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A PSEUDO-LIKELIHOOD ESTIMATION METHOD FOR NONHOMOGENEOUS GAMMA PROCESS MODEL WITH RANDOM EFFECTS

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Abstract: We study a semiparametric pseudo-likelihood inference for nonhomogeneous Gamma process with random effects for degradation data. The setting for degradation data is one in which n independent subjects, each with a nonhomogeneous Gamma process, are observed at possible different times. The random effects are used to represent heterogeneity of degradation paths. To obtain the maximum pseudo-likelihood estimator, we use the Pool Adjacent Violators Algorithm. We study the asymptotic properties for this estimator. A simulation study is conducted to validate the method and its application is illustrated by using degradation data of a civil engineering structure to estimate its reliability.

Key words and phrases: Degradation data, empirical process, greatest convex minorant, pseudo-likelihood, profile likelihood, Gamma process.

1. Introduction

Suppose $\{Y(t) : t \geq 0\}$ is a univariate nonhomogeneous Gamma process and Λ is a nondecreasing function. The nonhomogeneous Gamma process Y has the following properties: the increments $\Delta Y(t) = Y(t + \Delta t) - Y(t)$ are independent; $\Delta Y(t)$ has a Gamma$(\theta, \Delta\Lambda)$ distribution with mean $\theta\Delta\Lambda(t) = \theta(\Lambda(t+\Delta t) - \Lambda(t))$ and variance $\theta^2\Delta\Lambda(t)$. The nonhomogeneous Gamma process model has been found useful in the analysis of degradation data (e.g., Bagdonavicius and Nikulin (2001) and Lawless and Crowder (2004)). In some studies, subjects are put on test at time 0 and degrade over time; when the amount of degradation reaches a pre-specified critical level d, failure occurs. In practice, each subject has degradation measurements taken over time; the number of observation times and observation times themselves are allowed to vary across subjects. These data are referred to as regular *degradation data*. Examples are given by Davies (1998) and Meeker and Escobar (1998, Chaps. 14 and 21, and references therein). Degradation data are a rich source of reliability information and offer many advantages over time-to-failure data. Degradation data have applications in many fields, such as industrial reliability and HIV study; see Doksum (1991), Singpurwalla (1995), Meeker and Escobar (1998) and Wang (2007).

Component-to-component variability can be incorporated into the model to represent the heterogeneity in the degradation paths of different units. One such model can be specified by using a random scale parameter θ. We adopt the same model as Lawless and Crowder (2004), in which $\omega = 1/\theta$, and

$$\omega \sim \text{Gamma}(\gamma^{-1}, \delta), \qquad Y(t)|\omega \sim \text{Gamma Process}(\omega^{-1}, \Lambda(t)). \tag{1.1}$$

Then the marginal density of $Y(t)$ is given by

$$f(y) = \frac{\Gamma(\Lambda(t) + \delta)}{\Gamma(\Lambda(t))\Gamma(\delta)} \frac{\gamma^\delta y^{\Lambda(t)-1}}{(y+\gamma)^{\Lambda(t)+\delta}}. \tag{1.2}$$

Suppose we observe Y at times $t_1, \ldots, t_m$, yielding observations $y_1, \ldots, y_m$. Based on the conditional independence of the increments given the random effect ω, the joint density of $y_1, \ldots, y_m$ is

$$f(y_1, \ldots, y_m) = \frac{\gamma^\delta \Gamma(\delta + \sum_{j=1}^m \Delta\Lambda_j)}{\Gamma(\delta) \prod_{j=1}^m \Gamma(\Delta\Lambda_j)} \frac{\prod_{j=1}^m (\Delta y_j)^{\Delta\Lambda_j - 1}}{(\gamma + \sum_{j=1}^m \Delta y_j)^{\delta + \sum_{j=1}^m \Delta\Lambda_j}}, \tag{1.3}$$

where $\Delta y_j = y_j - y_{j-1}$ and $\Delta\Lambda_j = \Lambda(t_j) - \Lambda(t_{j-1})$. An extreme case of the model occurs when there is no random effect; this case can be realized by letting $\gamma \to \infty$ with $\delta/\gamma = 1/\theta$ fixed.

