Extended Tapered Block Bootstrap

Xiaofeng Shao

University of Illinois at Urbana-Champaign

Supplementary Material

This note contains proofs for Theorems 3.1 and 3.2.

Appendix

In the appendix, $C > 0$ denotes a generic constant that may vary from line to line. Denote by $\mathbb{Z}_+ = \{0, 1, 2 \cdots, \}$. For $v = (v_1, \cdots, v_p)' \in (\mathbb{Z}_+)^p$, $x \in \mathbb{R}^p$, write $x^v = \Pi_{i=1}^p x_i^{v_i}$, $v! = \Pi_{i=1}^p (v_i!)$. For a vector $x = (x_1, \cdots, x_d)' \in \mathbb{R}^d$, let $\|x\|_1 = \sum_{i=1}^d |x_i|$ denote the l^1 norm of x. Write $c_v = D^v H(\mu)/v!$. The following three statements correspond to the equations (4), (5) and (6) in the paper and are needed in our proofs.

$$\begin{aligned} \operatorname{cov}^*(\tilde{f}_t, \tilde{f}_s) &\approx O(1/k) + l\|w_l\|_1^{-2} \sum_{h=1+(t-s)\vee 0}^{l+(t-s)\wedge 0} w_l(h) w_l\{h-(t-s)\} \quad (1.1) \\ &\quad \times\{1+O(1/k)\}. \\ H(\bar{X}_N) &= H(\mu) + \nabla' N^{-1} \sum_{t=1}^N (X_t - \mu) + R_N. \quad (1.2) \\ H\left(\bar{X}_N^*\right) &= H(\mu) + \nabla'(\bar{X}_N^* - \mu) + R_N^*. \quad (1.3) \end{aligned}$$

Proof of Theorem 3.1: (i) We apply a Taylor expansion to $H(\bar{X}_N^*)$ around μ and write $H(\bar{X}_N^*) - H(\mu) = J_{0N} + J_{1N} + J_{2N}$, where

$$\begin{aligned} J_{0N} &= \sum_{\|v\|_1=1} c_v(\bar{X}_N^* - \mu)^v, \quad J_{1N} = \sum_{\|v\|_1=2} c_v(\bar{X}_N^* - \mu)^v, \\ J_{2N} &= 3 \sum_{\|v\|_1=3} (v!)^{-1} (\bar{X}_N^* - \mu)^v \int_0^1 (1-w)^2 D^v H\{\mu + w(\bar{X}_N^* - \mu)\} dw. \end{aligned}$$

In the sequel, we shall show

$$\begin{aligned}
\mathbb{E}\{N\mathrm{var}^*(J_{1N})\} &= O(1/N), & (1.4)\\
\mathbb{E}\{N\mathrm{var}^*(J_{2N})\} &= O(1/N^2), & (1.5)\\
\mathbb{E}\{N\mathrm{cov}^*(J_{0N},J_{1N})\} &= O(l/N), & (1.6)\\
\mathbb{E}\{M_l N\mathrm{var}^*(J_{0N})\} &= \sigma_F^2 + B_1 l^{-2} + o(l^{-2}). & (1.7)
\end{aligned}$$

If (1.4)-(1.7) hold, then by the Cauchy-Schwarz inequality, we get

$$N\mathbb{E}|\mathrm{cov}^*(J_{0N},J_{2N})| \le N\mathbb{E}^{1/2}\{\mathrm{var}^*(J_{0N})\}\mathbb{E}^{1/2}\{\mathrm{var}^*(J_{2N})\} = O(1/N) = o(l^{-2})$$

and $N\mathbb{E}|\mathrm{cov}^*(J_{1N},J_{2N})| = O(1/N^{3/2}) = o(l^{-2})$ under the assumption that $l = o(N^{1/3})$. Thus the conclusion follows.

