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PSEUDO-LIKELIHOOD ESTIMATION METHOD FOR NONHOMOGENEOUS GAMMA PROCESS MODEL WITH RANDOM EFFECTS

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Supplementary Material

The proofs of the theorems are established similarly to the nonhomogeneous Poisson process models discussed by Wellner and Zhang (2000, 2007). Let C or C_i, $i = 1, 2, \ldots$, stand for generic constants which may change from line to line in the proof. Let $\mathbb{P}_n$ denote the empirical measure and $\mathbb{G}_n$ denote the empirical process. Denote $\Lambda(T_{K,j})$ and $\Lambda_0(T_{K,j})$ by $\Lambda_{K,j}$ and $\Lambda_{0,K,j}$, respectively. Let $\alpha = (\Lambda, \gamma, \delta)$ and $\tilde{\alpha}_n = (\tilde{\Lambda}_n, \tilde{\gamma}_n, \tilde{\delta}_n)$. The true parameters are represented by Λ_0, γ_0 and δ_0.

Proof of Theorem 1. Denote

$$\tilde{m}_\alpha(X) = \sum_{j=1}^{K} \left[\log \frac{\Gamma(\Lambda_{k,j} + \delta)}{\Gamma(\Lambda_{k,j})\Gamma(\delta)} + \Lambda_{k,j} \log \frac{Y_{k,j}}{Y_{k,j} + \gamma} + \delta \log \frac{\gamma}{Y_{k,j} + \gamma} \right],$$

$\tilde{M}_n(\alpha) = \mathbb{P}_n \tilde{m}_\alpha(X)$ and $\tilde{M}(\alpha) = P\tilde{m}_\alpha(X)$. Then, the MPLE $\tilde{\alpha}_n = \arg\max_\alpha \tilde{M}_n(\alpha)$. Our proof of consistency will use the one-sided Glivenko-Cantelli theorem which is summarized as Theorem A.1 by Wellner and Zhang (2000). We first find the upper envelop function for the function class $\{\tilde{m}_\alpha(X) : \alpha \in \mathcal{F} \times \mathcal{R} \times \mathcal{R}\}$. Note that $H_\delta(x) = \log \Gamma(x+\delta) - \log \Gamma(x)$ is an increasing function of x. We have

$$\tilde{m}_\alpha(X) \leq \sum_{j=1}^{K} \log \frac{\Gamma(\Lambda_j + \delta)}{\Gamma(\Lambda_j)\Gamma(\delta)} \leq \sum_{j=1}^{K} [H_\delta(\Lambda(T)) - \log \Gamma(\delta)] \leq CK.$$

Next we show that $\tilde{\alpha}_n$ is uniformly bounded. $\tilde{\gamma}_n$ and $\tilde{\delta}_n$ is bounded because it is in the bounded compact set $\mathcal{R}$. We only need to show that $\tilde{\Lambda}_n$ is uniformly bounded. Since $\tilde{M}_n(\tilde{\alpha}_n) - \tilde{M}_n(\alpha_0) \geq 0$, it follows that

$$\mathbb{P}_n \sum_{j=1}^{K} \tilde{\Lambda}_{n,K,j} \leq \mathbb{P}_n \sum_{j=1}^{K} \tilde{\Lambda}_{n,K,j} \log \frac{Y_j + \tilde{\gamma}_n}{Y_{K,j}}$$

$$\leq \mathbb{P}_n \sum_{j=1}^{K} \log \frac{\Gamma(\tilde{\Lambda}_j + \tilde{\delta})}{\Gamma(\tilde{\Lambda}_{K,j})\Gamma(\tilde{\delta})} + \tilde{\delta} \log \frac{\tilde{\gamma}}{Y_{K,j} + \tilde{\gamma}} - \tilde{M}_n(\alpha_0)$$

$$\leq \mathbb{P}_n \sum_{j=1}^{K} H_{\tilde{\delta}}(\Lambda(T)) - \log \Gamma(\tilde{\delta}) - \tilde{M}_n(\alpha_0) \leq \mathbb{P}_n CK - \tilde{M}_n(\alpha_0),$$