Let T denote the failure time, $T = \inf\{t : Y(t) \geq d\}$. Noting from (1.2) that $\delta Y(t)/(\gamma\Lambda(t))$ has an F distribution $\mathcal{F}_{2\Lambda(t),2\delta}$, the time-to-failure distribution is

$$P(T < t) = P(Y(t) > d) = 1 - F(\frac{\delta d}{\gamma\Lambda(t)}), \tag{1.4}$$

where $F(\cdot)$ is the distribution function of $\mathcal{F}_{2\Lambda(t),2\delta}$. With different choices of Λ, (1.4) defines a wide class of failure time distribution functions.

The objective of this paper is to extend the parametric inference of this model by Lawless and Crowder (2004) to the semiparametric case. We would like to obtain a nonparametric estimate of Λ and an estimate of (γ, δ). It is natural to consider the maximum likelihood estimator of $(\Lambda, \gamma, \delta)$; however, we need to maximize the likelihood function under the monotonicity constraints of Λ and this cannot be done in a direct manner due to the complicated form (1.3), especially when the number and locations of observation times for each subject are different.

In this paper, we propose a pseudo-likelihood method to estimate the unknown parameters. The pseudo-likelihood estimator ignores the dependence between the degradation measurements at successive observation times, treating them as if they were independent random variables to form a pseudo-likelihood. When the nonhomogeneous Gamma process for each unit is observed at just one time (e.g., destructive degradation data), the pseudo-likelihood function becomes the regular likelihood function. We show that the maximum pseudo-likelihood estimator (MPLE) is consistent, and we also derive the convergence rate and asymptotic distribution of the MPLE. Simulation results suggest that this method works well, and we apply it to the degradation data of a civil engineering structure to estimate its reliability.

The rest of the paper is organized as follows. In Section 2, we introduce the algorithm to compute the MPLE. In Section 3, we establish the asymptotic results, including consistency, convergence rate, and asymptotic distribution of the MPLE. In Section 4, a simulation study is conducted to validate the method. In Section 5, we apply our method to degradation data of a civil engineering structure. The paper ends with discussion. The proofs of theorems are given in the Appendix.

2. Pseudo-Likelihood Estimation

Suppose we observe the degradation process Y at a random number K of random times $0 = T_{K,0} < T_{K,1} < \cdots < T_{K,K}$. Write $\underline{T}_K = (T_{K,1}, \ldots, T_{K,K})$ and $\underline{Y}_K = (Y_{K,1}, \ldots, Y_{K,K})$, where $Y_{K,j} = Y(T_{K,j})$. Assume $(K, \underline{T}_K)$ is independent of Y. Let $X = (\underline{Y}_K, \underline{T}_K, K)$ take value $x = (\underline{y}_k, \underline{t}_k, k)$. Suppose we observe n i.i.d. copies of X, $X_1, \ldots, X_n$, where $X_i = (\underline{Y}^{(i)}_{K_i}, \underline{T}^{(i)}_{K_i}, K_i)$ for $i = 1, \ldots, n$. Our goal is to estimate $(\Lambda(t), \gamma, \delta)$.

By ignoring the dependence of the data within each unit, from (1.2), we can form the log pseudo-likelihood function for $(\Lambda(t), \gamma, \delta)$, up to a constant, as

$$l_n(\Lambda, \gamma, \delta) = \sum_{i=1}^{n} \sum_{j=1}^{K_i} \Bigg[H_\delta(\Lambda(T^{(i)}_{K_i,j})) + \Lambda(T^{(i)}_{K_i,j}) \log \frac{Y^{(i)}_{K_i,j}}{Y^{(i)}_{K_i,j} + \gamma} + \delta \log \frac{\gamma}{Y^{(i)}_{K_i,j} + \gamma} - \log \Gamma(\delta) \Bigg], \tag{2.1}$$

where $H_\delta(t) = \log \Gamma(t+\delta) - \log \Gamma(t)$. Note that $h_\delta(t) = \partial H_\delta / \partial t = \psi_0(t+\delta) - \psi_0(t)$ is a positive decreasing function of t, where $\psi_0 = \Gamma'/\Gamma$ is the digamma function. Let $(\tilde{\Lambda}_n, \tilde{\gamma}_n, \tilde{\delta}_n)$ be the MPLE, so $(\tilde{\Lambda}_n, \tilde{\gamma}_n, \tilde{\delta}_n) = \arg\max_{\mathcal{F} \times \mathcal{R} \times \mathcal{R}} l_n(\Lambda, \gamma, \delta)$, where $\mathcal{R} \subset (0, \infty)$ is a compact set, and

$$\mathcal{F} = \{\Lambda : [0, \infty) \to [0, \infty) | \Lambda \text{ is a nondecreasing function with } \Lambda(0) = 0\}.$$

