 must be a multiple of 4 for A to be a design of resolution III, B_i to be column-orthogonal and $q_i \geq 3$. The maximum value of p and q_i in Construction 1 is $n_1 - 1$ and n_2 , respectively, for $i = 1, \dots, p$. Thus, given n_1 and n_2 , the largest robust design of variable resolution provided by Construction 1, in terms of the maximum number of columns, is a $d = (D_1, \dots, D_{n_1-1}) = RD\{n_1n_2, (n_2, \dots, n_2), (4, \dots, 4); 3\}$.

Example 1. Let A be a $D(8, 7, 3)$, and B_i be a Hadamard matrix of order 4, $i = 1, \dots, 7$. Construction 1 gives a $RD\{32, (4, 4, 4, 4, 4, 4, 4), (4, 4, 4, 4, 4, 4, 4); 3\}$. Now choosing A be a $D(4, 3, 3)$ and B_i be a Hadamard matrix of order 8, $i = 1, 2, 3$, we get a $RD\{32, (8, 8, 8), (4, 4, 4); 3\}$.

Construction 2. Let $A = (a_1, \dots, a_p)$ be a $D(n_1, p, 3)$ and B_i be a $D(n_2, q_i, 3)$ for $i = 1, \dots, p$. Suppose that B_i can be expressed as $B_i = (B_i^*, B_i \setminus B_i^*)$ such that B_i^* is a design of resolution V with m_i columns, where $B_i \setminus B_i^*$ represents the columns of B_i not in B_i^* . Let $E_i = (1_{n_2}, B_i)$ and $E_i^* = (1_{n_2}, B_i^*)$, where 1_{n_2} denotes a column of n_2 1's. For $i = 1, \dots, p$, let $D_i = a_i \otimes E_i^*$, $D_{p+i} = a_i \otimes (E_i \setminus E_i^*)$.

Proposition 2. Design $d = (D_1, \dots, D_p, D_{p+1}, \dots, D_{2p})$ in Construction 2 is a $RD\{n_1n_2, (m_1 + 1, \dots, m_p + 1, q_1 - m_1, \dots, q_p - m_p), (6, \dots, 6, 4, \dots, 4); 3\}$.

That Construction 2 provides a robust design of variable resolution can be readily verified using (3.1). Constructions 1 and 2 both use the Kronecker product of designs of resolution III and column-orthogonal designs. Construction 1 provides robust designs of variable resolution consisting of p groups of resolution IV, with q_i columns in the i th group. Construction 2 allows each of the p groups to be further partitioned into groups with one of resolution VI and the other of resolution IV. Both constructions can provide robust designs of variable resolution with the maximal number $(n_1 - 1)n_2$ of columns; this occurs for $p_1 = n_1 - 1$ and for $p_2 = n_2$ in Construction 1 and $p_1 = n_1 - 1$ and $p_2 = n_2 - 1$ in Construction 2. Example S1 in the Supplementary Materials is an application of Construction 2.

In addition to Constructions 1 and 2, several constructions from Lin (2012) yield robust designs of variable resolution. These constructions, and conditions

for ensuring the robustness property, can be found in the Supplementary Materials as Constructions S1 – S4.

3.2. Construction through clear two-factor interactions

A two-factor interaction is said to be clear if it is orthogonal to all main effects and all other two-factor interactions (Tang (2006)). Designs with clear two-factor interactions provide worry-free estimation of the clear two-factor interactions in the presence of non-negligible two-factor interactions. For designs of resolution V, all two-factor interactions are clear; for designs of resolution IV, not all two-factor interactions are clear since some two-factor interactions will be aliased with each other. If a design d of resolution IV has two groups of factors, $D_1 = \{a_1, \dots, a_{p_1}\}$ and $D_2 = \{b_1, \dots, b_{p_2}\}$, Ke, Tang, and Wu (2005) defined a clear compromise plan of Class 4 to be a design d with the property that the two-factor interactions between a_i 's and b_j 's are clear for $i = 1, \dots, p_1$ and $j = 1, \dots, p_2$. For notational simplicity, we denote the two-factor interactions between groups D_i and D_j by $D_i \times D_j$, for $i = 1, 2$ and $j = 1, 2$. Another class of designs with two groups is, a clear compromise plan of Class 3, in which the two-factor interactions $D_1 \times D_1$, and the two-factor interactions $D_1 \times D_2$ are clear (Ke, Tang, and Wu (2005)).