where the right hand side has finite limit by strong law of large number. On the other hand,

$$\limsup_{n\to\infty} \mathbb{P}_n \sum_{j=1}^{K} \tilde{\Lambda}_{n,K,j} \geq \limsup_{n\to\infty} \mathbb{P}_n \sum_{j=1}^{K} 1_{[b,T]}(T_{K,j})\tilde{\Lambda}_{n,K,j}$$

$$\geq \limsup_{n\to\infty} \tilde{\Lambda}_n(b) \sum_{j=1}^{K} 1_{[b,T]}(T_{K,j}) = \limsup_{n\to\infty} \tilde{\Lambda}_n(b)\mu([b,T]).$$

So, $\tilde{\Lambda}_n$ is uniformly bounded almost surely for $t \in [0,b]$ if $\mu([b,T]) > 0$ for some $0 < b < T$ or for $t \in [0,T]$ if $\mu(\{T\}) > 0$.

First consider the case when $\mu(\{T\}) > 0$ and the other case is similar. We have

$$\limsup_{n\to\infty} \tilde{\Lambda}_n(b) \leq \frac{C}{\mu(\{T\})} = M_T < \infty.$$

By Helly-Selection Theorem, $(\tilde{\Lambda}_n, \tilde{\gamma}_n, \tilde{\delta}_n)$ has a subsequence $(\tilde{\Lambda}_{n'}, \tilde{\gamma}_n, \tilde{\delta}_n)$ converges to $\alpha^+ = (\Lambda^+, \gamma^+, \delta^+)$ where Λ^+ is an increasing function on $[0,T]$ taking values in $[0, M_T]$. Consider the function class

$$\aleph = \{\tilde{m}_\alpha(X) : (\gamma, \delta) \in \mathcal{R} \times \mathcal{R}, \Lambda \in \mathcal{F}_T\},$$

where $\mathcal{F}_T = \{\Lambda \in \mathcal{F} : \Lambda(T) \leq M_T + 1\}$. Note that $\mathcal{F}_T$ is compact under d. Since $\tilde{M}_n(\alpha_0) \to \tilde{M}(\alpha_0)$ by strong law of large number and $\tilde{M}_n(\tilde{\alpha}_n) \geq \tilde{M}_n(\alpha_0)$, we have $\tilde{M}(\alpha_0) \leq \liminf_{n\to\infty} \tilde{M}_n(\tilde{\alpha}_n)$. Moreover, we showed that the function $\tilde{m}_\alpha(X)$ has an integrable envelope function. By one-sided Glivenko-Cantelli theorem, we have

$$\limsup_{n\to\infty} \sup_{\alpha} (\mathbb{P}_n - P)(\tilde{m}_\alpha) \leq 0, \quad a.s.$$

So, $\limsup_{n'\to\infty} \tilde{M}_{n'}(\tilde{\alpha}_{n'}) \leq \tilde{M}(\alpha^+)$.

Next, we show that α_0 is the unique maximum of $\tilde{M}(\alpha)$. Taylor expansion of $\log(Y_{K,j} + \gamma)$ at γ_0, we have

$$\log(Y_{K,j} + \gamma) = \log(Y_{K,j} + \gamma_0) + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(k-1)!}{(Y_{K,j} + \gamma_0)^k} (\gamma - \gamma_0)^k.$$

Then,

$$E[\log(Y_{K,j}+\gamma)] = \psi_0(\Lambda_{0,K,j}) - \psi_0(\delta_0) + \log\gamma_0 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(k-1)!\prod_{i=0}^{k-1}(\delta_0+i)}{\prod_{i=0}^{k-1}(\Lambda_{0,K,j}+\delta_0+i)} \Big(\frac{\gamma-\gamma_0}{\gamma_0}\Big)^k,$$

$$E\Big[\frac{1}{Y_{K,j}+\gamma}\Big] = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}k!\prod_{i=0}^{k-1}(\delta_0+i)}{\prod_{i=0}^{k-1}(\Lambda_{0,K,j}+\delta_0+i)} \frac{(\gamma-\gamma_0)^{k-1}}{\gamma_0^k}.$$

Direct calculation of $\tilde{M}(\alpha) = Pm_\alpha(X)$ yields that $\tilde{M}(\alpha)$ has its unique maxima when $\Lambda = \Lambda_0$, $\gamma = \gamma_0$ and $\delta = \delta_0$. Thus, $\alpha^+ = \alpha_0$, a.s. Finally, the dominated convergency theorem yields the consistency of $\tilde{\alpha}_n$ under d.