The algorithm to obtain the MPLE is carried out in two steps. First we solve for Λ while holding γ and δ fixed, then we construct a profile pseudo-likelihood function of γ and δ to get the estimator of (γ, δ). Let $t_1 < \cdots < t_m$ be the distinct observation time points in the set of all observation times $\{T^{(i)}_{K_i,j}, j = 1,\ldots,K_i, i = 1,\ldots,n\}$. For $l = 1,\ldots,m$, let

$$\omega_l = \sum_{i=1}^{n}\sum_{j=1}^{K_i} 1_{\{T^{(i)}_{K_i,j}=t_l\}}, \qquad u_l = \frac{1}{\omega_l}\sum_{i=1}^{n}\sum_{j=1}^{K_i}(\log \frac{Y^{(i)}_{K_i,j}}{Y^{(i)}_{K_i,j}+\gamma})1_{\{T^{(i)}_{K_i,j}=t_l\}}.$$

Given (γ,δ), up to a constant which does not depend on Λ, the pseudo-likelihood function (2.1) becomes

$$l_n(\Lambda) = \sum_{l=1}^{m}\omega_l\big[H_\delta(\Lambda_l) + \Lambda_l u_l\big]. \tag{2.2}$$

The MPLE $\tilde{\Lambda}_n$ can only be identified at t_i's and $\tilde{\Lambda}_n(\cdot;\gamma,\delta)$ can be taken as a nondecreasing piecewise linear function with possible knots at $t_1,\ldots,t_m$. The choice of making $\tilde{\Lambda}_n$ a piecewise linear function is arbitrary and other conventions are possible.

If we follow the same characterization procedure described in Groeneboom and Wellner (1992, p.35-40), the maximum of (2.2) can be written explicitly as

$$\tilde{\Lambda}_n(t_l;\gamma,\delta) = h_\delta^{-1}(-z_l), \qquad l = 1,\ldots,m, \tag{2.3}$$

where

$$z_l = \max_{r\le l}\min_{s\ge l}\frac{\omega_r u_r + \cdots \omega_s u_s}{\omega_r + \cdots + \omega_s}.$$

The computation of z_l is obtained using the well-known Pool Adjacent Violators Algorithm (PAVA) (see e.g., Robertson, Wright and Dykstra (1988)). The profile pseudo-likelihood of (γ,δ) is given by $l_n(\tilde{\Lambda}_n(\cdot;\gamma,\delta),\gamma,\delta)$ and $(\tilde{\gamma}_n,\tilde{\delta}_n) = \arg\max_{\mathcal{R}\times\mathcal{R}} l_n(\tilde{\Lambda}_n(\cdot;\gamma,\delta),\gamma,\delta)$, hence we have a nonlinear optimization problem that can be solved by standard Newton-Raphson methods.

3. Asymptotic Results

In this section, we show the consistency, convergence rate and asymptotic distribution of the MPLE. Let $\mathcal{B}$ denote the collection of Borel sets in $\mathbb{R}$, and let $\mathcal{B}_{[0,T]} = \{B\cap[0,T] : B\in\mathcal{B}\}$ for some fixed constant T. On $([0,T],\mathcal{B}_{[0,T]})$, for $B\in\mathcal{B}_{[0,T]}$, we define

$$\mu(B) = \sum_{k=1}^{\infty}\sum_{j=1}^{k}P(T_{K,j}\in B|K=k) = E_{K,\underline{T}_K}\big[\sum_{j=1}^{K}1_B(T_{K,j})\big].$$

Based on μ, for any $(\Lambda_1,\gamma_1,\delta_1)$ and $(\Lambda_2,\gamma_2,\delta_2)$ we define the L_2 metric

$$d^2((\Lambda_1,\gamma_1,\delta_1),(\Lambda_2,\gamma_2,\delta_2)) = (\gamma_1-\gamma_2)^2 + (\delta_1-\delta_2)^2 + \int(\Lambda_1(t)-\Lambda_2(t))^2 d\mu(t).$$

To establish the consistency of MPLE, assume the following regularity conditions.