To show (1.4), we note that for every $\|v\|_1 = 2$, there exist $v_1, v_2 \in (\mathbb{Z}^+)^m$, $\|v_1\|_1 = 1$, $\|v_2\|_1 = 1$ and $v = v_1 + v_2$. Then we have $(\bar{X}_N^* - \mu)^v = (\bar{X}_N^* - \mu)^{v_1}(\bar{X}_N^* - \mu)^{v_2}$ and

$$\begin{aligned}
&\mathrm{var}^*\left\{(\bar{X}_N^*-\mu)^v\right\} = \frac{1}{N^4}\mathrm{var}^*\left[\sum_{t_1,t_2=1}^N (X_{t_1}-\mu)^{v_1}(X_{t_2}-\mu)^{v_2}\{\tilde{f}_{t_1}\tilde{f}_{t_2} - \mathbb{E}(\tilde{f}_{t_1}\tilde{f}_{t_2})\}\right]\\
&\quad = \frac{1}{N^4}\sum_{t_1,t_2,t_3,t_4=1}^N (X_{t_1}-\mu)^{v_1}(X_{t_2}-\mu)^{v_2}(X_{t_3}-\mu)^{v_1}(X_{t_4}-\mu)^{v_2} \qquad (1.8)\\
&\quad \times\mathrm{cov}^*(\tilde{f}_{t_1}\tilde{f}_{t_2},\tilde{f}_{t_3}\tilde{f}_{t_4}).
\end{aligned}$$

It is straightforward to show that uniformly in (t_1,t_2,t_3,t_4),

$$\begin{aligned}
\mathrm{cov}^*(\tilde{f}_{t_1}\tilde{f}_{t_2},\tilde{f}_{t_3}\tilde{f}_{t_4}) &= \frac{N^4}{k^4\|w_l\|_1^4}\sum_{j_1,j_2,j_3,j_4=1}^k \sum_{h_1,h_2,h_3,h_4=1}^l w_l(h_1)w_l(h_2)w_l(h_3)w_l(h_4)\\
&= \mathrm{cov}^*\{\mathbf{1}(S_{j_1}=t_1-h_1, S_{j_2}=t_2-h_2), \qquad (1.9)\\
&\quad \mathbf{1}(S_{j_3}=t_3-h_3, S_{j_4}=t_4-h_4)\} = O(1).
\end{aligned}$$

Hence it follows from Assumption 3.2 (with $r=4$) that

$$\begin{aligned}
&\mathbb{E}[N\mathrm{var}^*\{(\bar{X}_N^*-\mu)^v\}] \le \frac{C}{N^3}\sum_{t_1,t_2,t_3,t_4=1}^N |\mathbb{E}\{(X_{t_1}-\mu)^{v_1}(X_{t_2}-\mu)^{v_2}(X_{t_3}-\mu)^{v_1}\\
&\quad (X_{t_4}-\mu)^{v_2}\}| \le \frac{C}{N^3}\sum_{t_1,t_2,t_3,t_4=1}^N \left\{|\mathrm{cov}(X_{t_1}^{v_1},X_{t_2}^{v_2})\mathrm{cov}(X_{t_3}^{v_1},X_{t_4}^{v_2})|\right.\\
&\quad +|\mathrm{cov}(X_{t_1}^{v_1},X_{t_3}^{v_1})\mathrm{cov}(X_{t_2}^{v_2},X_{t_4}^{v_2})| + |\mathrm{cov}(X_{t_1}^{v_1},X_{t_4}^{v_2})\mathrm{cov}(X_{t_2}^{v_2},X_{t_3}^{v_1})|\\
&\quad \left.+|\mathrm{cum}(X_{t_1}^{v_1},X_{t_2}^{v_2},X_{t_3}^{v_1},X_{t_4}^{v_2})|\right\} = O(N^{-1}).
\end{aligned}$$

Thus (1.4) holds. Under Assumption 3.1, we have that

$$\mathrm{var}^*(J_{2N}) \le C\mathbb{E}^*\{\|\bar{X}_N^* - \mu\|^6(1 + \|\mu\|^{2\kappa} + \|\bar{X}_N^* - \mu\|^{2\kappa})\}, \tag{1.10}$$

which implies (1.5) by Lemma 0.1. Next, to show (1.6), we write

$$\begin{aligned}\mathrm{cov}^*(J_{0N}, J_{1N}) &= \sum_{\|v_1\|_1=1}\sum_{\|v_2\|_1=1}\sum_{\|v_3\|_1=1} c_{v_1}c_{v_2+v_3}\\ &\quad\times\mathrm{cov}^*\{(\bar{X}_N^* - \mu)^{v_1}, (\bar{X}_N^* - \mu)^{v_2}(\bar{X}_N^* - \mu)^{v_3}\}.\end{aligned}$$