Proof of Theorem 2. We derive the rate of convergence of $(\tilde{\Lambda}_n, \tilde{\gamma}_n, \tilde{\delta}_n)$ by checking the conditions in Theorem 3.2.5 or Corollary 3.2.6 of van der Vaart and Wellner (1996). Since α_0 is the maximum of $\tilde{M}(\alpha)$, then the first derivative is zero at α_0 and the second derivative is negative definite. Thus, for α in a neighborhood of α_0, there there exists a constant C such that $\tilde{M}(\alpha) - \tilde{M}(\alpha_0) \le -Cd^2(\alpha, \alpha_0)$.

Let

$$\tilde{M}_\rho = \{\tilde{m}_\alpha(X) - \tilde{m}_{\alpha_0}(X) : d(\alpha, \alpha_0) < \rho\}$$

be a class of functions. To find the convergence rate, we need to find $\phi(\rho)$ such that

$$E \sup_{d(\alpha,\alpha_0)<\rho} \|\mathbb{G}_n\|_{\tilde{M}_\rho} \le C\phi(\rho).$$

We shall find the bracket entropy number for class $\tilde{M}_\delta$. Let $\mathcal{F}_\rho = \{\Lambda \in \mathcal{F} : \|\Lambda - \Lambda_0\|_\mu \le \rho\}$. Since $\mathcal{F}_\rho$ is the class of monotone function, it is well known that the set of all monotone functions possess a bracketing entropy of the order $1/\epsilon$. Therefore, for any $\epsilon > 0$, there exists a set of brackets $[\Lambda_1^l, \Lambda_1^u], \ldots, [\Lambda_q^l, \Lambda_q^u]$ with $q < \exp(M_1/\epsilon)$, such that for any $\Lambda \in \mathcal{F}_\rho$, $\Lambda_i^l(t) < \Lambda(t) < \Lambda_i^u(t)$ for all $t \in [T_l, T_u]$ for some i and $\|\Lambda_i^u - \Lambda_i^l\|_\mu^2 \le \epsilon^2$. From Lemma 8.2 in Wellner and Zhang (2005), we also can make these bracketing functions satisfying that $\Lambda_i^u - \Lambda_i^l \le \gamma_1 = 2\epsilon_2$ and $\Lambda_i^l \ge \gamma_2 = \Lambda_0(T_l) - \epsilon_2$ with $\epsilon_2 = (\sqrt{\epsilon^2 + \delta^2}/C)^{2/3}$ for all $t \in [T_l, T_u]$ and i for sufficient small ϵ and ρ.

Since γ and δ are in a compact set, we can construct an ϵ-net for both γ and δ, $\gamma_1, \ldots, \gamma_p$, with $p = M_2/\epsilon$ such that for any γ there is s such that $|\gamma_s - \gamma| \le \epsilon$. Similarly we have an ϵ-net for δ, $\delta_1, \ldots, \delta_p$. We can construct a set of brackets for $\tilde{M}_\rho$: $[\tilde{m}_{i,s}^l, \tilde{m}_{i,s}^u]$, $i = 1, \ldots, q$, $s = 1, \ldots, p$, where

$$\tilde{m}_{i,s}^l = \sum_{j=1}^{K} \Big[H_{\delta_{s_1}^*}(\Lambda_i^l(T_{K,j})) - \log\Gamma(\delta_{s_2}^*) + \Lambda_i^u(T_{K,j}) \log\frac{Y_{K,j}}{Y_{K,j}+\gamma_s+\epsilon}$$

$$+(\delta_s+\epsilon)\log\frac{\delta_s-\epsilon}{Y_{K,j}+\delta_s-\epsilon}\Bigg]$$