A1. The true parameter $\theta_0 = (\gamma_0, \delta_0)$ is in the interior of compact set $\mathcal{R} \times \mathcal{R}$.
A2. The observation times $T_{K,j}$, $j = 1, \ldots, K$ are random and take values in $[0, T]$ with $T < \infty$.
A3. $E(K) < \infty$.
A4. The true mean function at T, $\Lambda_0(T)$ is finite.

The above conditions usually hold in practice.

Theorem 1. (Consistency) *Suppose that* A1$-$A4 *hold. Then for every* $t < T$ *and* $\mu([t, T]) > 0$, $d((\tilde{\Lambda}_n 1_{[0,t]}, \tilde{\gamma}_n, \tilde{\delta}_n), (\Lambda_0 1_{[0,t]}, \gamma_0, \delta_0)) \to 0$, *almost surely as* $n \to \infty$. *If* $\mu(\{T\}) > 0$, *we also have* $d((\tilde{\Lambda}_n, \tilde{\gamma}_n, \tilde{\delta}_n), (\Lambda_0, \gamma_0, \delta_0)) \to 0$ *almost surely as* $n \to \infty$.

To obtain the convergence rate and asymptotic distribution of MPLE, we assume the following conditions.

B1. For some interval $[T_l, T_u]$ with $T_l > 0$ and $\Lambda_0(T_l) > 0$, $P(\cap_{j=1}^{K} T_{K,j} \in [T_l, T_u]) = 1$.
B2. $P(K < k_0) = 1$ for some $k_0 < \infty$.
B3. Let $G_{k,j}(t) = P(T_{K,j} < t | K = k)$. For a fixed $t_0 \in [T_l, T_u]$, there is a neighborhood of t_0 such that $G_{k,j}$ is differentiable, and $G'_{k,j}$ is continuous in this neighborhood, is positive and uniformly bounded for all $j = 1, \ldots, k, k = 1, 2, \ldots$.
B4. The first derivative Λ' is positive and has finite lower and upper bounds in the observation interval.

Condition B1 assumes $T_{K,j}$ is bounded away from zero, and B2 assumes K is finite almost surely. If $G(t) = \mu([0, t])$ is the distribution function corresponding the measure μ, we can write $G(t) = \sum_{k=1}^{\infty} P(K = k) \sum_{j=1}^{k} G_{k,j}(t)$. Condition B3 assures that $G'(t)$ exists. Condition B4 assumes that there is no flat part in Λ, that its derivative is bounded away from zero and infinity.

Theorem 2.(Convergence rate) *Suppose the conditions* A1$-$A4 *and* B1$-$B2 *hold. Then* $n^{1/3} d((\tilde{\Lambda}_n, \tilde{\gamma}_n, \tilde{\delta}_n), (\Lambda_0, \gamma_0, \delta_0)) = O_p(1)$.

Although the overall convergence rate for $\tilde{\Lambda}_n$, $\tilde{\gamma}_n$ and $\tilde{\delta}_n$ is of order $n^{-1/3}$, the rate of of convergence for $\tilde{\gamma}_n$ and $\tilde{\delta}_n$ is still $n^{-1/2}$.

Let $A = -(A_1 - A2)$, where

$$A_1 = E_{K,\underline{T}_K} \begin{bmatrix} -\sum_{j=1}^{K} \frac{\delta_0 \Lambda_0(T_{K,j})}{\gamma_0^2(\Lambda_0(T_{K,j})+\delta_0+1)} & \sum_{j=1}^{K} \frac{\Lambda_0(T_{K,j})}{\gamma_0(\Lambda_0(T_{K,j})+\delta_0)} \\ \sum_{j=1}^{K} \frac{\Lambda_0(T_{K,j})}{\gamma_0(\Lambda_0(T_{K,j})+\delta_0)} & \sum_{j=1}^{K} [\psi_1(\Lambda_0(T_{K,j}) + \delta_0) - \psi_1(\delta_0)] \end{bmatrix},$$

