A Semiparametrically Efficient Estimator of Single-index Varying Coefficient Cox Proportional Hazards Models

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Supplementary Material

Theorems 5–8.

Let $\mathcal{C}_0 = \{\boldsymbol{\delta}(v) = (\delta_1(v), \cdots, \delta_p(v)) : v \in [0,1], \delta_1(0) = 0, \boldsymbol{\delta}(v) \text{ is continuous on } [0,1]\}$, $\boldsymbol{\Theta} = (\boldsymbol{\theta}', \boldsymbol{\alpha}')'$, and $f(v;\boldsymbol{\alpha})$ be the conditional density function of $\mathbf{X}'\boldsymbol{\alpha}$. Denote $W(\boldsymbol{\alpha}, \boldsymbol{\delta}_2) = (\mathbf{Z}_1', \boldsymbol{\delta}_2(\mathbf{X}'\boldsymbol{\alpha})'\mathbf{Z}_2\mathbf{X}')'$,

$$
\begin{aligned}
&\mu_i = \int x^i K(x)dx, \quad \nu_i = \int x^i K^2(x)dx, \quad P(t \mid \mathbf{z}, \mathbf{x}) = \Pr(\mathcal{T} \geq t \mid \mathbf{Z} = \mathbf{z}, \mathbf{X} = \mathbf{x}),\\
&\Gamma(\mathbf{z}, \mathbf{x}) = \int_0^\tau P(t \mid \mathbf{z}, \mathbf{x})\lambda_0(t)dt, \quad s_{00}(t; \boldsymbol{\Theta}, \boldsymbol{\delta}_1) = E\left[P(t \mid \mathbf{Z}, \mathbf{X}) \exp\{\mathbf{Z}_1'\boldsymbol{\theta} + \boldsymbol{\delta}_1(\mathbf{X}'\boldsymbol{\alpha})'\mathbf{Z}_2\}\right],\\
&s_{10}(t; \boldsymbol{\Theta}, \boldsymbol{\delta}_1, v) = E\left[\mathbf{Z}_2 P(t \mid \mathbf{Z}, \mathbf{X}) \exp\{\mathbf{Z}_1'\boldsymbol{\theta} + \boldsymbol{\delta}_1(v)'\mathbf{Z}_2\} \mid \mathbf{X}'\boldsymbol{\alpha} = v\right] f(v; \boldsymbol{\alpha}),\\
&s_{20}(t; \boldsymbol{\Theta}, \boldsymbol{\delta}_1, v) = E\left[\mathbf{Z}_2\mathbf{Z}_2' P(t \mid \mathbf{Z}, \mathbf{X}) \exp\{\mathbf{Z}_1'\boldsymbol{\theta} + \boldsymbol{\delta}_1(v)'\mathbf{Z}_2\} \mid \mathbf{X}'\boldsymbol{\alpha} = v\right] f(v; \boldsymbol{\alpha}),\\
&s_{11}(t; \boldsymbol{\Theta}, \boldsymbol{\delta}_1, v) = E\left[\mathbf{Z}_2\mathbf{X}' P(t \mid \mathbf{Z}, \mathbf{X}) \exp\{\mathbf{Z}_1'\boldsymbol{\theta} + \boldsymbol{\delta}_1(v)'\mathbf{Z}_2\} \mid \mathbf{X}'\boldsymbol{\alpha} = v\right] f(v; \boldsymbol{\alpha}),\\
&r_0(t; \boldsymbol{\alpha}_1, \boldsymbol{\Theta}, \boldsymbol{\delta}_1, \boldsymbol{\delta}_2) = E\left[P(t \mid \mathbf{Z}, \mathbf{X}) W(\boldsymbol{\alpha}_1, \boldsymbol{\delta}_2) \exp\{\mathbf{Z}_1'\boldsymbol{\theta} + \boldsymbol{\delta}_1(\mathbf{X}'\boldsymbol{\alpha})'\mathbf{Z}_2\}\right],\\