and

$$\tilde{m}^u_{i,s}=\sum_{j=1}^{K}\Bigg[H_{\delta^*_{s_3}}(\Lambda^u_i(T_{K,j}))-\log\Gamma(\delta^*_{s4})+\Lambda^l_i(T_{K,j})\log\frac{Y_{K,j}}{Y_{K,j}+\gamma_s-\epsilon}$$
$$+(\delta_s-\epsilon)\log\frac{\delta_s+\epsilon}{Y_{K,j}+\delta_s+\epsilon}\Bigg],$$

where $\delta^*_{s_i}$, $i=1,2,3,4$ are constants in $[\delta_s-\epsilon,\delta_s+\epsilon]$. In the following, we show that $\|\tilde{m}^u_{i,s}-\tilde{m}^l_{i,s}\|_{P,B}\le C\epsilon^2$ where $\|\cdot\|_{P,B}$ is the "Bernstein norm" defined by

$$\|f\|_{P,B}=\sqrt{2P(e^{|f|}-1-|f|)}.$$

Since $2(e^x-1-x)\le x^2e^x$ for $x>0$, we have $\|f\|^2_{P,B}\le P(e^{|f|}|f|^2)$. With simple algebra, we can see that $\tilde{m}^u_{i,s}-\tilde{m}^l_{i,s}$ are all uniformly bounded and there exists a constant C such that

$$\|\tilde{m}^u_{i,s}-\tilde{m}^l_{i,s}\|_{P,B}\le C\epsilon^2.$$

This shows that the total number of ϵ-brackets for $\tilde{M}_\rho$ will be of order M_1/ϵ $\exp(CM_2/\epsilon)$ and

$$\log N_{[\ ]}(\epsilon,\tilde{M}_\rho,\|\cdot\|_{P,B})\le\frac{C}{\epsilon}.$$

Similarly, we can show that $P(\tilde{m}_\alpha(X)-\tilde{m}_{\alpha_0}(X))\le C\rho^2$ for any $\tilde{m}_\alpha(X)-\tilde{m}_{\alpha_0}(X)\in\tilde{M}_\rho(\alpha_0)$. By Lemma 3.4.3 of van der Vaart and Wellner (1996) or Lemma 8.3 of van der Vaart (2002),

$$E^*_P\|\mathbb{G}_n\|_{\tilde{M}_\rho}\le CJ_{[\ ]}(\rho,\tilde{M}_\rho,||\cdot||_{P,B})\left(1+\frac{J_{[\ ]}(\rho,\tilde{M}_\rho,\|\cdot\|_{P,B})}{\rho^2\sqrt{n}}\right),$$

where

$$J_{[\ ]}(\rho,\tilde{M}_\rho,\|\cdot\|_{P,B})=\int_0^\rho\sqrt{1+\log N_{[\ ]}(\epsilon,\tilde{M}_\rho(\alpha_0),\|\cdot\|_{P,B})}d\epsilon$$
$$=C\int_0^\rho\sqrt{1+\frac{1}{\epsilon}}d\epsilon\le C\int_0^\rho\epsilon^{-\frac{1}{2}}d\epsilon\le C\rho^{\frac{1}{2}}.$$

So, $\phi_n(\rho)=\rho^{1/2}(1+\rho^{1/2}/(\rho^2/\sqrt{n}))=\rho^{1/2}+\rho^{-1}/\sqrt{n}$, and $\phi_n(\rho)/\rho$ is a decreasing function of ρ, and $n^{2/3}\phi_n(n^{-1/3})=2n^{1/2}$. So, by Theorem 3.2.5 of van der Vaart and Wellner (1996), we have $n^{1/3}d(\tilde{\alpha}_n,\alpha_0)=O_p(1)$.