Note that for any three random variables X, Y, Z, $\mathrm{cov}(X, YZ) = \mathbb{E}[\{X-\mathbb{E}(X)\}\{Y-\mathbb{E}(Y)\}\{Z-\mathbb{E}(Z)\}] + \mathbb{E}(Y)\mathrm{cov}(X,Z) + \mathbb{E}(Z)\mathrm{cov}(X,Y)$. Then for each (v_1, v_2, v_3), we have $\mathrm{cov}^*\{(\bar{X}_N^* - \mu)^{v_1}, (\bar{X}_N^* - \mu)^{v_2}(\bar{X}_N^* - \mu)^{v_3}\} = W_{1N} + W_{2N} + W_{3N}$, where

$$\begin{aligned}W_{1N} &= \mathbb{E}^*[\{(\bar{X}_N^* - \mu)^{v_1} - \mathbb{E}^*((\bar{X}_N^* - \mu)^{v_1})\}\{(\bar{X}_N^* - \mu)^{v_2} - \mathbb{E}^*((\bar{X}_N^* - \mu)^{v_2})\}\\ &\quad\{(\bar{X}_N^* - \mu)^{v_3} - \mathbb{E}^*((\bar{X}_N^* - \mu)^{v_3})\}],\\ W_{2N} &= \mathbb{E}^*(\bar{X}_N^* - \mu)^{v_2}\mathrm{cov}^*\{(\bar{X}_N^* - \mu)^{v_1}, (\bar{X}_N^* - \mu)^{v_3}\},\\ W_{3N} &= \mathbb{E}^*(\bar{X}_N^* - \mu)^{v_3}\mathrm{cov}^*\{(\bar{X}_N^* - \mu)^{v_1}, (\bar{X}_N^* - \mu)^{v_2}\}.\end{aligned}$$

Write $W_{1N} = N^{-3}\sum_{t_1,t_2,t_3=1}^N (X_{t_1}-\mu)^{v_1}(X_{t_2}-\mu)^{v_2}(X_{t_3}-\mu)^{v_3}CF(t_1,t_2,t_3)$, where

$$\begin{aligned}CF(t_1,t_2,t_3) &:= \mathbb{E}^*[\{\tilde{f}_{t_1} - \mathbb{E}^*(\tilde{f}_{t_1})\}\{\tilde{f}_{t_2} - \mathbb{E}^*(\tilde{f}_{t_2})\}\{\tilde{f}_{t_3} - \mathbb{E}^*(\tilde{f}_{t_3})\}]\\ &= \frac{N^3}{k^3\|w_l\|_1^3}\sum_{j=1}^k\sum_{h_1,h_2,h_3=1}^l w_l(h_1)w_l(h_2)w_l(h_3)\\ &\quad\times\mathbb{E}[\{\mathbf{1}(S_j = t_1 - h_1) - P(S_j = t_1 - h_1)\}\{\mathbf{1}(S_j = t_2 - h_2)\\ &\quad-P(S_j = t_2 - h_2)\}\{\mathbf{1}(S_j = t_3 - h_3) - P(S_j = t_3 - h_3)\}].\end{aligned}$$

It is not hard to see that $|CF(t_1,t_2,t_3)| \le Cl$ uniformly over (t_1,t_2,t_3). So $|\mathbb{E}(W_{1N})| \le ClN^{-3}\sum_{t_1,t_2,t_3=1}^N |\mathrm{cum}\{(X_{t_1}-\mu)^{v_1}, (X_{t_2}-\mu)^{v_2}, (X_{t_3}-\mu)^{v_3}\}| = O(l/N^2)$ under Assumption 3.2. Similarly, we have

$$W_{2N} = N^{-3}\sum_{t_1,t_2,t_3=1}^N (X_{t_2}-\mu)^{v_2}(X_{t_1}-\mu)^{v_1}(X_{t_3}-\mu)^{v_3}\mathbb{E}^*(\tilde{f}_{t_2})\mathrm{cov}^*(\tilde{f}_{t_1}, \tilde{f}_{t_3})$$

and $|\mathbb{E}(W_{2N})| \le CN^{-2}$. The same argument yields $|\mathbb{E}(W_{3N})| = O(N^{-2})$ under Assumption 3.2. Therefore (1.6) holds.