$$A_2 = E_{K,\underline{T}_K} \begin{bmatrix} \sum_{j=1}^K \frac{\delta_0^2}{\gamma_0^2(\Lambda(T_{K,j})+\delta_0)^2 a_j} & -\sum_{j=1}^K \frac{\delta_0\psi_1(\Lambda_0(T_{K,j})+\delta_0)}{\gamma_0(\Lambda_0(T_{K,j})+\delta_0)b_j} \\ -\sum_{j=1}^K \frac{\delta_0\psi_1(\Lambda_0(T_{K,j})+\delta_0)}{\gamma_0(\Lambda_0(T_{K,j})+\delta_0)b_j} & \sum_{j=1}^K \frac{\psi_1^2(\Lambda_0(T_{K,j})+\delta_0)}{[\psi_1(\Lambda_0(T_{K,j})+\delta_0)-\psi_1(\delta_0)]} \end{bmatrix},$$

and $a_j = \psi_1(\Lambda_0(T_{K,j}) + \delta_0) - \psi_1(\Lambda_0(T_{K,j}))$, $b_j = \psi_1(\Lambda_0(T_{K,j}) + \delta_0) - \psi_1(\delta_0)$. Let $B = E_{K,\underline{T}_K}(B_1 B_1')$, where

$$B_1 = \begin{pmatrix} \sum_{j=1}^K \left[\frac{\delta_0}{\gamma_0} - \frac{\Lambda_0(T_{K,j})+\delta_0}{Y_{K,j}+\delta_0} - (\psi_0(\Lambda_0(T_{K,j}) + \delta_0) - \right. \\ \left. \psi_0(\Lambda_0(T_{K,j})) + \log\frac{Y_{K,j}}{Y_{K,j}+\gamma_0})h_{1j}^*\right] \\ \sum_{j=1}^K \left[\psi_0(\Lambda(T_{K,j}) + \delta_0) - \psi_0(\Lambda_0(T_{K,j})) + \log\frac{\gamma_0}{Y_{K,j}+\gamma_0} - \right. \\ \left. (\psi_0(\Lambda(T_{K,j}) + \delta_0) - \psi_0(\Lambda_0(T_{K,j})) + \log\frac{Y_{K,j}}{Y_{K,j}+\gamma_0})h_{2j}^*\right] \end{pmatrix},$$

$$h_{1j}^* = -\frac{\delta_0}{\gamma_0(\Lambda_0(T_{K,j}) + \delta_0)(\psi_1(\Lambda_0(T_{K,j}) + \delta_0) - \psi_1(\Lambda_0(T_{K,j})))},$$

$$h_{2j}^* = \frac{\psi_1(\Lambda_0(T_{K,j}) + \delta_0)}{\psi_1(\Lambda_0(T_{K,j}) + \delta_0) - \psi_1(\Lambda_0(T_{K,j}))},$$

and $\sigma(t_0)^2 = \text{var}[\log(Y(t_0)/(Y(t_0) + \gamma_0))]$, with $\psi_1 = \psi_0'$. Take $\mathbb{Z}$ to denote a two-sided Brownian motion process, starting from zero.

Theorem 3.(Asymptotic Distributions) *Under the conditions of Theorem* 2, *if* $\tilde{\theta}_n = (\tilde{\gamma}_n, \tilde{\delta}_n)'$ *and* $\theta_0 = (\gamma_0, \delta_0)'$ *we have*

$$\sqrt{n}(\tilde{\theta}_n - \theta_0) \xrightarrow{D} N\left[0,\ A^{-1}B(A^{-1})'\right]. \tag{3.1}$$

If, moreover B3 *and* B4 *hold, we have*

$$\begin{aligned} & n^{\frac{1}{3}}(\tilde{\Lambda}_n(t_0) - \Lambda_0(t_0)) \\ & \xrightarrow{D} \left[\frac{\sigma(t_0)^2}{2G'(t_0)[\psi_1(\Lambda_0(t_0)) - \psi_1(\Lambda_0(t_0) + \delta_0)]^2}\right]^{\frac{1}{3}} 2\arg\max_h\{\mathbb{Z}(h) - h^2\}. \end{aligned} \tag{3.2}$$

The proofs of the theorems are established similarly by following the arguments in Wellner and Zhang (2000, 2007, and references therein). For more details of the proofs, see Supplemental Materials.