Proof of Theorem 3. We first show the asymptotic normal distribution of $\tilde{\theta}_n = (\tilde{\gamma}_n, \tilde{\delta}_n)'$ with convergency rate $\sqrt{n}$. This is done by checking conditions A1-A6 of Theorem 7.1 in Wellner and Zhang (2005), which is a generalization of Huang (1996). Let $\alpha = (\Lambda, \gamma, \delta)$, $\theta = (\gamma, \delta)$, $\Lambda_t(y) = \int_0^y (1 + th(x))d\Lambda(x)$ and $h_j = \int_0^{T_{K,j}} h(x)d\Lambda(x)$. Denote

$$m(\alpha) = \sum_{j=1}^{K} \left[\log \frac{\Gamma(\Lambda_{K,j} + \delta)}{\Gamma(\Lambda_{K,j})\Gamma(\delta)} + \Lambda_{K,j} \log \frac{Y_{K,j}}{Y_{K,j} + \gamma} + \delta \log \frac{\gamma}{Y_{K,j} + \gamma} \right],$$

$$m_1(\alpha) = \frac{\partial m}{\partial \theta} = \left(\sum_{j=1}^{K} \Big[\frac{\delta}{\gamma} - \frac{\Lambda_{K,j} + \gamma}{Y_{K,j} + \gamma} \Big], \sum_{j=1}^{K} \Big[\psi_0(\Lambda_{K,j} + \delta) - \psi_0(\delta) + \log \frac{\gamma}{Y_{K,j} + \gamma} \Big] \right),$$

$$m_2(\alpha)[h] = \frac{\partial m(\Lambda_t, \gamma, \delta)}{\partial t}|_{t=0} = \sum_{j=1}^{K} \Big[\psi_0(\Lambda_{K,j} + \delta) - \psi_0(\delta) + \log \frac{\gamma}{Y_{K,j} + \gamma} \Big] h_j,$$

$$m_{11}(\alpha) = \nabla_\theta^2 m = \begin{bmatrix} \sum_{j=1}^{K} [-\frac{\delta}{\gamma^2} + \frac{\Lambda_{K,j}+\delta}{(Y_{K,j}+\gamma)^2}] & \sum_{j=1}^{K} [\frac{1}{\gamma} - \frac{1}{Y_{K,j}+\gamma}] \\ \sum_{j=1}^{K} [\frac{1}{\gamma} - \frac{1}{Y_{K,j}+\gamma}] & \sum_{j=1}^{K} [\psi_1(\Lambda_{K,j} + \delta) - \psi_1(\delta)] \end{bmatrix},$$

$$m_{12}(\alpha)[h] = \frac{\partial m_2(\alpha)[h]}{\partial \theta} = \left(\sum_{j=1}^{K} \Big[- \frac{h_j}{Y_{K,j} + \gamma} \Big], \quad \sum_{j=1}^{K} [\psi_1(\Lambda_{K,j} + \delta) h_j] \right)$$

$$m_{22}(\alpha)[h^1, h^2] = \frac{\partial m_2(\Lambda_t, \gamma, \delta)}{\partial t}\Big|_{t=0} = \sum_{j=1}^{K} \Big[\psi_1(\Lambda_{K,j} + \delta) - \psi_1(\Lambda_{K,j}) \Big] h_j^1 h_j^2.$$

Let $\dot{S}_{11} = Pm_{11}$, $\dot{S}_{12} = Pm_{21} = Pm_{12}$ and $\dot{S}_{22} = Pm_2$. The least favorable directions h_1^* and h_2^* are defined as $\dot{S}_{12} - \dot{S}_{22}[h^*, h] = 0$ for all h. After straightforward algebra, we may choose

$$h_{1j}^* = -\frac{\delta}{\gamma(\Lambda_{K,j} + \delta)(\psi_1(\Lambda_{K,j} + \delta) - \psi_1(\Lambda_{K,j}))},$$

$$h_{2j}^* = \frac{\psi_1(\Lambda_{K,j} + \delta)}{\psi_1(\Lambda_{K,j} + \delta) - \psi_1(\Lambda_{K,j})},$$

for $j = 1, \ldots, K$.