It remains to show (1.7). Let $Y_t = \nabla'(X_t - \mu)$. Then

$$M_l N\mathrm{var}^*(J_{0N}) = \frac{M_l}{N}\sum_{t,s=1}^N Y_tY_s\mathrm{cov}^*(\tilde{f}_t, \tilde{f}_s). \tag{1.11}$$

In view of (1.1), the above expression is the same as the TBB variance estimator (PP (2001)) except for the mean correction, so (1.7) basically follows from the argument in the proof of Theorem 1 in PP (2001). We omit the details.

(ii). Following the proof of Part (i), the result follows from the following statements:

$$\begin{aligned} \mathrm{var}\{N\mathrm{var}^*(J_{1N})\} &= O(N^{-2}), & (1.12)\\ \mathrm{var}\{N\mathrm{var}^*(J_{2N})\} &= O(N^{-4}), & (1.13)\\ \mathrm{var}\{N\mathrm{var}^*(J_{0N})\} &= B_2 l/N + o(l/N), & (1.14) \end{aligned}$$

since if (1.12)-(1.14) hold, then by the Cauchy-Schwarz inequality and (1.4)-(1.7),

$$\mathrm{var}\{N\mathrm{cov}^*(J_{0N}, J_{1N})\} = O(1/N),\ \ \mathrm{var}\{N\mathrm{cov}^*(J_{0N}, J_{2N})\} = O(N^{-2})$$

and $\mathrm{var}\{N\mathrm{cov}^*(J_{1N}, J_{2N})\} = o(l/N)$.

To show (1.12), we note from (1.8) that for each $v = v_1 + v_2$, $\|v_1\|_1 = 1$, $\|v_2\|_1 = 1$,

$$\begin{aligned} \mathrm{var}[\mathrm{var}^*\{(\bar{X}_N^* - \mu)^v\}] = \frac{1}{N^8} \sum_{t_j=1, j=1,\cdots,8}^{N} &\mathrm{cov}\{(X_{t_1} - \mu)^{v_1}(X_{t_2} - \mu)^{v_2}(X_{t_3} - \mu)^{v_1} \\ &(X_{t_4} - \mu)^{v_2}, (X_{t_5} - \mu)^{v_1}(X_{t_6} - \mu)^{v_2}(X_{t_7} - \mu)^{v_1}(X_{t_8} - \mu)^{v_2}\} \\ &\times \mathrm{cov}^*(\tilde{f}_{t_1}\tilde{f}_{t_2}, \tilde{f}_{t_3}\tilde{f}_{t_4})\mathrm{cov}^*(\tilde{f}_{t_5}\tilde{f}_{t_6}, \tilde{f}_{t_7}\tilde{f}_{t_8}). \end{aligned}$$

By Theorem 2.3.2 in Brillinger (2001), the major summand that involves the covariance of X_ts can be expressed as linear combinations of product of cumulants up to the 8-th order. Assumption 3.2 (with $r = 8$), in conjunction with (1.9), implies that $\mathrm{var}[\mathrm{var}^*\{(\bar{X}_N^* - \mu)^v\}] = O(N^{-4})$, which results in (1.12). According to (1.10) and Lemma 0.1, we have

$$\mathrm{var}\{N\mathrm{var}^*(J_{2N})\} \le CN^2 \mathbb{E}\{\mathbb{E}^*\|\bar{X}_N^* - \mu\|^6 + \mathbb{E}^*\|\bar{X}_N^* - \mu\|^{6+2\kappa}\}^2 \le CN^{-4}.$$

Finally, (1.14) follows from (1.11) and the argument in Theorem 2 of PP (2001). The proof is complete.

◊

Proof of Theorem 3.2: Let $\Sigma_F = \sum_{k=-\infty}^{\infty} \mathrm{cov}(X_0, X_k)$. Since $\sqrt{N}(\bar{X}_N - \mu) \to_D N(0, \Sigma_F)$, we have $\sqrt{N}\{H(\bar{X}_N) - H(\mu)\} \to_D N(0, \sigma_F^2)$ by the delta method.