&r_1(t; \boldsymbol{\alpha}_1, \boldsymbol{\Theta}, \boldsymbol{\delta}_1, \boldsymbol{\delta}_2, v) = E\left[P(t \mid \mathbf{Z}, \mathbf{X}) W(\boldsymbol{\alpha}_1, \boldsymbol{\delta}_2) \exp\{\mathbf{Z}_1'\boldsymbol{\theta} + \boldsymbol{\delta}_1(v)'\mathbf{Z}_2\}\mathbf{Z}_2' \mid \mathbf{X}'\boldsymbol{\alpha} = v\right] f(v; \boldsymbol{\alpha}),\\
&m_1(t) = E\left[P(t \mid \mathbf{Z}, \mathbf{X}) \exp\{\mathbf{Z}_1'\boldsymbol{\theta}_0 + \boldsymbol{\beta}(\mathbf{X}'\boldsymbol{\alpha}_0)'\mathbf{Z}_2\} W(\boldsymbol{\alpha}_0, \dot{\boldsymbol{\beta}})\right],\\
&m_2(t) = E\left[P(t \mid \mathbf{Z}, \mathbf{X}) \exp\{\mathbf{Z}_1'\boldsymbol{\theta}_0 + \boldsymbol{\beta}(\mathbf{X}'\boldsymbol{\alpha}_0)'\mathbf{Z}_2\} W(\boldsymbol{\alpha}_0, \dot{\boldsymbol{\beta}}) W(\boldsymbol{\alpha}_0, \dot{\boldsymbol{\beta}})'\right],\\
&\kappa(t, v) = E\left[P(t \mid \mathbf{Z}, \mathbf{X})\mathbf{Z}_2 W(\boldsymbol{\alpha}_0, \dot{\boldsymbol{\beta}})' \exp\{\mathbf{Z}_1'\boldsymbol{\theta}_0 + \mathbf{Z}_2'\boldsymbol{\beta}(v)\} \mid \mathbf{X}'\boldsymbol{\alpha}_0 = v\right] f(v; \boldsymbol{\Theta}_0),\\
&\mathbf{D}_1 = \int_0^\tau \left\{\frac{r_0(t)m_1(t)'}{s_{00}(t)} - m_2(t)\right\}\lambda_0(t)dt, \quad \mathbf{D}_2(v) = \int_0^\tau \left\{\frac{s_{10}(t;v)m_1(t)'}{s_{00}(t)} - \kappa(t, v)\right\}\lambda_0(t)dt,\\
&\mathbf{D}_3(v) = \int_0^\tau \left\{\frac{r_0(t)s_{10}(t;v)'}{s_{00}(t)} - r_1(t;v)\right\}\lambda_0(t)dt, \quad \boldsymbol{\Sigma}(v) = \int_0^\tau s_{20}(t;v)\lambda_0(t)dt,\\
&\Psi(w;v) = \int_0^\tau \left[\frac{s_{10}(t;v)s_{10}(t;w)'}{s_{00}(t)} - \mathbf{D}_2(v)\mathbf{D}_1^{-1}\left\{\frac{r_0(t)s_{10}(t;w)'}{s_{00}(t)} - r_1(t;w)\right\}\right]\lambda_0(t)dt,\\
&r_0(t) = r_0(t; \boldsymbol{\alpha}_0, \boldsymbol{\Theta}_0, \boldsymbol{\beta}, \dot{\boldsymbol{\beta}}), \quad r_1(t;v) = r_1(t; \boldsymbol{\alpha}_0, \boldsymbol{\Theta}_0, \boldsymbol{\beta}, \dot{\boldsymbol{\beta}}, v), \quad s_{00}(t) = s_{00}(t; \boldsymbol{\Theta}_0, \boldsymbol{\beta}),
\end{aligned}
$$

$s_{10}(t;v) = s_{10}(t; \boldsymbol{\Theta}_0, \boldsymbol{\beta}, v)$, $s_{20}(t;v) = s_{20}(t; \boldsymbol{\Theta}_0, \boldsymbol{\beta}, v)$ and $s_{11}(t;v) = s_{11}(t; \boldsymbol{\Theta}_0, \boldsymbol{\beta}, v)$.

Let g satisfy the following integral equation in $\mathcal{C}_0$: $\mathbf{D}_3(w) = g(w)\boldsymbol{\Sigma}(w) - \int_0^1 g(v)\Psi(w;v)dv$.