To verify A4, we need check $\mathbb{P}_n m_1(\tilde{\alpha}_n) = o_p(n^{-1/2})$ and $\mathbb{P}_n m_2(\tilde{\alpha}_n)[h^*] = o_p(n^{-1/2})$. The first part holds since $\tilde{\alpha}_n$ satisfies the pseudo-score function and $\mathbb{P}_n m_1(\tilde{\alpha}_n) = 0$. Since $(\tilde{\Lambda}_n, \tilde{\gamma}_n, \tilde{\delta}_n)$ maximizes $\mathbb{P}_n m(\Lambda, \gamma, \delta)$ over the feasible region, consider the path $(\tilde{\Lambda}_n + \epsilon h, \tilde{\gamma}_n, \tilde{\delta}_n)$ for nondecreasing h. Then,

$$\lim_{\epsilon \downarrow 0} \frac{d}{d\epsilon} \mathbb{P}_n m(\tilde{\Lambda}_n + \epsilon h, \tilde{\gamma}_n, \tilde{\delta}_n) = \mathbb{P}_n m_2(\tilde{\alpha}_n)[h] = 0.$$

We may choose $h = h_1^*$ and $h = -h_2^*$ to show the second part, where h_1^* is an increasing function and h_2^* is a decreasing function.

To verify A5, note that

$$\sqrt{n}(\mathbb{P}_n - P)(m_1(\alpha; X) - m_1(\alpha_0; X)) = \mathbb{G}_n a_\alpha(X),$$
$$\sqrt{n}(\mathbb{P}_n - P)(m_2(\alpha; X)[h^*] - m_2(\alpha_0; X)[h^*]) = \mathbb{G}_n b_\alpha(X),$$

where

$$a_\alpha(X) = \Bigg(\sum_{j=1}^K \Big[\frac{\delta}{\gamma} - \frac{\delta_0}{\gamma_0} - \frac{\Lambda_{K,j} + \gamma}{Y_{K,j} + \gamma} + \frac{\Lambda_{0,K,j} + \gamma_0}{Y_{K,j} + \gamma_0} \Big], \quad \sum_{j=1}^K \Big[\psi_0(\Lambda_{K,j} + \delta) - \psi_0(\delta) - \psi_0(\Lambda_{0,K,j} + \delta_0) + \psi_0(\delta_0) + \log \frac{\gamma}{Y_{K,j} + \gamma} - \log \frac{\gamma_0}{Y_{K,j} + \gamma_0} \Big] \Bigg),$$

$$b_\alpha(X) = \Bigg(\sum_{j=1}^K \Big[\psi_0(\Lambda_{K,j} + \delta) - \psi_0(\Lambda_{0,K,j} + \delta_0) - \psi_0(\delta) + \psi_0(\delta_0) + \log \frac{\gamma}{Y_{K,j} + \gamma} - \log \frac{\gamma_0}{Y_{K,j} + \gamma_0} \Big] h_{1j}^*, \quad \sum_{j=1}^K \Big[\psi_0(\Lambda_{K,j} + \delta) - \psi_0(\Lambda_{0,K,j} + \delta_0) - \psi_0(\delta) + \psi_0(\delta_0) + \log \frac{\gamma}{Y_{K,j} + \gamma} - \log \frac{\gamma_0}{Y_{K,j} + \gamma_0} \Big] h_{2j}^* \Bigg).$$

For any $\eta > 0$, define $A(\eta) = \{a_\alpha : d(\alpha, \alpha_0) \le \eta\}$ and $B(\eta) = \{b_\alpha : d(\alpha, \alpha_0) \le \eta\}$. Then by applying the same bracketing argument with the rate of convergence proof, we can show that both classes $A(\eta)$ and $B(\eta)$ are P-Donsker. We also can show that $\sup_{a \in A(\eta)} \rho_P(a_\alpha(X)) \to 0$ and $\sup_{b \in B(\eta)} \rho_P(b_\alpha(X)) \to 0$. Then, by Corollary 2.3.12 of van der Vaart and Wellner (1996), we have

$$\sup_{|\theta - \theta_0| \le \epsilon_n, \|\Lambda - \Lambda_0\|_\mu \le C n^{-\frac{1}{3}}} |\mathbb{G}_n a_\alpha(X)| = o_P(1)$$

and

$$\sup_{|\theta - \theta_0| \le \epsilon_n, \|\Lambda - \Lambda_0\|_\mu \le C n^{-\frac{1}{3}}} |\mathbb{G}_n b_\alpha(X)| = o_P(1).$$