Let $\Phi(x)$ be the standard normal cumulative distribution function. By Polyā's theorem,

$$\sup_{x\in\mathbb{R}} |P[\sqrt{N}\{H(\bar{X}_N) - H(\mu)\} \leq x] - \Phi(x/\sigma_F)| = o(1) \text{ as } N \to \infty.$$

Then the first assertion follows if we can show that

$$\sup_{x\in\mathbb{R}} |P^*[\sqrt{N}\{H(\bar{X}_N^*) - \mathbb{E}^*(H(\bar{X}_N^*))\} \leq x] - \Phi(x/\sigma_F)| = o_p(1) \text{ as } N \to \infty.$$

Recall the notation $Y_t = \nabla'(X_t - \mu)$. Based on (1.3), we have

$$H(\bar{X}_N^*) - \mathbb{E}^*\{H(\bar{X}_N^*)\} = N^{-1}\sum_{t=1}^{N} Y_t\{\tilde{f}_t - \mathbb{E}^*(\tilde{f}_t)\} + R_N^* - \mathbb{E}^*(R_N^*).$$

Since $\sqrt{N}\mathbb{E}^*|R_N^*| = o_p(1)$, which is to be shown below, it suffices in view of Lemma 4.1 of Lahiri (2003) to show that

$$\sqrt{M_l/(kl)}\sum_{t=1}^{N} Y_t\{\tilde{f}_t - \mathbb{E}^*(\tilde{f}_t)\} \to_D N(0, \sigma_F^2)$$

in probability. Note that

$$(kl)^{-1}\sqrt{M_l}\sum_{t=1}^{N} Y_t\tilde{f}_t = (kl)^{-1}\sum_{j=1}^{k}\sum_{h=1}^{l} w_l(h)\frac{\sqrt{l}}{\|w_l\|_2}Y_{S_j+h},$$

which is identical to the bootstrap sample mean delivered by the TBB applied to the series Y_t (PP (2001)) except for a mean correction. Thus the remaining proof basically follows the argument in the proof of PP's (2001) Theorem 3. We omit the details.

By Slutsky's theorem, the second assertion follows from $\sqrt{N}[\mathbb{E}^*\{H(\bar{X}_N^*)\} - H(\bar{X}_N)] = o_p(1)$. In view of (1.2) and (1.3), it suffices to show that

$$\sqrt{N}\mathbb{E}^*|R_N^*| = o_p(1), \tag{1.15}$$

$$N^{-1/2}\sum_{t=1}^{N} Y_t\{\mathbb{E}^*(\tilde{f}_t) - 1\} = o_p(1), \tag{1.16}$$

$$\sqrt{N}R_N = o_p(1). \tag{1.17}$$

The assertion (1.15) is true since

$$\begin{aligned}
\mathbb{E}\{\mathbb{E}^*|J_{1N}|\} &\leq C\mathbb{E}\{\mathbb{E}^*\|\bar{X}_N^* - \mu\|^2\} \leq C/N, \\
\mathbb{E}\{\mathbb{E}^*|J_{2N}|\} &\leq C\mathbb{E}[\mathbb{E}^*\{\|\bar{X}_N^* - \mu\|^3(1 + \|\mu\|^\kappa + \|\bar{X}_N^* - \mu\|^\kappa)\}] \leq CN^{-3/2},
\end{aligned}$$

where we have applied Lemma 0.1. Further, since $\mathbb{E}^*(\tilde{f}_t) = N/(N-l+1)$ when $l \le t \le N-l+1$, and is bounded, (1.16) follows. Finally, a Taylor expansion of $H(\bar{X}_N)$ around μ yields

$$R_N = 2\sum_{\|v\|_1=2}(v!)^{-1}(\bar{X}_N-\mu)^v\int_0^1(1-w)D^vH\{\mu+w(\bar{X}_N-\mu)\}dw.$$

Under Assumption 3.1 on $H(\cdot)$, it is straightforward to derive that $\max\{|D^vH(x)| : |v|=2\} \le C(1+\|x\|^{\kappa+1})$, so

$$\mathbb{E}(|R_N|) \le C\mathbb{E}\{\|\bar{X}_N-\mu\|^2(1+\|\mu\|^{\kappa+1}+\|\bar{X}_N-\mu\|^{\kappa+1})\} \le CN^{-1},$$

by Lemma 3.2 of Lahiri (2003). Thus (1.17) holds and this completes the proof. ♢

LEMMA **0.1**. *Assume* $X_t \in \mathcal{L}^{r+\delta}$, $\delta > 0$ *for* $r > 2$, $r \in \mathbb{N}$ *and* $\Delta(\lfloor (r+1)/2 \rfloor; 1) < \infty$. *Then* $\mathbb{E}\{\mathbb{E}^*\|\bar{X}_N^* - \mu\|^r\} \le CN^{-r/2}$.