Construction 3. Let (D_1, D_2) be an n -run clear compromise plan of Class 3 or Class 4, where $D_1 = \{a_1, \dots, a_{p_1}\}$ and $D_2 = \{b_1, \dots, b_{p_2}\}$. For $i = 1, \dots, p_1$, let $D_{i+2} = (a_i b_1, \dots, a_i b_{p_2})$, and $d = (D_1, \dots, D_{p_1+2})$.

Proposition 3. *Design d in Construction 3 is a $RD\{n, (p_1, p_2, p_2, \dots, p_2), (4, \dots, 4); 3\}$ if (D_1, D_2) is a clear compromise plan of Class 4.*

Proposition 4. *Design d in Construction 3 is a $RD\{n, (p_1, p_2, p_2, \dots, p_2), (5, 4, \dots, 4); 3\}$ if (D_1, D_2) is a clear compromise plan of Class 3.*

Proposition 3 follows by the definition of a clear compromise plan of Class 4. Specifically, the J -characteristics $J(a_{i_1}, a_{i_2}, b_{j_1})$, $J(a_{i_1}, b_{j_1}, b_{j_2})$, $J(b_{j_1}, b_{j_2}, b_{j_3})$, $J(a_{i_1}, a_{i_2}, b_{j_1}, b_{j_2})$, $J(a_{i_1}, a_{i_2}, a_{i_3}, b_{j_1})$, and $J(a_{i_1}, b_{j_1}, b_{j_2}, b_{j_3})$ are all zero for $i_1, i_2, i_3 = 1, \dots, p_1$ and $j_1, j_2, j_3 = 1, \dots, p_2$, implying (2.4) and (2.5) are zero. A similar argument applies to verify Proposition 4. In addition to being robust designs of variable resolution, designs in both propositions enjoy the property that the two-factor interactions between any two factors in D_1 are orthogonal to the two-factor interactions within other groups. If two-factor interactions within D_1 are known to be negligible, this property can be used to add additional factors to D_1 by using any unique two-factor interactions as the new factors. This makes the sub-design of factors in D_1 resolution III, implying that $B_{(1,3)}$ in (2.4) will not be zero. However, if the two-factor interactions in D_1 are negligible, the quantity $B_{(1,3)}$ is removed from (2.3) and thus the value of C_3 remains 0. Designs in

Proposition 3 and those in Proposition 4 have different properties, in that D_1 in the latter is of resolution V because of the use of a compromise plan of Class 3: all main effects and the two-factor interactions of D_1 in Proposition 4 are orthogonal to each other and to the two-factor interactions within other groups. Thus, the two-factor interactions of D_1 in Proposition 4 are estimable in the presence of non-negligible two-factor interactions. They are referred to as partially clear two-factor interactions by Lekivetz and Tang (2011). Example 2 illustrates the use of Construction 3 to obtain robust designs of variable resolution.

Example 2. Ke, Tang, and Wu (2005) provided a clear compromise plan of Class 4 with 64 runs in which D_1 and D_2 have 6 factors each. Using Construction 3, we obtain a $RD\{64, (6, 6, 6, 6, 6, 6, 6), (4, 4, 4, 4, 4, 4, 4); 3\}$. Using the clear compromise plan of Class 3 of 64 runs with 5 factors in D_1 and 4 factors in D_2 provided by Ke, Tang, and Wu (2005), we obtain a $RD\{64, (5, 4, 4, 4, 4, 4, 4), (5, 4, 4, 4, 4, 4, 4); 3\}$ with the two-factor interactions of D_1 being partially clear.