To verify A6, direct algebra yields

$$P\Big\{ m_1(\alpha; X) - m_1(\alpha_0; X) - m_{11}(\alpha_0; X)(\theta - \theta_0) - m_{12}(\alpha_0; X)[\Lambda - \Lambda_0] \Big\}$$
$$\le C(\|\theta - \theta_0\|^2 + \|\Lambda - \Lambda_0\|_\mu^2) = o(|\gamma - \gamma_0|) + o(|\delta - \delta_0|) + O(\|\Lambda - \Lambda_0\|_\mu^2).$$

Similarly, we can show that

$$P\Big\{ m_2(\alpha; X)[h^*] - m_2(\alpha_0; X)[h^*] - m_{21}(\alpha_0; X)[h^*](\theta - \theta_0)$$

$$-m_{22}(\alpha_0; X)[h^*, \Lambda - \Lambda_0]\}$$
$$\leq C(\|\theta - \theta_0\|^2 + \|\Lambda - \Lambda_0\|_\mu^2).$$

So we finish the proof of the first part of Theorem 3.

Recall that

$$u_l = u_l(\gamma) = \frac{1}{\omega_l} \sum_{i=1}^{n} \sum_{j=1}^{K_i} \Big(\log \frac{Y_{K_i,j}}{Y_{K_i,j} + \gamma} \Big) 1_{\{T_{K_i,j}^{(i)} = t_l\}}, \quad l = 1, \ldots, m,$$

and the isotonic version of $u_1(\tilde{\gamma}_n), \ldots, u_m(\tilde{\gamma}_n)$, say $\tilde{\Upsilon}_n(t; \tilde{\gamma}_n)$, is the estimate of function $\Upsilon(t) = -h_{\delta_0}(\Lambda_0(t))$. Since $\tilde{\Lambda}_n(t) = h_{\tilde{\delta}_n}^{-1}(-\tilde{\Upsilon}_n(t; \tilde{\gamma}_n))$, we first derive the asymptotic distribution of $\tilde{\Upsilon}_n(t; \tilde{\gamma}_n)$ and then use the Δ-method to obtain the asymptotic distribution of $\tilde{\Lambda}_n(t_0)$. Define two stochastic processes:

$$V_n(t; \tilde{\gamma}_n) = \sum_{l=1}^{m} \omega_l u_l(\tilde{\gamma}_n) 1_{\{T_{K_i,j}^{(i)} = t_l\}}, \qquad U_n(t) = \sum_{l=1}^{m} \omega_l 1_{\{T_{K_i,j}^{(i)} = t_l\}}.$$

Following the same arguments of Theorem 4.3 in Wellner and Zhang (2000), we have

$$P\Big[n^{\frac{1}{3}}(\tilde{\Upsilon}_n(t_0; \tilde{\gamma}_n) - \Upsilon(t_0)) \leq x\Big]$$
$$= P\Big[\arg\min_h \{V_n(t_0 + n^{-\frac{1}{3}}h; \tilde{\gamma}_n) - (\Upsilon(t_0) + n^{-\frac{1}{3}}x) U_n(t_0 + n^{-\frac{1}{3}}h)\} \geq 0\Big]. \quad (1)$$

Now rewrite V_n and U_n as

$$V_n(t; \gamma) = \sum_{i=1}^{n} \sum_{j=1}^{K_i} (\log \frac{Y_{K_i,j}}{Y_{K_i,j} + \gamma}) 1_{\{T_{K_i,j}^{(i)} \leq t\}} = n\mathbb{P}_n \sum_{j=1}^{K} (\log \frac{Y_{K,j}}{Y_{K,j} + \gamma}) 1_{\{T_{K,j} \leq t\}},$$

$$U_n(t) = \sum_{i=1}^{n} \sum_{j=1}^{K_i} 1_{\{T_{K_i,j}^{(i)} \leq t\}} = n\mathbb{P}_n \sum_{j=1}^{K} 1_{\{T_{K,j} \leq t\}}.$$