Proof of Lemma 0.1: It suffices to show that for any v_j, $j = 1, \cdots, m$, where v_j is a m-dimensional unit vector with j-th element being 1 and 0 otherwise, $\mathbb{E}\{\mathbb{E}^*|(\bar{X}_N^*-\mu)^{v_j}|^r\} \le CN^{-r/2}$. Denote by $Z_t = Z_t(j) = (X_t-\mu)^{v_j}$. Under our moment and mixing assumptions, by Lemma 3.2 of Lahiri (2003), we have

$$\mathbb{E}\left|\sum_{t=1}^N Z_t\right|^r \le CN^{r/2}. \quad (1.18)$$

Let $H_j := \sum_{h=1}^l w_l(h)\sum_{t=1}^N Z_t\{\mathbf{1}(S_j = t-h) - P(S_j = t-h)\}$. Note that

$$\begin{aligned}\mathbb{E}^*|(\bar{X}_N^*-\mu)^{v_j}|^r &= N^{-r}\mathbb{E}^*\left|\sum_{t=1}^N Z_t\tilde{f}_t\right|^r \le CN^{-r}\mathbb{E}^*\left|\sum_{j=1}^k H_j\right|^r \\ &\quad + CN^{-r}\left|\sum_{j=1}^k\sum_{h=1}^l w_l(h)\sum_{t=1}^N Z_tP(S_j = t-h)\right|^r \\ &= CN^{-r}(V_1+V_2).\end{aligned}$$

It is easy to see that

$$\begin{aligned}|\mathbb{E}(V_2)| &\le C\mathbb{E}\left|\sum_{t=1}^N Z_t\right|^r \\ &\quad + \frac{C}{N^r}\mathbb{E}\left|\sum_{j=1}^k\sum_{h=1}^l w_l(h)\sum_{t=1}^N Z_t\mathbf{1}(t-h<0 \text{ or } t-h>N-l)\right|^r\end{aligned}$$

which is bounded by $CN^{r/2}$ in view of (1.18). Regarding V_1, we apply Burkholder's inequality and get

$$V_1 \leq C\mathbb{E}^* \left| \sum_{j=1}^{k} H_j^2 \right|^{r/2} \leq C \left(\sum_{j=1}^{k} \mathbb{E}^{*2/r} |H_j|^r \right)^{r/2} \leq Ck^{r/2-1} \sum_{j=1}^{k} \mathbb{E}^* |H_j|^r,$$

whereas by (1.18),

$$\begin{aligned} \mathbb{E}^*|H_j|^r &\leq C\mathbb{E}^* \left| \sum_{h=1}^{l} w_l(h) \sum_{t=1}^{N} Z_t \mathbf{1}(S_j = t-h) \right|^r + \frac{C}{N^r} \left| \sum_{h=1}^{l} w_l(h) \sum_{t=h}^{N-l+h} Z_t \right|^r \\ &\leq (N-l+1)^{-1} \sum_{g=0}^{N-l} \left| \sum_{h=1}^{l} w_l(h) Z_{g+h} \right|^r + \frac{C}{N^r} \left| \sum_{h=1}^{l} w_l(h) \sum_{t=h}^{N-l+h} Z_t \right|^r . \end{aligned}$$

By a variant of Lemma 3.2 of Lahiri (2003), $\mathbb{E}|\sum_{h=1}^{l} w_l(h) Z_h|^r \leq Cl^{r/2}$. Thus we can derive $|\mathbb{E}(V_1)| \leq Ck^{r/2} l^{r/2}$ and $\mathbb{E}\{\mathbb{E}^*|(\bar{X}_N^* - \mu)^{v_j}|^r\} \leq CN^{-r/2}$. The conclusion is established.

$\diamondsuit$