It is immediate that Theorem 3 provides a way to construct large sample confidence intervals for (γ, δ), and pointwise confidence band for Λ. The distribution $\arg\max_h\{\mathbb{Z}(h) - h^2\}$ is known as Chernoff's distribution and its quantiles are listed in Groeneboom and Wellner (2001).

4. Simulations

A Monte Carlo study is used to validate the method. Let $\{\underline{Y}_{K_i}, \underline{T}_{K_i}, K_i\}$, $i = 1, \ldots, n$, be a random sample. We choose $K_i \in \{9, 10, 11, 12\}$ and $P(K_i = k)$

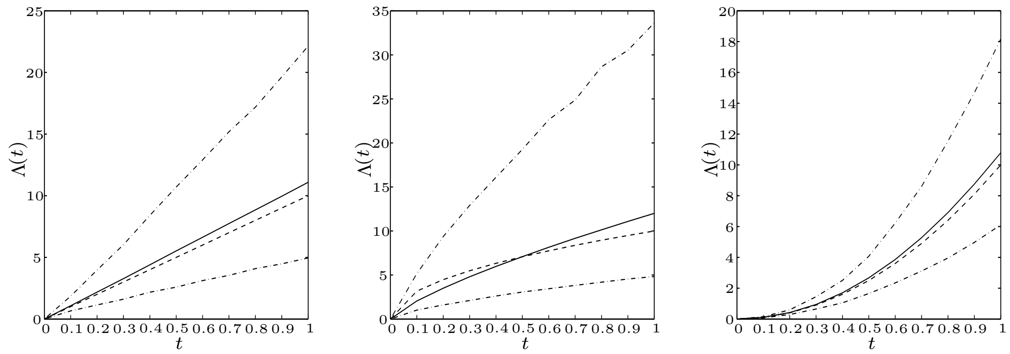

Figure 3.1. The MPLE of the $\Lambda(t)$ when the true function is $10t, 10t^2, 10t^{3/4}$, respectively. The solid line is the estimate from sample size $n = 40$; the dash-dotted lines are 95% pointwise confidence bands; the dashed lines are the true functions.

Table 4.1. Results of Monte Carlo study for γ and δ estimates based on $1,000$ repeated samples for data generated from Gamma process with random effects.

		MPLE of γ		MPLE of δ	
		$n = 40$	$n = 80$	$n = 40$	$n = 80$
$\Lambda(t) = 10t$	BIAS	0.2577	0.1474	0.6338	0.4146
	SD	1.1810	0.9598	2.1028	1.7719
$\Lambda(t) = 10t^2$	BIAS	0.1359	0.0651	0.4262	0.3288
	SD	0.9482	0.6500	1.8975	1.5028
$\Lambda(t) = 10t^{3/4}$	BIAS	0.3441	0.1318	0.7580	0.4078
	SD	1.3066	0.9116	2.0591	1.6135

$= 1/4$ for $k = 9, \ldots, 12$. Then the $\underline{T}_{K_i}$ are chosen as the order statistics of K_i random observations from uniform$(0, 1)$. The time points are rounded to the first decimal point to make the observation times possibly tied. The degradation measurements $\underline{Y}_{K_i}$ are generated from a Gamma process with random effects, that is, $\omega \sim Gamma(\delta, \gamma^{-1})$ and $Y_{K_i,j} - Y_{K_i,j-1}|\omega \sim Gamma(\omega^{-1}, \Lambda(T_{K_i,j}) - \Lambda(T_{K_i,j-1}))$, where $\gamma = 1$, $\delta = 8$, and $\Lambda(t)$ is chosen as one of the functions $10t, 10t^2, 10t^{3/4}$. The number of subjects is $n = 40$ and 80. We carry out a Monte Carlo study by repeating the simulation $1,000$ times. Figure 3.1 displays the MPLE of Λ, along with 95% pointwise confidence bands when $n = 40$. The bias, standard error and mean squared error for the MPLE of γ and δ are given in Table 4.1. The table has the sample bias and standard error for both $\tilde{\gamma}_n$ and $\tilde{\delta}_n$ within reasonable ranges, and that such bias and standard deviation have small effect on the degradation paths and the resulting first passage time distribution.