Then the argmin term in right hand side of (1) can be rewritten as

$$\arg\min_h \Big\{ V_n(t_0 + n^{-\frac{1}{3}}h; \tilde{\gamma}_n) - (\Upsilon(t_0) + n^{-\frac{1}{3}}x) U_n(t_0 + n^{-\frac{1}{3}}h) \Big\}$$
$$= \arg\min_h \Big\{ n^{\frac{2}{3}} \mathbb{P}_n \sum_{j=1}^{K} \Big(\log \frac{Y_{K,j} + \gamma_0}{Y_{K,j} + \tilde{\gamma}_n} \Big) \Big(1_{\{T_{K,j} \leq t + n^{-\frac{1}{3}}h\}} - 1_{\{T_{K,j} \leq t\}} \Big)$$
$$+ n^{-\frac{1}{3}} V_n(t_0 + n^{-\frac{1}{3}}h; \gamma_0) - n^{-\frac{1}{3}} \Big(\Upsilon(t_0) + n^{-\frac{1}{3}}x \Big) U_n\Big(t_0 + n^{-\frac{1}{3}}h \Big) \Big\}.$$

By applying the same bracketing argument with the rate of convergence proof, we also have, for $\epsilon_n = O_p(n^{-1/2})$,

$$\sup_{|\gamma-\gamma_0|\le epsilon_n} n^{\frac{2}{3}}\mathbb{P}_n \sum_{j=1}^{K} \Big(\log \frac{Y_{K,j}+\gamma_0}{Y_{K,j}+\gamma}\Big)\Big(1_{\{T_{K,j}\le t+n^{-\frac{1}{3}}h\}} - 1_{\{T_{K,j}\le t\}}\Big) = o_p(1).$$

Wellner and Zhang (2000) also showed that

$$\begin{aligned} &n^{-\frac{1}{3}}V_n(t_0+n^{-\frac{1}{3}}h;\gamma_0) - n^{-\frac{1}{3}}(\Upsilon(t_0)+n^{-\frac{1}{3}}x)U_n(t_0+n^{-\frac{1}{3}}h)\\ &\to^D \sqrt{\sigma^2(t_0)G'(t_0)}\mathbb{Z}(h) + \frac{1}{2}\Upsilon'(t_0)G'(t_0)h^2 - G'(t_0)xh. \end{aligned}$$

Thus, combining the above results, by the *Argmax Continuous Mapping Theorem* (Van der Varr and Wellner 1996, page 286), we have the following limiting process:

$$\begin{aligned} &\arg\min_h \Big\{V_n(t_0+n^{-\frac{1}{3}}h;\tilde{\gamma}_n) - (\Upsilon(t_0)+n^{-\frac{1}{3}}x)U_n(t_0+n^{-\frac{1}{3}}h)\Big\}\\ &\to^D \arg\min_h \Big\{\sqrt{\sigma^2(t_0)G'(t_0)}\mathbb{Z}(h) + \frac{1}{2}\Upsilon'(t_0)G'(t_0)h^2 - G'(t_0)xh\Big\}. \end{aligned}$$

Hence,

$$n^{\frac{1}{3}}\Big(\tilde{\Upsilon}_n(t_0;\tilde{\gamma}_n) - \Upsilon(t_0)\Big) \to^d \Big[\frac{\sigma(t_0)^2\Upsilon_0'(t_0)}{2G'(t_0)}\Big]^{\frac{1}{3}} 2\arg\max_h\{\mathbb{Z}(h)-h^2\},$$

where $\sigma^2(t_0) = \mathrm{var}(\log\frac{Y(t_0)}{Y(t_0)+\gamma_0})$. Further, since the convergence rate for $\tilde{\delta}_n$ is $\sqrt{n}$,

$$n^{\frac{1}{3}}\Big[h^{-1}_{\tilde{\delta}_n}(\tilde{\Upsilon}_n(t_0;\tilde{\gamma}_n)) - h^{-1}_{\delta_0}(\tilde{\Upsilon}_n(t_0;\tilde{\gamma}_n))\Big] = o_p(1).$$

Finally, by Δ-method, we have the proof of the second part of Theorem 3.

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