4. Existence Results

This section establishes existence results of two-level robust designs of variable resolution. The problem we aim to address is whether or not a $RD(n, (p_1, \dots, p_k), (r_1, \dots, r_k); 3)$ exists given $n, (r_1, \dots, r_k)$ and $r = 3$, and if it exists, what the maximum value of p_i 's is for $i = 1, \dots, k$.

Proposition 5. *There does not exist a $RD(n, (p_1, p_2), (r_1, r_2); r)$ with $r_1 \geq 4, r_2 \geq 4$, and $r = 3$.*

Proof. Lin (2012) showed an alternative derivation of C_3 in (2.3) is

$$C_3 = B_3 + 2 \sum_{i=1, p_i > 2}^k B_{(i,3)} - \sum_{k \geq 3, i \neq j \neq l} B_{(i,1)(j,1)(l,1)}, \tag{4.1}$$

where $B_3 = n^{-2} \sum_{c_1, c_2, c_3 \in (D_1, \dots, D_k)} J(c_1, c_2, c_3)^2$,

$$B_{(i,1)(j,1)(l,1)} = n^{-2} \sum_{c_1 \in D_i, c_2 \in D_j, c_3 \in D_l} J(c_1, c_2, c_3)^2, \tag{4.2}$$

and $B_{(i,3)}$ is given in (2.4). If $r_1 \geq 4, r_2 \geq 4$, and $r = 3$, then $B_{(1,3)} = B_{(2,3)} = 0$ and $B_3 > 0$. Since $k < 3$, we have $C_3 = B_3 > 0$; for designs of variable resolution with two groups and the overall resolution III, we have $C_3 > 0$. Thus, there is not such a robust design of variable resolution.

Proposition 5 leads to Corollaries 1 and 2. Based on the fact that the maximum number of columns in an n -run two-level design of resolution IV is $n/2$ (Wu and Hamada (2011)), we have $p_1 + p_2 \leq n/2, p_1 + p_3 \leq n/2$, and $p_2 + p_3 \leq n/2$.

Corollary 1. For a $RD(n, (p_1, p_2, p_3), (r_1, r_2, r_3); 3)$, removing any group of factors results in a design of resolution IV or higher.

Corollary 2. A $RD(n, (p_1, p_2, p_3), (4, 4, 4); 3)$ has $p_1 + p_2 + p_3 \leq 3n/4$.

Proposition 6 considers a special case of two-level robust designs of variable resolution with three groups each of which has resolution IV or higher.

Proposition 6. Let $d = (D_1, D_2, D_3)$ be a $RD(n, (p_1, p_2, p_3), (r_1, r_2, r_3); 3)$ with $r_i \geq 4$, and d_{ij_i} be the j_i th column of D_i for $j_i = 1, \dots, p_i$ and $i = 1, 2, 3$. If there exist j_1, j_2 , and j_3 such that $d_{1j_1}d_{2j_2} = d_{3j_3}$, we have $p_i \leq n/4$ for $i = 1, 2, 3$.

Proof. For a $RD(n, (p_1, p_2, p_3), (r_1, r_2, r_3); 3)$ with $r_i \geq 4$, suppose that there are j_1, j_2 , and j_3 such that $d_{1j_1}d_{2j_2} = d_{3j_3}$. Without loss of generality, we assume $d_{11}d_{21} = d_{31}$. Using a similar argument as in the proof of Proposition 1 in Tang (2006), the following columns are mutually orthogonal and orthogonal to the column of all ones: $d_{11}, \dots, d_{1p_1}, d_{21}, d_{31}, d_{11}d_{12}, \dots, d_{11}d_{1p_1}, d_{21}d_{12}, \dots, d_{21}d_{1p_1}, d_{31}d_{12}, \dots, d_{31}d_{1p_1}$. That these columns are pairwise orthogonal can be verified with their J -characteristics. For example, given $x, y \in 1, \dots, p_1$, $J(d_{21}d_{1x}, d_{31}d_{1y}) = J(d_{21}d_{31}d_{1x}d_{1y}) = J(d_{11}d_{1x}d_{1y}) = 0$, since the resolution of the first group of factors, $r_1 \geq 4$. Taking these $p_1 + 2 + 3(p_1 - 1)$ mutually orthogonal columns, we have