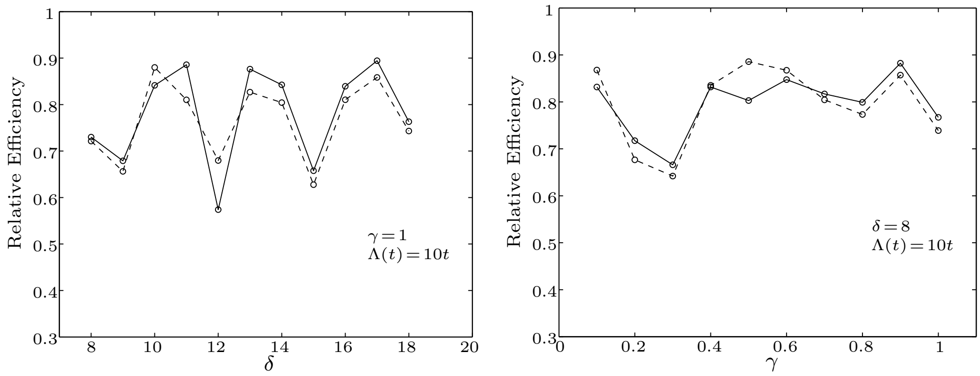

Figure 4.2. The relative efficiency of MLE vs. MPLE. The solid line is the estimated relative efficiency for δ and the dashed line is for γ.

The ratio of the standard error for the two sample sizes is close to $\sqrt{2}$, which suggests that the convergence rate of $(\tilde{\gamma}_n, \tilde{\delta}_n)$ is $\sqrt{n}$.

Although the MPLE is consistent and its implementation is easier and requires less computing time than the MLE, it should be noted that the MLE is apparently more efficient than the MPLE. The complete algorithm to compute the MLE is currently under investigation. Here, a simple Monte Carlo simulation based on $1,000$ runs for $n = 80$ is studied to evaluate the efficiency loss of estimating γ and δ when $\Lambda(t) = 10t$. The left panel of Figure 4.2 displays the estimated relative efficiency of the MLE with respect to the MPLE when $\gamma_0 = 1$ and $\delta_0 = 8, 9, \ldots, 18$; the right panel of Figure 4.2 displays the case when $\delta_0 = 8$ and $\gamma_0 = 0.1, 0.2, \ldots, 1$; the solid line is the estimated relative efficiency for δ, the dashed line is for γ. As can be seen from Figure 4.2, the estimated relative efficiency is about 70%. We expect more efficiency loss when Λ is unknown and needs to be estimated.

5. Application: Bridge Beams Data

We apply our method to the bridge beams data given by Elsayed and Liao (2004). This dataset includes the degradation of bridge beams due to chloride ion ingression for a sample of size 20. The field data considered are bivariate $(t_j, y_{i,j})$, in which $y_{i,j}$ is the measurement of loss of strength of the component i at time t_j, $i = 1, \ldots, 20$, $j = 1, \ldots, 40$. The left panel of Figure 5.3 shows the bridge beams' strength loss from 10 to 40 years. Assume failure occurs when the loss of strength exceeds 400pst. Based on the random effects nonhomogeneous Gamma process model, $\tilde{\gamma}_n = 0.1580$, $\tilde{\delta}_n = 9.3184$, and the estimated $\tilde{\Lambda}_n$ is the solid line given in the right panel of Figure 5.3. The estimated time-to-failure distribution function of bridge beams is shown in the left panel of Figure 5.4 by