Theorem 5 *Under Conditions (C1)–(C7) stated in Appendix with* $\mathbf{Z} = (\mathbf{Z}_1', \mathbf{Z}_2')'$, *we have*

$$\|\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}\| \to 0, \quad \|\widehat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}\| \to 0, \quad \sup_{0<w<1} \|\widehat{\boldsymbol{\beta}}(w) - \boldsymbol{\beta}(w)\| \to 0,$$

in probability.

Theorem 6 *Denote* $\boldsymbol{\Theta} = (\boldsymbol{\theta}', \boldsymbol{\alpha}')'$. *Under the conditions of Theorem 5, if* $nh^4 = o(1)$, *as* $n \to \infty$

$$\sqrt{n}\left(\widehat{\boldsymbol{\Theta}} - \boldsymbol{\Theta}_0\right) \to N(0, \Delta), \tag{S0.1}$$

where

$$\Delta = \mathbf{D}_1^{-1} \int_0^\tau E\left[\xi_i^2(t) P(t|\mathbf{Z}_{1i}, \mathbf{Z}_{2i}, \mathbf{X}_i) \exp\{\boldsymbol{\theta}'\mathbf{Z}_{1i} + \boldsymbol{\beta}(\mathbf{X}_i'\boldsymbol{\alpha}_0)'\mathbf{Z}_{2i}\}\right] \lambda_0(t) dt (\mathbf{D}_1^{-1})',$$

$$\xi_i(t) = \int_0^1 g(v) \frac{s_{10}(t;v)}{s_{00}(t)} dv + \left\{\int_0^1 g(v)\mathbf{D}_2(v)\mathbf{D}_1^{-1} dv - I\right\}\left\{W_i - \frac{r_0(t)}{s_{00}(t)}\right\} - \mathbf{Z}_{2i} g(\mathbf{X}_i'\boldsymbol{\alpha}_0),$$

$W_i = (\mathbf{Z}_{1i}', \dot{\boldsymbol{\beta}}(\mathbf{X}_i'\boldsymbol{\alpha}_0)'\mathbf{Z}_{2i}\mathbf{X}_i')'$, $\mathbf{D}_1$, $\mathbf{D}_2$, $P(t|\mathbf{z}, \mathbf{x})$, $g(\cdot)$, $s_{10}(\cdot)$, $s_{00}(\cdot)$, $r_0(\cdot)$ *are defined in the Supplementary Material.*

Theorem 7 *Under the conditions of Theorem 5, if* $nh^5 = O(1)$, *for* $v \in (0, 1)$,

$$(nh)^{1/2}\left\{\widehat{\boldsymbol{\beta}}(v) - \boldsymbol{\beta}(v) - \frac{1}{2}h^2\mu_2(I - \mathcal{A})^{-1}(\ddot{\boldsymbol{\beta}})(v)\right\} \xrightarrow{\mathcal{D}} N(0, \mathbf{V}(v)), \tag{S0.2}$$

where I, $\mathcal{A}$ *and* $\mathbf{V}(v)$ *are defined similarly with those in Theorem 3 except that* $\boldsymbol{\Sigma}(v)$ *and* $\Psi(w; v)$ *are defined in the Supplementary Material.*

For any function $\phi(w) = \{\phi_1', \phi_2(w)'\}'$, which has a continuous second derivative on $[0, \tau]$, let $\phi_1'\widehat{\boldsymbol{\Theta}} + \int_0^\tau \phi_2'(w)\widehat{\boldsymbol{\beta}}(w)dw$ be the proposed estimator of $\phi_1'\boldsymbol{\Theta} + \int_0^\tau \phi_2'(w)\boldsymbol{\beta}(w)dw$. Theorem 8 below shows that $\widehat{\boldsymbol{\Theta}}$ is an efficient estimator of $\boldsymbol{\Theta}$ and $\widehat{\boldsymbol{\beta}}(\cdot)$ is a semiparametrically efficient estimator of $\boldsymbol{\beta}(\cdot)$.

Theorem 8 *Under the conditions of Theorem 5, if* $nh^4 = o(1)$, *then, as* $n \to \infty$,

$$\phi_1'\widehat{\boldsymbol{\Theta}} + \int_0^\tau \phi_2'(w)\widehat{\boldsymbol{\beta}}(w)dw \ \text{ is an efficient estimator of } \ \phi_1'\boldsymbol{\Theta} + \int_0^\tau \phi_2'(w)\boldsymbol{\beta}(w)dw.$$