$$p_1 + 2 + 3(p_1 - 1) = 4p_1 - 1 \leq n - 1,$$

which upon rearranging yields the result $p_1 \leq n/4$. Similarly, we obtain $p_2 \leq n/4$ and $p_3 \leq n/4$.

Proposition 7 places a bound on the number of factors in a group when the number of factors in each group is the same. Its proof is given in the Supplementary Materials.

Proposition 7. For a robust design of variable resolution with n runs and p_1 groups of p_2 factors, we have that $p_2 \leq n/(p_1 + 1)$.

Corollary 3. Using Construction 1, if A is a saturated design with n_1 runs and $p_1 = n_1 - 1$ columns, and B_i is a saturated design with n_2 runs and $p_2 = n_2$ columns for $i = 1, \dots, p_1$, then all p_2 's in the design d in Construction 1 achieve the upper bound given in Proposition 7.

Proposition 8 presents a situation in which there is no two-level robust design of variable resolution with the overall resolution being III and the resolution in each group being IV or higher. The situation lies in the fold-over designs. A two-level design d is fold-over if d can be represented by $d = (d_0^T, -d_0^T)^T$.

Proposition 8. There does not exist a $RD(n, (p_1, \dots, p_k), (r_1, \dots, r_k); r)$ with $r_i \geq 4$ and $r = 3$ if all $D(n, m, 4)$'s are fold-overs for $m \leq n/2$.

Corollary 4. *There does not exist a $RD(24, (p_1, \dots, p_k), (r_1, \dots, r_k); r)$ with $r_i \geq 4$ and $r = 3$.*

The proof of Proposition 8 is given in the Supplementary Materials. Corollary 4 follows from the fact that all 24-run designs of resolution IV are fold-overs (Schoen, Eendebak, and Nguyen (2010)).

5. Discussion

This article extends the concept of variable resolution by ensuring designs are robust to non-negligible two-factor interactions that occur within groups. Robust designs of variable resolution allow worry-free estimation of the main effects in the presence of non-negligible interactions occurring within groups of factors. While these designs do allow for the estimation of some two-factor interactions, there is still potential confounding between non-negligible interactions. If one needs to ensure the estimation of a particular set of interactions, the so-called requirement set problem (Greenfield (1976)) will generally require either more runs or fewer factors to be considered.

An application of variable resolution designs that requires further study is for use with multiple response variables. If each response is believed to be driven by one of the groups, not only can the main effects of the factors be accounted for, but the higher resolution within each group allows for better estimation of the interaction effects within that group for the particular response. Two main advantages of using variable resolution designs in such a situation are that it allows for the study of additional factors that might not be considered if each response was experimented on separately, and that specifying all model effects to be estimated among the total set of responses may result in a substantially larger run size in comparison to using a variable resolution design.

While our focus has been on robust designs of variable resolution, the constructions can be generally useful for maintaining certain properties within the groups of factors. For example, one may want to ensure all main effects and two-factor interactions within a group are estimable. In Constructions 1 and 2, if all B_i 's have such a property, each group of the resulting design does as well. Using this group structure to maintain certain properties within each group is a topic of future research.

Supplementary Materials

The online supplementary materials include an example of Construction 2, additional construction methods, and the proofs of Propositions 7 and 8.

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