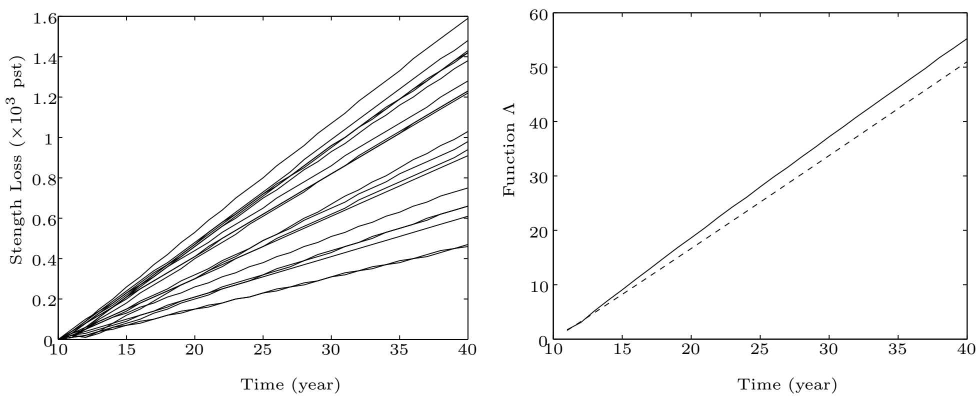

Figure 5.3. The left panel is the bridge beams' strength loss over time. The right panel is the estimated $\tilde{\Lambda}_n$, where the solid line is the maximum pseudo-likelihood estimator and the dashed line is the parametric estimator.

the solid line, along with the Kaplan-Meier estimate, computed from follow-up of each unit until it crosses the failure threshold. We then evaluated the reliability function under three additional threshold levels: $d = 600$, $d = 800$, $d = 1,000$; see the solid lines in the right panel of Figure 5.4. This illustrates the sensitivity of reliability to the threshold levels.

For well-understood failure mechanisms, one may have a parametric form for Λ based on a physical/chemical theory that describes the underlying degradation process. Without such information, the nonparametric estimate of Λ can be used to suggest a parametric form for $\Lambda(t)$. If we assume Λ has a power function form, $\Lambda(t; a, b) = a(t - 10)^b$, the estimated unknown parameters are $\hat{a} = 1.5976$, $\hat{b} = 1.0183$, $\hat{\gamma} = 0.1810$ and $\hat{\delta} = 9.6687$. These estimated values of γ and δ are close to those obtained from the pseudo-likelihood method. The right panel of Figure 5.3 shows the parametric estimate of Λ and its proximity to the pseudo-likelihood estimator. The parametric estimate of the reliability functions under different values of the threshold, shown in Figure 5.4, almost coincide with the pseudo-likelihood estimators.

6. Discussion

An interesting extension of the current model allows covariate information. For example, in an accelerated degradation experiment, temperature might be a covariate. Bagdonavicius and Nikulin (2001) incorporated covariates by replacing $\Lambda(t)$ with $\Lambda(te^{x^T\beta})$, where x is the covariate vector. Lawless and Crowder (2004) treated the scale parameter θ as a function of x to accommodate the covariate. Similar to the Cox model, one can study the proportional mean model,

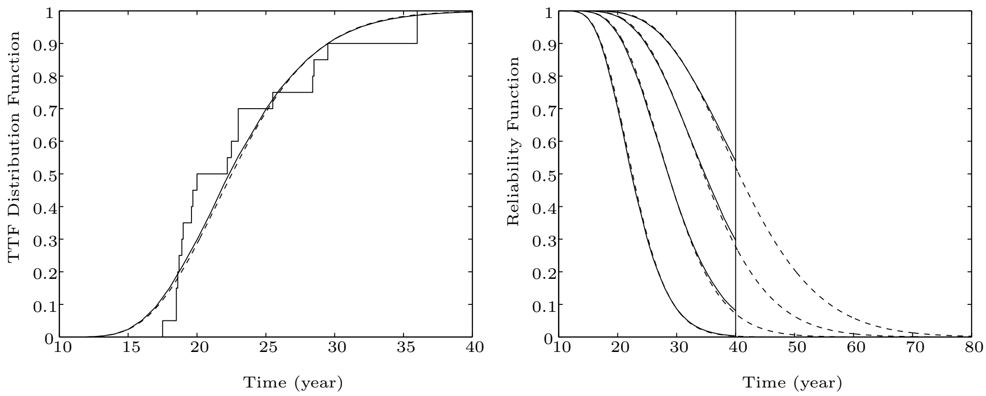

Figure 5.4. The left panel is the estimated time-to-failure distribution function of bridge beams for $d = 400$. The right panel is the reliability estimation for bridge beam at four different threshold levels, $d = 400, 600, 800, 1000$. The solid lines are maximum pseudo-likelihood estimators and the dashed lines are the parametric estimators.

replacing $\Lambda(t)$ by $\Lambda(t)e^{x^T\beta}$ to incorporate the covariate information. For all these models, we can study the pseudo-likelihood inference in a similar way.

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